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Asymptotic Distributions of Return for Several Stocks with Trading Volume *

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Abstract

In this paper, we present a nonlinear statistical model which describes directly the interaction relationship between stock's return and itself historical volumes and prices and other stocks' volumes and prices. Further, we prove that a sequence of the returns can converge in distribution to an exponentially Lévy stable distribution or Lévy stable distribution, depending on different value of parameter.

Keywords: Return and trading volume, stable laws, asymptotic distributions.

AMS Subject Classification: 62E20.

§1. Introduction

There is an extensive research into the theoretical and empirical aspects of the stock price and trading volume relationship (see Gallant et al. (1992)). Theoretical models, such as the "MDH" (mixture of distribution hypothesis) model of Tauchen (1983), "SIF" (the sequential information flow) model of Copeland (1976), bivariate model of Glosten (1985), and so on, suggest that volume and price are jointly determined. Relying on the motivation of these models, most of the empirical literatures (see Gallant et al. (1992), Chordia et al. (2000)) test and consistently find evidence for a positive contemporaneous correlation between volume and the price variability. These studies using trading volume all employ some measure of nonevent related normal trading in order to isolate and analyze abnormal trading activity.

Most of these models use indirect variable to explain the price-volume relationship, and therefore, the original data are fully changed in these models.

Recently, Chen (2003) give a model to show price-volume relationship directly:

$$r_{i+1} = r_i + cv_ir_i + \varepsilon_{i+1} \tag{1.1}$$

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where r_i is the return of the ith day, v_i denotes the relative rate of the volume change of the ith day, and the residues, $\varepsilon_i \sim N(0,\sigma^2)$. However, the model (1.1) considers that the return depends on only one day's volume. It is not consistent with practical situation. We know that the price-volume relation should base on the amount accumulation of the volumes. Based on the model (1.1), Qiu and Han (2005) present the following model:

$$r_{i+1} = r_i + c\frac{1}{i}s_i r_i + \varepsilon_{i+1} \tag{1.2}$$

where $s_i = \sum\limits_{j=1}^{i} v_j$ and c is a constant which represents the influence strength of rate of the average volume change on tomorrow's return.

The model (1.2) improves the model (1.1) in two aspects. First, substituting average value of v_i, $\sum\limits_{j=1}^{i} v_i/i$ for v_i. Second, making the hypothesis that v_i and ε_i is subject to the Lévy stable distribution, respectively.

We know that the distributions of financial data hold the "fat tail" property, this can be tested by a large number of empirical results (see Peters (1994)). Qiu and Han (2005) also give a test by the data of the trading volume per day of the index of Shanghai stock market from January 1998 to December 2002. The results show that the exponent of v_i closes to 1.7 (positive tail), and the exponent of ε_i closes to 1.5 (positive tail).

It is well-known that a stock's price can be effected not only by itself historical volumes and prices, but also by the other stocks' volumes and prices, which is true of securities market's practical situation. So we put forward a general non-linear statistical model, and study the asymptotic distribution of a sequence of the return by analyzing the relationship between the return, relative rate of trading volume and its residues. Next section, we describe the model and main results. The proof of several Lemmas and Theorem 1 will be given in section 3. In section 4, we present two special examples. We closed our paper with a problem in section 5.

§2. The Model and Main Results

For a portfolio composed by N stocks, let $r_i^{(j)}$, $V_i^{(j)}$ denote jth stock's return and trading volume at ith day (or week), respectively, and $v_i^{(j)} = (V_{i+1}^{(j)} - V_i^{(j)})/V_i^{(j)}$ denote the rate of trading volume of jth stock. We assume that the distributions of $v_i^{(j)}$ are subject to the stable laws, this is, $v_i^{(j)}$ are sequence of i.i.d. stable random variable, and write $v_i^{(j)} \sim \mathcal{L}(\alpha_j)$, $\alpha_j \in (1,2]$ where $\mathcal{L}(\alpha_j)$ denotes the Lévy stable distribution with the characteristic exponent α_j.

According to the theory of stable distribution (see Uchaikin (1999)), we know that the density function $p_X(x)$ and distribution function $F_X(x)$ of a stable random variable

X satisfy

$$1-F_X(x)=\int_x^{\infty}p_X(s)\mathrm{d}s\approx cx^{-\alpha},\qquad p_X(x)=F_X(x)'\approx c\alpha x^{-\alpha-1} \tag{2.1}$$

as $x\to\infty$, where the parameter α is the characteristic exponent of X, and c is a positive constant.

We put forward the following model:

$$R(n+1)=R(n)+CR(n)+E(n+1)$$

where

$$R(n)=\begin{pmatrix} r_n^{(1)}\\ r_n^{(2)}\\ \vdots\\ r_n^{(N)}\end{pmatrix},\qquad E(n)=\begin{pmatrix} \varepsilon_n^{(1)}\\ \varepsilon_n^{(2)}\\ \vdots\\ \varepsilon_n^{(N)}\end{pmatrix},$$

$$C=\begin{pmatrix} \sum\limits_{i=1}^{N}a_{i1}c_ns_n^{(i)} & 0 & \cdots & 0\\ 0 & \sum\limits_{i=1}^{N}a_{i2}c_ns_n^{(i)} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \sum\limits_{i=1}^{N}a_{iN}c_ns_n^{(i)}\end{pmatrix}.$$

Its jth component is

$$r_{n+1}^{(j)}-r_n^{(j)}=(a_{1j}c_ns_n^{(1)}+a_{2j}c_ns_n^{(2)}+\cdots+a_{Nj}c_ns_n^{(N)})r_n^{(j)}+\varepsilon_{n+1}^{(j)} \tag{2.2}$$

where $\varepsilon_n^{(j)}$ denote the residues, $s_n^{(i)}=\sum\limits_{j=n-[1/c_n]+1}^{n}v_j^{(i)}$ are the cumulative sums of the rate of trading volumes, a_{ij} denotes the influence strength of ith stock's cumulative rate of trading volumes to jth stock's return, c_n denote the structure factors which satisfy $[1/c_n]\le n$, where $[x]$ is the biggest integral smaller than x. Especially, when $c_n=1/n$,

$$c_ns_n^{(i)}=\frac{1}{n}\sum_{j=1}^{n}v_j^{(i)}$$

denotes the ith stock's average rate of trading volumes. We suppose that

$$c_n=f(n+1)-f(n)+\mathrm{o}(1) \tag{2.3}$$

trend to zero decreasingly, and $f(n)$ satisfies the following condition:

There exist $0\le\beta\le1$ and the integrable function $g(n)$ such that

$$f(n)-f(k)=g\Big(\frac{k}{n}\Big)n^{\beta} \tag{2.4}$$

and

$$f(n)=\mathrm{O}(n^{\beta}). \tag{2.5}$$

Next, we mention the main theorem of this paper.

