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On the Limit Behaviour of the Population-Size-Dependent Bisexual Branching Processes*

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Abstract

In this paper, a bisexual Galton-Watson branching process with the law of offspring distribution dependent on the population size is investigated. Under a suitable assumption on the offspring distribution, for the supercritical case, the limit behaviours on almost sure convergence of the process are established.

Keywords: Bisexual Galton-Watson branching processes, population-size-dependent branching processes, almost sure convergence.

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§1. Introduction

The bisexual Galton-Watson process was first introduced by Daley (see [1]) as a two-type branching model which is a modification of the standard Galton-Watson branching process. This model has received much attention in the literature (see for example [2]–[6]). In Daley's model the offspring reproduction laws are independent and identical distribution. Recently, Xing and Wang (see [9]) have introduced a bisexual Galton-Watson process whose offspring reproduction laws depend on the size of population, i.e. population-size-dependent bisexual Galton-Watson process (PSDBP). The biological background is that population size governs the reproduction laws. The mathematical model can be described as follows:

Definition 1.1 A bisexual Galton-Watson process $\{Z_n\}_{n=0}^{\infty}$ is called population-size-dependent bisexual branching process if it satisfies that

$$Z_0 = N, \tag{1.1}$$

$$(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (\xi_{n,i}^{(Z_n)}, \eta_{n,i}^{(Z_n)}), \qquad n = 0, 1, 2, \cdots, \tag{1.2}$$

$$Z_{n+1} = L(F_{n+1}, M_{n+1}), \qquad n = 0, 1, 2, \cdots, \tag{1.3}$$

where N is a positive integer, the empty sum is regarded as $(0, 0)$, and we make assumption that $(\xi_{n,i}^{(k)}, \eta_{n,i}^{(k)})$ $(n = 0, 1 \cdots;\ k, i = 1, 2, \cdots)$ are independent of each other, and for each $k = 1, 2, \cdots$,

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$(\xi_{n,i}^{(k)},\eta_{n,i}^{(k)})$ has the same distribution as $(\xi_{0,1}^{(k)},\eta_{0,1}^{(k)})$ for all $n,i=1,2,\cdots$, and the mating function $L:R^+\times R^+\longrightarrow R^+$ is assumed to be non-decreasing in each argument, integer-valued for integer-valued arguments with $L(x,y)\le xy$.

Intuitively, when the population size in the nth generation Z_n is given, then $\xi_{n,i}^{(Z_n)}$ and $\eta_{n,i}^{(Z_n)}$ represent the respective numbers of the female and the male produced by the ith mating unit in the nth generation which depend on population size in the nth generation. F_n and M_n denote the respective numbers of the female and the male in the nth generation. By some mating rule, they produce Z_n mating units ($Z_n=L(F_n,M_n)$) and then each mating unit produces the new generation independently.

It is easy to check that $\{(F_n,M_n)\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ are Markov chains with stationary transition probabilities and 0 is an absorbing state.

Definition 1.2 A PSDBP is called *superadditive* if for all positive integers $n\ge 1$, the mating function $L(\cdot,\cdot)$ satisfies

$$L\Big(\sum_{i=1}^{n}x_i,\sum_{i=1}^{n}y_i\Big)\ge\sum_{i=1}^{n}L(x_i,y_i),\qquad x_i,y_i\in R^+,\ i=1,\cdots,n. \tag{1.4}$$

As usual, we assume L is superadditive throughout this paper.

In this paper, we shall consider the supercritical PSDBP with superadditive mating function and investigate the asymptotic behaviour under the following Assumption A i.e. research the almost sure convergence of the sequences $\{r^{-n}Z_n\}_{n=0}^{\infty}$, $\{r^{-n}F_n\}_{n=0}^{\infty}$ and $\{r^{-n}M_n\}_{n=0}^{\infty}$.