Theorem 2.1 Suppose that $v_i^{(j)}\sim\mathcal{L}(\alpha_v^{(j)})$, $\varepsilon_i^{(j)}\sim\mathcal{L}(\alpha_\varepsilon^{(j)})$, $i=1,2,\cdots,N$, $\alpha_v^{(j)}\in(1,2)$, $\alpha_\varepsilon^{(j)}\in(1,2)$, and $\mathsf{P}(\omega;r_1^{(j)}(\omega)=0)=0$. Let $\mathsf{E}v_i^{(j)}=\mu^{(j)}$, $\rho_j=1+a_{1j}\mu^{(1)}+a_{2j}\mu^{(2)}+\cdots+a_{1N}\mu^{(N)}$ and $\alpha_{\min}=\min(\alpha_v^{(1)},\alpha_v^{(2)},\cdots,\alpha_v^{(N)})$, $I_{\min}=\{i|\alpha_v^{(i)}=\alpha_{\min}\}$.

(i) If $|\rho_j|>1$, then

$$\Big(\frac{|r_{n+1}^{(j)}|}{|\rho_j|^n}\Big)^{1/A_n^{(j)}}\xrightarrow{d}\exp\Big\{\Big[\Big(\sum_{i\in I_{\min}}|a_{ij}|^{\alpha_{\min}}\Big)^{1/\alpha_{\min}}\Big/|\rho_j|\Big]\cdot\mathcal{L}(\alpha_{\min})\Big\} \tag{2.6}$$

where $A_n^{(j)}=\Big(\sum\limits_{k=1}^{n}(g(k/n)n^{\beta})^{\alpha_v^{(j)}}\Big)^{1/\alpha_v^{(j)}}$, and "$\xrightarrow{d}$" denotes the convergence in distribution, as $n\to+\infty$.

(ii) If $|\rho_j|<1$, $\rho_j\neq0$, then

$$r_n^{(j)}\xrightarrow{d}\Big(\frac{1}{1-|\rho_j|^{\alpha_\varepsilon^{(j)}}}\Big)^{1/\alpha_\varepsilon^{(j)}}\mathcal{L}(\alpha_\varepsilon^{(j)}). \tag{2.7}$$

(iii) If $|\rho_j|=1$, then

$$|r_{n+1}^{(j)}|^{1/A_n^{(j)}}\geq\exp\Big\{\Big(\sum_{i\in I_{\min}}|a_{ij}|^{\alpha_{\min}}\Big)^{1/\alpha_{\min}}\mathcal{L}(\alpha_{\min})\Big\}:=\xi \tag{2.8}$$

in distribution, this is, $F_n(x):=\mathsf{P}(|r_{n+1}^{(j)}|^{1/A_n^{(j)}}\leq x)\leq F(x):=\mathsf{P}(\xi\leq x)$, as $n\to\infty$.

Remark This Theorem shows that the limit distributions of a sequence of the returns have different forms in different parameters. When $|\rho_j|>1$, the limit distribution of r_n is of exponential Lévy stable distribution. For $N=1$, (2.6) can be rewritten as $r_{n+1}\sim\rho^n e^{|a/\rho|A_n\mathcal{L}(\alpha_1)}:=\rho^n e^{\mathcal{L}(\alpha_1)\widetilde{A}_n}$. We know that if the rate of return of riskless is r, if $t=kh$, as $h\to0$, the return $\widetilde{r}=(1+rh)^k\to e^{rt}$. The conclusion of Theorem 2.1(i) can be regarded as its general case with risk and random interest rate $\mathcal{L}(\alpha_1)$. So do comprehension when $|\rho_j|<1$.

§3. The Proof of Theorem

We only prove Theorem 2.1 in the case of $N=2$ since the theorem can be proved similarly in other cases. Fixed $j=1$, the model (2.2) can be rewritten as

$$r_{n+1}^{(1)}=(1+a_{11}c_ns_n^{(1)}+a_{21}c_ns_n^{(2)})r_n^{(1)}+\varepsilon_{n+1}^{(1)}. \tag{3.1}$$

Further, for writing more simply in the following proof, let us suppose $a=a_{11}$, $b=a_{21}$, and $\alpha_v^{(i)}=\alpha_i$, $i=1,2$, $\alpha_\varepsilon^{(1)}=\alpha_3$.

To prove Theorem 2.1, we first give several lemmas.

Lemma 3.1 (i) $(1/A_n)\cdot\sum_{k=1}^{n} g(k/n)n^{\beta}(v_j-\mu)\xrightarrow{d}\mathcal{L}(\alpha)$, where α is of the characteristic exponent of stable random variables v_j, and $\mu=\mathsf{E}(v_j)$, $A_n=\Big(\sum_{k=1}^{n}(g(k/n)n^{\beta})^{\alpha}\Big)^{1/\alpha}$,

(ii) $A_n=\mathrm{O}(n^{1/\alpha+\beta})$.

Proof The idea of the proof is analogous with Qiu and Han (2005)'s. 1cm□

Lemma 3.2 Let $\delta>0$ such that $|\rho|-\delta>1$, then

$$\mathsf{P}\Big(\omega;\bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty}|\varepsilon_{j+1}(\omega)|\geq(|\rho|-\delta)^j\Big)=0$$

where ε_j are stable random variables with characteristic exponent α.

Proof See Qiu and Han (2005). 1cm□

Lemma 3.3 Under the assumption of Lemma 3.2,

$$\left|1+\frac{\varepsilon_{n+1}+\sum\limits_{k=2}^{n}\varepsilon_k\prod\limits_{j=k}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})}{r_1^{(1)}\prod\limits_{j=1}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})}\right|^{1/A_n}\to 1\qquad \text{a.s..}$$

Proof Let $w_j=ac_js_j^{(1)}+bc_js_j^{(2)}$ and

$$U_n=\frac{\varepsilon_{n+1}+\sum\limits_{k=2}^{n}\varepsilon_k\prod\limits_{j=k}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})}{r_1^{(1)}\prod\limits_{j=1}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})}=\frac{\varepsilon_{n+1}+\sum\limits_{k=2}^{n}\varepsilon_k\prod\limits_{j=k}^{n}(1+w_j)}{r_1^{(1)}\prod\limits_{j=1}^{n}(1+w_j)},\tag{3.2}$$

then, by $|1+U_n|^{1/A_n}=\exp(\log|1+U_n|/A_n)$, we only need to prove $\log|1+U_n|/A_n\to 0$ a.s..

Since

$$\mathsf{P}(\omega;r_1^{(1)}(\omega)=0)=0,\tag{3.3}$$

and, by the law of large numbers, $c_j(s_j^{(i)}-[1/c_j]\mu_i)\to 0$, so when $j\to\infty$,

$$\begin{aligned}1+w_j&=1+ac_js_j^{(1)}+bc_js_j^{(2)}\\&=1+a\mu_1+b\mu_2+ac_j\Big(s_j^{(1)}-\Big[\frac{1}{c_j}\Big]\mu_1\Big)+bc_j\Big(s_j^{(2)}-\Big[\frac{1}{c_j}\Big]\mu_2\Big)\to\rho,\quad\text{a.s..}\end{aligned}\tag{3.4}$$

$$\mathsf{P}(\omega;1+w_j(\omega)=0,\forall j>0)=0,\qquad\text{a.s..}$$

There exist two zero measure sets, $\aleph_1$ and $\aleph_2$, such that $\forall\,\omega\in\Omega-\aleph_1$, $r_1^{(1)}(\omega)\neq 0$ and $\forall\,\omega\in\Omega-\aleph_2$, $1+w_j(\omega)\neq 0$, $\forall\,j>0$.

Similarly, it follows from Lemma 3.2 and $1+w_j\to\rho$ a.s. that there exist two zero measure sets $\aleph_3$ and $\aleph_4$ such that $\forall\,\omega\in\Omega-\aleph_3$, $\exists\,K_1(\omega)$, such that $\forall\,j\geq K_1(\omega)$, $|\varepsilon_{j+1}(\omega)|<(|\rho|-\delta)^j$ and $\forall\,\omega\in\Omega-\aleph_4$, $\exists\,K_2(\omega)$, such that $\forall\,j>K_2(\omega)$, $|1+w_j(\omega)|\geq|\rho|-\delta$.

Let $\widetilde{\Omega}=\Omega-\bigcup_{i=1}^{4}\aleph_i$ and $K(\omega)=\max\{K_1(\omega),K_2(\omega)\}$. Then $\forall\,\omega\in\widetilde{\Omega}$, we have

$$\begin{aligned}
&|U_n(\omega)|\\
\le\ & \Big|\varepsilon_{n+1}(\omega)\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{n}(1+w_j(\omega))\Big]\Big|+\Big|\varepsilon_n(\omega)\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{n-1}(1+w_j(\omega))\Big]\Big|\\
&+\cdots+\Big|\varepsilon_2(\omega)\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{1}(1+w_j(\omega))\Big]\Big|\\
\le\ & \Big|(|\rho|-\delta)^n\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{K(\omega)-1}(1+w_j(\omega))(|\rho|-\delta)^{n-K(\omega)+1}\Big]\Big|\\
&+\cdots+\Big|(|\rho|-\delta)^{K(\omega)}\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{K(\omega)-1}(1+w_j(\omega))(|\rho|-\delta)\Big]\Big|\\
&+\Big|\varepsilon_{K(\omega)}(\omega)\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{K(\omega)-1}(1+w_j(\omega))\Big]\Big|+\cdots+\Big|\varepsilon_2(\omega)\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{1}(1+w_j(\omega))\Big]\Big|\\
=\ & (n-K(\omega)+1)\Big|(|\rho|-\delta)^{K(\omega)-1}\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{K(\omega)-1}(1+w_j(\omega))\Big]\Big|\\
&+\Big|\varepsilon_{K(\omega)}(\omega)\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{K(\omega)-1}(1+w_j(\omega))\Big]\Big|+\cdots+\Big|\varepsilon_2(\omega)\Big/\Big[r_1^{(1)}(\omega)\prod_{j=1}^{1}(1+w_j(\omega))\Big]\Big|.
\end{aligned}$$

Let

$$M(\omega)=\max\left\{\left|\frac{(|\rho|-\delta)^{K(\omega)-1}}{r_1^{(1)}(\omega)\prod_{j=1}^{K(\omega)-1}(1+w_j(\omega))}\right|,\left|\frac{\varepsilon_i(\omega)}{r_1^{(1)}(\omega)\prod_{j=1}^{i-1}(1+w_j(\omega))}\right|,i=2,3,\cdots,K(\omega)\right\},$$

then $|U_n(\omega)|\le n\cdot M(\omega)$ and

$$\frac{\log|1+U_n(\omega)|}{A_n}\le\frac{\log(1+|U_n(\omega)|)}{A_n}\le\frac{\log(1+nM(\omega))}{A_n}.$$

Because $\log(1+nM(\omega))=\mathrm{O}(\log n)$, $A_n=\mathrm{O}(n^{1/\alpha+\beta})$, we have

$$\frac{\log|1+U_n(\omega)|}{A_n}\to 0,\qquad \forall\,\omega\in\widetilde{\Omega},$$

that is

$$\frac{\log|1+U_n|}{A_n}\to 0\qquad \text{a.s..}\qquad\square$$

Lemma 3.4 Let $r_{n+1}^{(1)}(\rho)=\sum_{k=2}^{n+1}\rho^{n+1-k}\varepsilon_k^{(1)}$, then

$$r_{n+1}^{(1)}(\rho)\xrightarrow{d}\Big(\frac{1}{1-|\rho|^{\alpha_3}}\Big)^{1/\alpha_3}\mathcal{L}(\alpha_3),$$

where $|\rho|<1$.

Proof By Stability Theorem (see Uchaikin et al. (1999)) and the definition of the characteristic function, it can be proved easily. 1cm□

Lemma 3.5 $\forall\, \varepsilon > 0$, $x_n \sim \mathcal{L}(\alpha)$, then $|x_n| < n^{\varepsilon}$, a.s. $n \to \infty$.

Proof When n is large enough,

$$\mathsf{P}(|x_n| \ge n^{\varepsilon}) \approx \int_{n^{\varepsilon}}^{+\infty} a x^{-(\alpha+1)} \mathrm{d}x + \int_{-\infty}^{-n^{\varepsilon}} a(-x)^{-(\alpha+1)} \mathrm{d}x = 2\frac{a}{\alpha} n^{-\varepsilon\alpha} \to 0,$$

so $|x_n| < n^{\varepsilon}$, a.s.. 1cm□

Lemma 3.6 The density function $f(x)$ of U_n in (3.2) is bounded, and

$$|1 + U_n| > \frac{e^{-n^{\epsilon}}}{n^k}, \qquad \text{a.s.} \qquad n \to \infty$$

where $0 < \epsilon < 1/\alpha_1 + \beta$, $k \ge 2$.