§ 2. Basic Assumptions and Preliminaries

In this section, we make the following assumption and state some preliminary results on the sequence of bivariate random variables $\{(\xi_{0,1}^{(k)},\eta_{0,1}^{(k)})\}_k$.

Assumption A The sequence $\{(\xi_{0,1}^{(k)},\eta_{0,1}^{(k)})\}_k$ satisfies

$$\mathsf{E}g(\xi_{0,1}^{(k+1)},\eta_{0,1}^{(k+1)})\le\mathsf{E}g(\xi_{0,1}^{(k)},\eta_{0,1}^{(k)}),\qquad k=0,1,\cdots \tag{2.1}$$

for every bounded component-wise increasing function $g(\cdot,\cdot)$.

Under Assumption A, the following results (see e.g. [7]) are useful for our later purpose.

Proposition 2.1 There exist random variables $(\xi^{(k)},\eta^{(k)})^\star$, $(\xi^{(k+1)},\eta^{(k+1)})^\star$ and $(\xi^{(k,k+1)},\eta^{(k,k+1)})$ defined on the same probability space, with the former two bivariate random variables having the same respective distributions as $(\xi_{0,1}^{(k)},\eta_{0,1}^{(k)})$ and $(\xi_{0,1}^{(k+1)},\eta_{0,1}^{(k+1)})$, such that

$$(\xi^{(k)},\eta^{(k)})^\star=(\xi^{(k+1)},\eta^{(k+1)})^\star+(\xi^{(k,k+1)},\eta^{(k,k+1)}),\qquad k=0,1,\cdots \tag{2.2}$$

for non-negative integer-valued random variables $(\xi^{(k,k+1)},\eta^{(k,k+1)})$.

Next, by an abuse of notation, we will use $(\xi^{(k)},\eta^{(k)})$ instead of $(\xi^{(k)},\eta^{(k)})^\star$ and use $\{Z_n\}_{n=0}^{\infty}$ instead of the process $\{Z_n^\star\}_{n=0}^{\infty}$ corresponding to the sequence $(\xi^{(k)},\eta^{(k)})^\star$. Thus we have

Proposition 2.2 Under Assumption A, we have that

(1) $\{(\xi^{(k)},\eta^{(k)})\}_k$ is a monotonic non-increasing sequence and converges almost surely to a pair of nonnegative, integer-valued random variables (ξ,η).

(2) $\{\mathsf{E}g(\xi^{(k)},\eta^{(k)})\}_k$ is a monotonic non-increasing sequence and converges to $\mathsf{E}g(\xi,\eta)$, where $g(\cdot,\cdot)$ is defined as above in Assumption A, where the bivariate random variables $(\xi^{(k)},\eta^{(k)})$ have the same distribution as $(\xi_{0,1}^{(k)},\eta_{0,1}^{(k)})$.

Let $\mu_1^{(k)}:=\mathsf{E}\xi^{(k)}$, $\mu_2^{(k)}:=\mathsf{E}\eta^{(k)}$ and $\mu_1:=\mathsf{E}\xi$, $\mu_2:=\mathsf{E}\eta$. Take $g(x,y)=x$, or y, then we have that

$$\lim_{n\to\infty}\mu_1^{(n)}=\mu_1,\qquad \lim_{n\to\infty}\mu_2^{(n)}=\mu_2.$$

An important factor in the study of PSDBP is the mean growth rate per mating unit, which was defined in [5] for the bisexual Galton-Watson process, by $r_k:=(1/k)\cdot\mathsf{E}[Z_{n+1}|Z_n=k]$, $k=1,2,\cdots$.

Under Assumptions A, Xing and Wang [9] proved that $r:=\lim\limits_{k\to\infty}r_k$ exists and showed that $\mathsf{P}(Z_n\to 0)+\mathsf{P}(Z_n\to\infty)=1$. Moreover, if $r<1$, then $\mathsf{P}(Z_n\to 0|Z_0=j)=1$; and if $r>1$, then $\mathsf{P}(Z_n\to 0|Z_0=j)<1$, $j=1,2,\cdots$.