Proof If $|f(x)| \le M$, then

$$\begin{aligned} &\sum_{n=1}^{N} \mathsf{P}\Big(|1 + U_n| \le \frac{e^{-n^{\epsilon}}}{n^k}\Big) = \sum_{n=1}^{N} \mathsf{P}\Big(-1 - \frac{e^{-n^{\epsilon}}}{n^k} \le U_n \le -1 + \frac{e^{-n^{\epsilon}}}{n^k}\Big) \\ = &\sum_{n=1}^{N} \int_{-1-e^{-n^{\epsilon}}/n^k}^{-1+e^{-n^{\epsilon}}/n^k} f(x)\mathrm{d}x \le 2M \sum_{n=1}^{N} \frac{e^{-n^{\epsilon}}}{n^k} \le 2M \sum_{n=1}^{N} \frac{1}{n^k} < \infty, \end{aligned}$$

this has denoted that when $n \to \infty$, $\mathsf{P}(|1+U_n| \le e^{-n^{\epsilon}}/n^k) \to 0$, this is, $|1+U_n| > e^{-n^{\epsilon}}/n^k$, a.s..

Let

$$\widetilde{U}_n = \Big[\varepsilon_{n+1}^{(1)} + \sum_{k=2}^{n} \varepsilon_k^{(1)} \prod_{j=k}^{n} (1 + w_j)\Big] \Big/ \prod_{j=1}^{n} (1 + w_j) = \sum_{k=1}^{n} \Big[\varepsilon_{k+1} \Big/ \prod_{j=1}^{k} (1 + w_j)\Big]$$

by (3.3), equivalently, we only need to prove that the density function $f(x)$ of $\widetilde{U}_n$ is bounded.

Let $F(y)$ is the distribution function of U_n.

$$F(y) = \mathsf{P}(\widetilde{U}_n \le y) = \int \cdots \int p_v(v_j^{(1)} \in \mathrm{d}x_j^{(1)}, v_j^{(2)} \in \mathrm{d}x_j^{(2)}, j = 1, 2, \cdots, n) \mathsf{P}_{\varepsilon}\Big(\sum_{k=1}^{n} \frac{\varepsilon_{k+1}}{B_k} \le y\Big)$$

where $B_k = \prod_{j=1}^{k} \Big(1 + a c_j \sum_{i=1}^{j} x_i^{(1)} + b c_j \sum_{i=1}^{j} x_i^{(2)}\Big)$. Since ε_k are the sequence of i.i.d. stable random variable, there exists a random variable ε with the same distribution as ε_k such that

$$\sum_{k=1}^{n} \frac{\varepsilon_{k+1}}{B_k} \overset{d}{=} \frac{\varepsilon}{\widetilde{B}_n}.$$

So the density function $f(x)$ of $\widetilde{U}_n$

$$f(y) = F'(y) = \int \cdots \int p_v(v_j^{(1)} \in \mathrm{d}x_j^{(1)}, v_j^{(2)} \in \mathrm{d}x_j^{(2)}, j = 1, 2, \cdots, n) \widetilde{B}_n p_{\varepsilon}(\widetilde{B}_n y).$$

Since $p_\varepsilon(\cdot)$, $p_v(\cdot)$ are the density function of stable random variable, they are absolutely continuous. When x is large enough, $p(x)=\mathrm{O}(x^{-(\alpha+1)})$. So

$$\begin{aligned} f(y) \ \leq\ & \int\cdots\int_{|\widetilde{B}_n|\leq M_1} M_1 p_\varepsilon(\widetilde{B}_n y) p_v(v_j^{(1)}\in \mathrm{d}x_j^{(1)})^*\cdot p_v(v_j^{(2)}\in \mathrm{d}x_j^{(2)})^* \\ & + \int_{|\widetilde{B}_n|>M_1} |\widetilde{B}_n| p_\varepsilon(\widetilde{B}_n y) p_v(v_j^{(1)}\in \mathrm{d}x_j^{(1)})^*\cdot p_v(v_j^{(2)}\in \mathrm{d}x_j^{(2)})^*. \end{aligned}$$

The first term is bounded, and the second term is of equivalence to $\mathrm{O}(x^{-\sum(\alpha_v^{(j)}+\alpha_\varepsilon)})\to 0$, so $f(y)$ is bounded. 1cm$\square$

Now we can prove the Theorem 2.1.

Proof of Theorem 2.1(i) By (3.1) and (3.4), we have

$$r_{n+1}^{(1)}=r_1^{(1)}\prod_{j=1}^n(1+ac_js_j^{(1)}+bc_js_j^{(2)})+\varepsilon_{n+1}^{(1)}+\sum_{k=2}^n\varepsilon_k^{(1)}\prod_{j=k}^n(1+ac_js_j^{(1)}+bc_js_j^{(2)}) \tag{3.5}$$

and

$$1+w_j=1+ac_js_j^{(1)}+bc_js_j^{(2)}=\rho\Big[1+\frac{a}{\rho}c_j\Big(s_j^{(1)}-\Big[\frac{1}{c_j}\Big]\mu_1\Big)+\frac{b}{\rho}c_j\Big(s_j^{(2)}-\Big[\frac{1}{c_j}\Big]\mu_2\Big)\Big],$$

so

$$\begin{aligned} \Big|\frac{r_{n+1}^{(1)}}{\rho^n}\Big| \ =\ & |r_1^{(1)}|\Big|\prod_{j=1}^n\Big[1+\frac{a}{\rho}c_j\Big(s_j^{(1)}-\Big[\frac{1}{c_j}\Big]\mu_1\Big)+\frac{b}{\rho}c_j\Big(s_j^{(2)}-\Big[\frac{1}{c_j}\Big]\mu_2\Big)\Big]\Big| \\ & \cdot\Big|1+\Big[\varepsilon_{n+1}^{(1)}+\sum_{k=2}^n\varepsilon_k^{(1)}\prod_{j=k}^n(1+w_j)\Big]\Big/\Big[r_1^{(1)}\prod_{j=1}^n(1+w_j)\Big]\Big|. \end{aligned}$$

It follows from Lemma 3.3 that

$$\Big|1+\Big[\varepsilon_{n+1}^{(1)}+\sum_{k=2}^n\varepsilon_k^{(1)}\prod_{j=k}^n(1+w_j)\Big]\Big/\Big[r_1^{(1)}\prod_{j=1}^n(1+w_j)\Big]\Big|^{1/A_n}\to 1 \qquad \text{a.s..}$$

Furthermore, by Lemma 3.3 and Lemma 3.1(ii), we have

$$\forall\,\omega\in\widetilde{\Omega},\qquad |r_1^{(1)}(\omega)|^{1/A_n}\to 1. \tag{3.6}$$