A PSDBP defined by (1.1), (1.2) and (1.3) is called *subcritical*, *critical*, or *supercritical* respectively according to $r<1$, $=1$, or >1.

§3. The Almost Sure Convergence of the Normed Sequences

In this section, we aim to investigate the a.s. convergence of the sequences $\{r^{-n}Z_n\}_{n=0}^{\infty}$ and $\{r^{-n}F_n\}_{n=0}^{\infty}$ ($\{r^{-n}M_n\}_{n=0}^{\infty}$ as well).

Let

$$r_k':=\frac{1}{k}\mathsf{E}\Big[L\Big(\sum_{i=1}^{k}(\xi_{n,i},\eta_{n,i})\Big)\Big],$$

where $(\xi_{n,i},\eta_{n,i})$ $(i=1,2,\cdots;\ n=0,1,\cdots)$ are i.i.d. nonnegative, integer-valued random variables and have the same probability distribution as (ξ,η), ((ξ,η) is the same as that of Proposition 2.2(1)). Let $\varepsilon_k:=r-r_k$, $k=1,2,\cdots$, then $\varepsilon_k\to 0$ as $k\to\infty$. Define $W_n:=r^{-n}Z_n$.

Theorem 3.1 If $r_1'>0$ and $\{|\varepsilon_k|\}_{k=1}^{\infty}$ is a decreasing sequence satisfying $\sum\limits_{k=1}^{\infty}k^{-1}|\varepsilon_k|<\infty$, then

(1) $a:=\lim\limits_{n\to\infty}\mathsf{E}[W_n]$ exists and $0<a<\infty$;

(2) there exists an a.s. finite random variable W such that $\lim\limits_{n\to\infty}W_n=W$ a.s..

Proof (1) Clearly,

$$\mathsf{E}[W_{n+1}|\mathcal{F}_n]=r^{-(n+1)}\mathsf{E}[Z_{n+1}|Z_n]=W_n-r^{-(n+1)}Z_n\varepsilon_{Z_n}\qquad\text{a.s.,}\tag{3.1}$$

where $\mathcal{F}_n:=\sigma(Z_0,\cdots,Z_n)$, $n=0,1,\cdots$, then we have

$$\mathsf{E}[W_{n+1}]=\mathsf{E}[W_n]-r^{-(n+1)}\mathsf{E}[Z_n\varepsilon_{Z_n}].\tag{3.2}$$

Define

$$\widehat{\varepsilon}(x) := |\varepsilon_1| I_{[0,1)}(x) + x^{-1}\Big(|\varepsilon_1| + \int_1^x \varepsilon(t)\mathrm{d}t\Big) I_{[1,\infty)}(x),$$

where $\varepsilon(t) := |\varepsilon_1| I_{[0,1)}(t) + |\varepsilon_{[t]}| I_{[1,\infty)}(t)$, of which $[x]$ is the largest integer not greater than x and $I_A(u)$ is the indicator function.

It is verified that $|\varepsilon_n| \le \widehat{\varepsilon}(n)$, $\sum\limits_{n=1}^{\infty} n^{-1}\widehat{\varepsilon}(n) < \infty$ and $x\widehat{\varepsilon}(x)$ is a concave function on R^+. It follows immediately from (3.2) and Jensen's inequality that

$$|\mathsf{E}[W_{n+1}] - \mathsf{E}[W_n]| \le r^{-(n+1)}\mathsf{E}[Z_n\widehat{\varepsilon}(Z_n)] \le r^{-1}\mathsf{E}[W_n]\widehat{\varepsilon}(\mathsf{E}[r^n W_n]).$$