The remain task is to show the convergence of the following

$$\Big|\prod_{j=1}^n\Big[1+\frac{a}{\rho}c_j\Big(s_j^{(1)}-\Big[\frac{1}{c_j}\Big]\mu_1\Big)+\frac{b}{\rho}c_j\Big(s_j^{(2)}-\Big[\frac{1}{c_j}\Big]\mu_2\Big)\Big]\Big|^{1/A_n}. \tag{3.7}$$

When $\alpha_1<\alpha_2$, for all $1<r\leq\alpha_1$, $\mathsf{E}|v_j^{(1)}-\mu_1|^r<\infty$ and $\mathsf{E}|v_j^{(2)}-\mu_2|^r<\infty$. For our purpose, we fix $r\in(2\alpha_1/(\alpha_1+\alpha_1\beta+1),\alpha_1)$. By Marcinkiewicz-Zygmund's strong law of large numbers (see Chow (1988)), we have

$$c_j\Big(s_j^{(i)}-\Big[\frac{1}{c_j}\Big]\mu_1\Big)=\mathrm{o}\Big(\Big[\frac{1}{c_j}\Big]^{1/r-1}\Big)\to 0,\qquad \text{a.s.}\qquad i=1,2,$$

so

$$\widetilde{\varepsilon}_j \triangleq \frac{a}{\rho}c_j\Big(s_j^{(1)}-\Big[\frac{1}{c_j}\Big]\mu_1\Big)+\frac{b}{\rho}c_j\Big(s_j^{(2)}-\Big[\frac{1}{c_j}\Big]\mu_2\Big)\to 0, \qquad \text{a.s.}$$

and

$$\sum_{j=1}^{n}\widetilde{\varepsilon}_j^2=\mathrm{o}\Big(n\Big[\frac{1}{c_n}\Big]^{2/r-2}\Big)\le \mathrm{o}(n^{2/r-1}). \tag{3.8}$$

Since $2/r-1<1/\alpha_1+\beta$, $A_n=\mathrm{O}(n^{1/\alpha_1+\beta})$, thus $(1/A_n)\cdot\sum\limits_{j=1}^{n}\widetilde{\varepsilon}_j^2\to 0$.

Therefore

$$\begin{aligned}
&\Big|\prod_{j=1}^{n}\Big(1+\frac{a}{\rho}c_j\Big(s_j^{(1)}-\Big[\frac{1}{c_j}\Big]\mu_1\Big)+\frac{b}{\rho}c_j\Big(s_j^{(2)}-\Big[\frac{1}{c_j}\Big]\mu_2\Big)\Big)\Big|^{1/A_n}\\
=&\Big|\prod_{j=1}^{n}\exp\Big(\frac{\widetilde{\varepsilon}_j+\mathrm{O}(\widetilde{\varepsilon}_j^2)}{A_n}\Big)\Big|=\exp\Big[\Big(\sum_{j=1}^{n}\widetilde{\varepsilon}_j\Big)\Big/A_n+\mathrm{O}\Big(\sum_{j=1}^{n}\widetilde{\varepsilon}_j^2\Big)\Big/A_n\Big]\\
=&\exp\Big[\Big(\sum_{j=1}^{n}\widetilde{\varepsilon}_j\Big)\Big/A_n+\mathrm{o}(1)\Big].
\end{aligned}\tag{3.9}$$

Let $\overline{v}_j^{(i)}=v_j^{(i)}-\mu_i$, $\overline{s}_j^{(i)}=s_j^{(i)}-[1/c_j]\mu_i=\sum\limits_{k=j-[1/c_j]+1}^{j}\overline{v}_k^{(i)}$, $i=1,2$, then

$$\begin{aligned}
&\sum_{j=1}^{n}c_j\Big(s_j^{(i)}-\Big[\frac{1}{c_j}\Big]\mu_i\Big)\\
=&\sum_{j=1}^{n}(f(j+1)-f(j))(\overline{s}_j^{(i)}+\overline{v}_{j+1}^{(i)}-\overline{v}_{j+1}^{(i)})\\
=&\sum_{j=1}^{n}f(j+1)\overline{s}_{j+1}^{(i)}-f(j)\overline{s}_j^{(i)}-f(j+1)\overline{v}_{j+1}^{(i)}\\
=&f(n+1)\overline{s}_{n+1}^{(i)}-f(1)\overline{v}_1^{(i)}-\sum_{j=1}^{n}f(j+1)\overline{v}_{j+1}^{(i)}\\
=&f(n+1)\sum_{j=n+1-[1/c_{n+1}]+1}^{n+1}\overline{v}_j^{(i)}-\sum_{j=1}^{n+1}f(j)\overline{v}_j^{(i)}\\
=&\sum_{k=1}^{n+1}(f(n+1)-f(k))\overline{v}_k^{(i)}-\sum_{j=1}^{n+1-[1/c_{n+1}]}\overline{v}_j^{(i)}f(n+1)\\
=&\sum_{k=1}^{n+1}g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(i)}-\sum_{j=1}^{n+1-[1/c_{n+1}]}\overline{v}_j^{(i)}f(n+1)+\mathrm{o}(1).
\end{aligned}\tag{3.10}$$

Since

$$\sum_{j=1}^{n+1-[1/c_{n+1}]}\overline{v}_j^{(i)}=\overline{s}_{n+1-[1/c_{n+1}]}^{(i)}=\mathrm{o}((n+1)^{1/\alpha_i})$$

and $f(n)=\mathrm{O}(n^{\beta})$, so

$$\Big[\sum_{j=1}^{n+1-[1/c_{n+1}]}\overline{v}_j^{(i)}f(n+1)\Big]\Big/A_n=\frac{\mathrm{o}((n+1)^{1/\alpha_1+\beta})}{\mathrm{O}(n^{1/\alpha_1+\beta})}=\mathrm{o}(1). \tag{3.11}$$

By Lemma 3.1 and $\alpha_1 < \alpha_2$

$$\begin{aligned}\Big[\sum_{k=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(2)}\Big]\Big/A_n &= \Big[\sum_{j=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(2)}\Big]\Big/\mathrm{O}((n+1)^{1/\alpha_2+\beta}) \\ &\quad\cdot\frac{\mathrm{O}((n+1)^{1/\alpha_2+\beta})}{\mathrm{O}((n+1)^{1/\alpha_1+\beta})}\cdot\frac{\mathrm{O}((n+1)^{1/\alpha_1+\beta})}{\mathrm{O}((n)^{1/\alpha_1+\beta})}.\end{aligned}$$

Let

$$x_n = \Big[\sum_{j=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(2)}\Big]\Big/\mathrm{O}((n+1)^{1/\alpha_2+\beta}),$$

then, by Lemma 3.1, $x_n \xrightarrow{d} \mathcal{L}(\alpha_2)$, therefore, by Lemma 3.5, $|x_n| < n^{\varepsilon}$. Taking $0 < \varepsilon < 1/\alpha_1 - 1/\alpha_2$, then

$$\Big[\sum_{k=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(2)}\Big]\Big/A_n \to 0. \tag{3.12}$$

So, by (3.11) and (3.12)

$$\begin{aligned}\frac{1}{A_n}\sum_{j=1}^{n}\widetilde{\varepsilon}_j &= \frac{1}{A_n}\Big[\frac{a}{\rho}c_j\Big(s_j^{(1)} - \Big[\frac{1}{c_j}\Big]\mu_1\Big) + \frac{b}{\rho}c_j\Big(s_j^{(2)} - \Big[\frac{1}{c_j}\Big]\mu_2\Big)\Big] \\ &= \frac{1}{A_n}\Big[\frac{a}{\rho}\sum_{k=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(1)} + \frac{b}{\rho}\sum_{k=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(2)} + \mathrm{o}(1)\Big] \\ &= \frac{1}{A_n}\Big[\frac{a}{\rho}\sum_{k=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(1)} + \mathrm{o}(1)\Big].\end{aligned}$$

Thus, by applying Lemma 3.1 and the properties of characteristic functions, we know when $\alpha_1 < \alpha_2$, (2.6) hold.

When $\alpha = \alpha_1 = \alpha_2$, $v_i^{(1)} \sim v_i^{(2)}$, then

$$\frac{1}{A_n}\sum_{j=1}^{n}\widetilde{\varepsilon}_j = \frac{1}{A_n}\Big[\frac{a}{\rho}\sum_{k=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(1)} + \frac{b}{\rho}\sum_{k=1}^{n+1} g\Big(\frac{k}{n+1}\Big)(n+1)^{\beta}\overline{v}_k^{(1)} + \mathrm{o}(1)\Big]. \tag{3.13}$$

Since the characteristic function of $aY_1 + bY_2$ is

$$f_{aY_1+bY_2}(t) = f(at)\cdot f(bt) = f(t)^{|a|^{\alpha}} f(t)^{|b|^{\alpha}} = f(t)^{(|a|^{\alpha}+|b|^{\alpha})}$$

and it is also the characteristic function of $(|a|^{\alpha} + |b|^{\alpha})^{1/\alpha}\mathcal{L}(\alpha)$, so by (3.12), we can immediately obtain (2.6).

Proof of Theorem 2.1(ii) To (3.5), by Lemma 3.4, we only need to prove $r_{n+1}^{(1)} - r_{n+1}^{(1)}(\rho) \to 0$, a.s..

Let $\widetilde{\rho}$ and $\widetilde{a}$ be two constants such that $|\rho| < |\widetilde{\rho}| < 1$ and

$$\widetilde{a} > \frac{1}{\log(1/|\widetilde{\rho}|)} = \frac{1}{-\log|\widetilde{\rho}|} > 0.$$

Again let

$$n' = [(\widetilde{a}+1)\log n], \tag{3.14}$$

then

$$\begin{aligned} r_{n+1}^{(1)} - r_{n+1}^{(1)}(\rho) &= r_1^{(1)} \prod_{j=1}^{n} (1 + ac_j s_j^{(1)} + bc_j s_j^{(2)}) \\ &\quad + \sum_{k=2}^{n-n'} \Big[\varepsilon_k^{(1)} \prod_{j=k}^{n} (1 + ac_j s_j^{(1)} + bc_j s_j^{(2)}) - \varepsilon_k^{(1)} \rho^{n+1-k} \Big] \\ &\quad + \sum_{k=n-n'+1}^{n} \varepsilon_k^{(1)} \Big[\prod_{j=k}^{n} (1 + ac_j s_j^{(1)} + bc_j s_j^{(2)}) - \rho^{n+1-k} \Big] \\ &= I_0(n) + I_1(n) + I_2(n). \end{aligned}$$

(I) Since $1 + ac_j s_j^{(1)} + bc_j s_j^{(2)} \to 1 + a\mu_1 + b\mu_2 = \rho$ a.s., we can find a zero measure set $\aleph_1$ such that $\forall\, \omega \in \Omega - \aleph_1$, there exists a number $K_1(\omega)$ such that $\forall\, j \geq K_1(\omega)$, $|1 + ac_j s_j^{(1)}(\omega) + bc_j s_j^{(2)}(\omega)| \leq |\widetilde{\rho}| < 1$. By (3.3) we have a zero measure set $\aleph_2$ such that $\forall\, \omega \in \Omega - \aleph_2$, $r_1^{(1)}(\omega) \neq 0$. Thus, for $\forall\, \omega \in \Omega - \aleph_1 \cup \aleph_2$, we have

$$\begin{aligned} |I_0(n)| &= \Big| r_1^{(1)}(\omega) \prod_{j=1}^{n} (1 + ac_j s_j^{(1)}(\omega) + bc_j s_j^{(2)}(\omega)) \Big| \\ &\leq \Big| r_1^{(1)}(\omega) \prod_{j=1}^{K_1(\omega)} (1 + ac_j s_j^{(1)}(\omega) + bc_j s_j^{(2)}(\omega)) \Big| \cdot |\widetilde{\rho}|^{n+1-K_1(\omega)} \to 0, \qquad n \to \infty. \end{aligned}$$

(II) $|I_1(n)| \leq \sum\limits_{k=2}^{n-n'} |\varepsilon_k^{(1)}| \Big| \prod\limits_{j=k}^{n} (1 + ac_j s_j^{(1)} + bc_j s_j^{(2)}) \Big| + \sum\limits_{k=2}^{n-n'} |\varepsilon_k^{(1)}| |\rho|^{n+1-k}$,

by the property of the stable laws

$$\sum_{k=1}^{\infty} \mathsf{P}(|\varepsilon_k^{(1)}| > k) = \mathrm{O}\Big(\sum_{k=1}^{\infty} \frac{1}{k^{\alpha_3}} \Big) < \infty,$$

so we can find a zero measure set $\aleph_3$ such that $\forall\, \omega \in \Omega - \aleph_3$ there exists a number $K_2(\omega)$ such that $\forall\, k \geq K_2(\omega)$, $|\varepsilon_k^{(1)}(\omega)| \leq k$. Let $K(\omega) = \max\{K_1(\omega), K_2(\omega)\}$ and

$$M(\omega) = \sum_{k=2}^{K(\omega)} |\varepsilon_k^{(1)}(\omega)| \prod_{j=k}^{K(\omega)} |1 + ac_j s_j^{(1)}(\omega) + bc_j s_j^{(2)}(\omega)|.$$