By Proposition 2.2

$$\frac{1}{k}\mathsf{E}\Big[L\Big(\sum_{i=1}^{k}\xi_{n,i}^{(k)}, \sum_{i=1}^{k}\eta_{n,i}^{(k)}\Big)\Big] \ge \frac{1}{k}\mathsf{E}\Big[L\Big(\sum_{i=1}^{k}\xi_{n,i}, \sum_{i=1}^{k}\eta_{n,i}\Big)\Big].$$

This shows $r_k \ge r'_k$, since $r'_k \ge r'_1$, let $\alpha = \inf\limits_k r_k$, then

$$\mathsf{E}[Z_{n+1}] = \mathsf{E}[Z_n r_{Z_n}] \ge \alpha\mathsf{E}[Z_n] \ge \alpha^{n+1} Z_0 \ge (r'_1)^{n+1} N > 0, \qquad n = 0, 1, \cdots.$$

So $\mathsf{E}[W_n] = \mathsf{E}[r^{-n} Z_n] > 0$, $n = 0, 1, \cdots$, which satisfy the conditions of Lemma 1 in [8], so $a := \lim\limits_{n\to\infty} \mathsf{E}[W_n]$ exists and $a > 0$ by Lemma 1 and Theorem 5 of [8].

(2) Let $Y_{n+1} = W_{n+1} + r^{-1}\sum\limits_{k=0}^{n} W_k\varepsilon_{Z_k}$, $n = 0, 1, \cdots$, then we have from (3.1) that

$$\mathsf{E}(Y_{n+1}|\mathcal{F}_n) = \mathsf{E}(W_{n+1}|\mathcal{F}_n) + r^{-1}\sum_{k=0}^{n} W_k\varepsilon_{Z_k} = W_n + r^{-1}\sum_{k=0}^{n-1} W_k\varepsilon_{Z_k} = Y_n.$$

Hence $\{Y_n, \mathcal{F}_n\}_{n=0}^{\infty}$ is a martingale.

To prove a.s. convergence of $\{W_n\}_{n=0}^{\infty}$, we shall discuss the martingale $\{Y_n\}_{n=0}^{\infty}$. Due to concavity of $x\widehat{\varepsilon}(x)$ we have that $|\mathsf{E}W_n\varepsilon_{Z_n}| \le \mathsf{E}W_n\widehat{\varepsilon}(\mathsf{E}[r^n W_n])$. By Lemma 1 in [8], we have, from convergence of the series $\sum\limits_{n=1}^{\infty}\widehat{\varepsilon}(n)/n$ and bound of $\{\mathsf{E}W_n\}_n$, that

$$\mathsf{E}\sum_{n=0}^{\infty}|W_n\varepsilon_{Z_n}| = \sum_{n=0}^{\infty}\mathsf{E}|W_n\varepsilon_{Z_n}| \le \sum_{n=0}^{\infty}\mathsf{E}W_n\widehat{\varepsilon}(\mathsf{E}[r^n W_n]) < \infty.$$

This implies that

$$\sum_{n=0}^{\infty}|W_n\varepsilon_{Z_n}| < \infty \qquad \text{a.s..}$$

On the other hand, since

$$\sup_n \mathsf{E}|Y_n| \le \sup_n \mathsf{E}W_n + r^{-1}\mathsf{E}\sum_{n=0}^{\infty}|W_n\varepsilon_{Z_n}| < \infty,$$

the martingale convergence theorem shows that $\{Y_n\}_{n=0}^{\infty}$ converges a.s. to a finite random variable Y. Thus $\{W_n\}_{n=0}^{\infty}$ converges a.s. to $W = Y - \sum\limits_{n=0}^{\infty} W_n\varepsilon_{Z_n}$, and by Fatou's lemma $\mathsf{E}[W] \le \lim\limits_{n\to\infty}\mathsf{E}[W_n] < \infty$. #