Then $\forall\, \omega \in \Omega - \aleph_1 \cup \aleph_3$, we have

$$\begin{aligned} & \sum_{k=2}^{n-n'} |\varepsilon_k^{(1)}(\omega)| \prod_{j=k}^{n} |1 + ac_j s_j^{(1)}(\omega) + bc_j s_j^{(2)}(\omega)| \\ \leq\ & \sum_{k=2}^{K(\omega)} |\varepsilon_k^{(1)}(\omega)| \prod_{j=k}^{K(\omega)} |1 + ac_j s_j^{(1)}(\omega) + bc_j s_j^{(2)}(\omega)| \cdot |\widetilde{\rho}|^{n-K(\omega)} \\ & + \sum_{k=K(\omega)+1}^{n-n'} |\varepsilon_k^{(1)}(\omega)| \cdot |\widetilde{\rho}|^{n-k+1} \\ \leq\ & M(\omega) |\widetilde{\rho}|^{n-K(\omega)} + \sum_{k=K(\omega)+1}^{n-n'} k \cdot |\widetilde{\rho}|^{n-k+1} \\ \leq\ & M(\omega) |\widetilde{\rho}|^{n-K(\omega)} + \frac{|\widetilde{\rho}|^{n'+1}}{1 - |\widetilde{\rho}|} (n - n'). \end{aligned}$$

Because of $|\widetilde{\rho}|^{n'+1}(n-n') \le |\widetilde{\rho}|^{n'} \cdot n = \exp(n'\log|\widetilde{\rho}|)n = \exp(-n'\log(1/|\widetilde{\rho}|)) \cdot n$, by (3.14) we have

$$\begin{aligned}
\exp\Big(-n'\log\frac{1}{|\widetilde{\rho}|}\Big)\cdot n &= \exp\Big(\frac{-[(\widetilde{a}+1)\log n]}{\widetilde{a}}\widetilde{a}\log\frac{1}{|\widetilde{\rho}|}\Big)\cdot n \\
&= (1+\mathrm{o}(1))\exp\Big(-\Big(1+\frac{1}{\widetilde{a}}\Big)\widetilde{a}\log\frac{1}{|\widetilde{\rho}|}\log n\Big)\cdot n \\
&= (1+\mathrm{o}(1))\frac{n}{n^{(1+1/\widetilde{a})\widetilde{a}\log(1/|\widetilde{\rho}|)}} \to 0
\end{aligned}$$

and $M(\omega)|\widetilde{\rho}|^{n-K(\omega)} \to 0$, so

$$\sum_{k=2}^{n-n'}|\varepsilon_k^{(1)}|\prod_{j=k}^{n}|1+ac_js_j^{(1)}(\omega)+bc_js_j^{(2)}(\omega)| \to 0 \qquad \text{a.s..}$$

Similarly, we have

$$\sum_{k=2}^{n-n'}|\varepsilon_k^{(1)}|\cdot|\rho|^{n+1-k} \to 0 \qquad \text{a.s..}$$

(III) As we know that $\{\varepsilon_k^{(1)}\}$ is a sequence of i.i.d. random variables, by the strong law of large numbers we can find zero measure set $\aleph_4$, $\forall\,\omega \in \Omega - \aleph_4$,

$$\Big[\sum_{k=n-n'+1}^{n}\varepsilon_k^{(1)}(\omega)\Big]\Big/n' \to \mu_3. \tag{3.15}$$

Note that $\widetilde{\varepsilon}_j = (a/\rho)\cdot c_j(s_j^{(1)}-[1/c_j]\mu_1)+(b/\rho)\cdot c_j(s_j^{(2)}-[1/c_j]\mu_2) = \mathrm{o}(j^{1/r-1}) \to 0$, so

$$\begin{aligned}
|I_2(n)| &\le \sum_{k=n-n'+1}^{n}|\varepsilon_k^{(1)}|\cdot|\rho|^{n+1-k}\Big|\prod_{j=k}^{n}(1+\widetilde{\varepsilon}_j)-1\Big| \\
&= \sum_{k=n-n'+1}^{n}|\varepsilon_k^{(1)}|\cdot|\rho|^{n+1-k}\Big|\sum_{j=k}^{n}\widetilde{\varepsilon}_j+\mathrm{O}\Big(\sum_{k\le i<j\le n}\widetilde{\varepsilon}_i\widetilde{\varepsilon}_j\Big)\Big| \\
&= (1+\mathrm{o}(1))\sum_{k=n-n'+1}^{n}|\varepsilon_k^{(1)}|\cdot|\rho|^{n+1-k}\cdot\Big|\sum_{j=k}^{n}\widetilde{\varepsilon}_j\Big| \\
&\le (1+\mathrm{o}(1))\sum_{k=n-n'+1}^{n}|\varepsilon_k|\cdot|\rho|^{n+1-k}\cdot n'\cdot(n-n')^{1/r-1} \\
&\le (1+\mathrm{o}(1))\sum_{k=n-n'+1}^{n}\frac{|\varepsilon_k^{(1)}|}{n'}\cdot|c|\cdot\frac{(n')^2}{(n-n')^{1-1/r}}
\end{aligned}$$

and therefore, by $n' = \mathrm{O}(\log n)$ and (3.15),

$$\sum_{k=n-n'+1}^{n}\frac{|\varepsilon_k^{(1)}|}{n'}\cdot|c|\cdot\frac{(n')^2}{(n-n')^{1-1/r}} \to 0 \qquad \text{a.s..}$$

Thus

$$r_{n+1}-r_{n+1}(\rho) \to 0 \qquad \text{a.s..}$$

Proof of Theorem 2.1(iii) Since $|\rho|=1$, it follows from (3.5) that

$$|r_{n+1}^{(1)}|=|r_1^{(1)}|\cdot\Big|\prod_{j=1}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})\Big|\cdot\left|1+\frac{\varepsilon_{n+1}^{(1)}+\sum\limits_{k=2}^{n}\varepsilon_k^{(1)}\prod\limits_{j=k}^{n}(1+w_j)}{r_1^{(1)}\prod\limits_{j=1}^{n}(1+w_j)}\right|.$$

Apparently, (3.6) still hold, and we can prove by the same method with the proof of Theorem 2.1(i) that $\Big|\prod\limits_{j=1}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})\Big|^{1/A_n}$ possesses analogical result as Theorem 2.1(i). The remain task is to prove

$$|1+U_n|=\left|1+\frac{\varepsilon_{n+1}^{(1)}+\sum\limits_{k=2}^{n}\varepsilon_k^{(1)}\prod\limits_{j=k}^{n}(1+w_j)}{r_1^{(1)}\prod\limits_{j=1}^{n}(1+w_j)}\right|^{1/A_n}\geq 1\qquad \text{a.s..}\tag{3.16}$$

By Lemma 3.6,

$$\frac{\log|1+U_n|}{A_n}\geq\frac{-n^{\epsilon}-k\log n}{n^{1/\alpha_1+\beta}}=-n^{\epsilon-1/\alpha_1-\beta},$$

when $\epsilon-1/\alpha_1-\beta<0$, this is, $0<\epsilon<1/\alpha_1+\beta$,

$$\frac{\log|1+U_n|}{A_n}\geq 0\qquad (n\to\infty),$$

so $e^{\log|1+U_n|/A_n}\geq 1$, this is (3.16).

This completes the proof of the theorem. □

§4. Application of Theorem

We give two special examples.

Example 1 In the model (2.2), let $C_n=1/n$, therefore $c_ns_n^{(i)}=\sum\limits_{j=1}^{n}v_j^{(i)}/n$ are the average of rate of volumes. Since $\log(n+1)-\log(n)=\log(1+1/n)=1/n+o(1/n)$, we can take $f(n)=\log(n)$, then $f(n)-f(k)=\log(n/k)=-\log(k/n)$. Taking $g(k/n)=-\log(k/n)$, apparently, $g(x)=-\log(x)$ is of integrable function on interval $[0,1]$. In the situation, A_n in the Theorem is

$$A_n^{(j)}=\Big(\sum_{i=1}^{n}\Big(\log\frac{n+1}{i}\Big)^{\alpha_v^{(j)}}\Big)^{1/\alpha_v^{(j)}}.$$

Example 2 In the model (2.2), let $C_n=1/n^{\delta}$. When $\delta=0$, $s_n^{(i)}=\sum\limits_{j=n-1+1}^{n}v_j^{(i)}=v_n^{(i)}$. The model (1.1) is its special case.

For $0<\delta<1$, since

$$(n+1)^{1-\delta}-n^{1-\delta}=n^{1-\delta}[(1+1/n)^{1-\delta}-1]=n^{1-\delta}[(1-\delta)(1/n)+\mathrm{O}(1/n^2)]=(1-\delta)(1/n^{\delta}),$$

therefore let $f(n)=[1/(1-\delta)]n^{1-\delta}$,

$$f(n)-f(k)=[1/(1-\delta)](n^{1-\delta}-k^{1-\delta})=[1/(1-\delta)]n^{1-\delta}(1-(k/n)^{1-\delta})=g(k/n)n^{1-\delta}$$

where $g(k/n)=[1/(1-\delta)](1-(k/n)^{1-\delta})$, $0<\beta=1-\delta<1$. Apparently, $g(x)=(1/\beta)\cdot(1-x^{\beta})$ is an integrable function on interval $[0,1]$, and $f(n)=\mathrm{O}(n^{\beta})$.

This model shows that we can only consider the influence of near n^{β} days' volumes acting on the return. This is in accordance with the practice situation.

§5. Summary and Discussion

In the proof of Theorem 2.1(i), $A_n^{(j)}$ play an important role. Generally, $A_n^{(j)}$ can be written as

$$A_n^{(j)}=\Big[\int_0^1 g^{\alpha_v^{(j)}}(x)\mathrm{d}x\cdot n^{\beta\alpha_v^{(j)}+1}+\mathrm{o}(1)\Big]^{1/\alpha_v^{(j)}}=G^{(j)}\cdot n^{\beta+1/\alpha_v^{(j)}}+\mathrm{o}(1)$$

where

$$G^{(j)}=\Big[\int_0^1 g^{\alpha_v^{(j)}}(x)\mathrm{d}x\Big]^{1/\alpha_v^{(j)}}.$$

For $C_n=1/n$, since (see Qi (2003))

$$\frac{\sum\limits_{j=1}^{n}\Big(\log\dfrac{n+1}{j}\Big)^{\alpha_v^{(j)}}}{n}\to\int_0^{\infty}x^{\alpha_v^{(j)}}e^{-x}=\Gamma(\alpha_v^{(j)}+1)$$

so

$$A_n^{(j)}=\Big(\sum_{i=1}^{n}\Big(\log\frac{n+1}{i}\Big)^{\alpha_v^{(j)}}\Big)^{1/\alpha_v^{(j)}}\to\Gamma(\alpha_v^{(j)}+1)^{1/\alpha_v^{(j)}}\cdot n^{1/\alpha_v^{(j)}}.$$

$A_n^{(j)}$ satisfies the power law.

Furthermore, when $|\rho_j|>1$ the stock price fluctuates more,

$$\begin{aligned}\log\frac{|r_{n+1}^{(j)}|}{|\rho_j|^n}&\to A_n^{(j)}\cdot\frac{(\sum|a_{ij}|^{\alpha_{\min}})^{1/\alpha_{\min}}}{|\rho_j|}\cdot\mathcal{L}(\alpha_{\min})\\&\to G^{(j)}\cdot n^{\beta+1/\alpha_{\min}}\cdot\frac{(\sum|a_{ij}|^{\alpha_{\min}})^{1/\alpha_{\min}}}{|\rho_j|}\cdot\mathcal{L}(\alpha_{\min}),\end{aligned}$$

$|r_{n+1}^{(j)}|$ also satisfies the power law.

For $|\rho|=1$, we can only give the lower boundary. When $\rho=1$, $\mathsf{E}(ac_ns_n^{(1)}+bc_ns_n^{(2)})=0$, $ac_ns_n^{(1)}+bc_ns_n^{(2)}$ can be either positive or negative, so $\prod\limits_{n=1}^{N}(1+ac_ns_n^{(1)}+bc_ns_n^{(2)})$ trends to either 0 or ∞. From the proof of Theorem 2.1, we can know that when $|\rho|>1$, the first term of (3.5) $r_1^{(1)}\prod\limits_{j=1}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})$ plays the important role and when $|\rho|<1$, the behind part of (3.7) $\sum\limits_{k=2}^{n}\varepsilon_k^{(1)}\prod\limits_{j=k}^{n}(1+ac_js_j^{(1)}+bc_js_j^{(2)})$ plays the key role. We guess when $|\rho|=1$, the two parts should all make effect.

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考虑成交量的股票收益率的渐近分布

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在本文中, 我们提出了一类直接描述多只股票收益率受股票本身的量价影响及其它股票的量价影响下的非线性统计模型. 我们证明了在不同的参数取值下, 收益率序列分别依分布收敛于Lévy指数稳定分布和Lévy稳定分布.

关 键 词: 收益率和成交量, 稳定律, 渐近分布.

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