Remark 1 If the mating function $L(x,y)=x$, then

$$r_k=\frac{1}{k}\mathsf{E}[Z_{n+1}|Z_n=k]=\frac{1}{k}\mathsf{E}\Big[L\Big(\sum_{i=1}^{k}(\xi_{n,i}^{(k)},\eta_{n,i}^{(k)})\Big)\Big]=\frac{1}{k}\mathsf{E}\Big[\sum_{i=1}^{k}\xi_{n,i}^{(k)}\Big]=\mathsf{E}[\xi_{n,i}^{(k)}]=\mu_1^{(k)}$$

from Proposition 2.2(2), we get that $\mu_1^{(k)}$ is a monotonic non-increasing sequence and converges to μ_1, so $|\varepsilon_n|=\mu_1^{(k)}-\mu_1$ is a decreasing sequence.

Lemma 3.1 Under Assumption A, for the sequence $\{(\xi_i^{(k)},\eta_i^{(k)}):i=1,2,\cdots;k=0,1,\cdots\}$ of independent bivariate random variables with finite expectation, for $k=0,1,\cdots$, $\{(\xi_i^{(k)},\eta_i^{(k)})\}$ has the same distribution as $\{(\xi^{(k)},\eta^{(k)})\}$, for all $i=1,2,\cdots$ and a mating function L satisfies the superadditivity condition (1.4), then

$$\begin{aligned}\frac{1}{k}L\Big(\sum_{i=1}^{k}\xi_i^{(k)},\sum_{i=1}^{k}\eta_i^{(k)}\Big)\ &\to\ \lim_{k\to\infty}k^{-1}L(k\mathsf{E}\xi,k\mathsf{E}\eta)\qquad \text{a.s.}\\ &=\ r(\mathsf{E}\xi,\mathsf{E}\eta),\end{aligned}$$

where $\mathsf{E}\xi=\lim\limits_{k\to\infty}\mathsf{E}\xi_i^{(k)}$, $\mathsf{E}\eta=\lim\limits_{k\to\infty}\mathsf{E}\eta_i^{(k)}$, $i=1,2,\cdots$.

Proof For every $m\geq 1$, applying Lemma 2.3 in [2], we have

$$\begin{aligned}\frac{1}{k}L\Big(\sum_{i=1}^{k}\xi_i^{(m)},\sum_{i=1}^{k}\eta_i^{(m)}\Big)\ &\to\ \lim_{k\to\infty}k^{-1}L(k\mu_1^{(m)},k\mu_2^{(m)})\qquad \text{a.s.}\\ &=\ r(\mu_1^{(m)},\mu_2^{(m)}),\end{aligned}$$

where $\mu_1^{(m)}=\mathsf{E}\xi_i^{(m)}$, $\mu_2^{(m)}=\mathsf{E}\eta_i^{(m)}$, $i=1,2,\cdots$.

By Proposition 2.2, the continuity of $r(x,y)$ in every (x,y) (see Proposition 3.2 in [3]) shows that

$$\lim_{m\to\infty}r(\mu_1^{(m)},\mu_2^{(m)})=r(\mu_1,\mu_2)=\lim_{k\to\infty}k^{-1}L(k\mu_1,k\mu_2),\tag{3.3}$$

where $\mu_1=\lim\limits_{m\to\infty}\mu_1^{(m)}=\mathsf{E}\xi_i$, $\mu_2=\lim\limits_{m\to\infty}\mu_2^{(m)}=\mathsf{E}\eta_i$, $i=1,2,\cdots$.

For each $k\geq 1$, by Proposition 2.2, it is easy to check

$$\frac{1}{k}L\Big(\sum_{i=1}^{k}\xi_i^{(k)},\sum_{i=1}^{k}\eta_i^{(k)}\Big)\geq\frac{1}{k}L\Big(\sum_{i=1}^{k}\xi_i,\sum_{i=1}^{k}\eta_i\Big).\tag{3.4}$$

From (3.4) and (3.3), we have

$$\begin{aligned}\liminf_{k\to\infty}\frac{1}{k}L\Big(\sum_{i=1}^{k}\xi_i^{(k)},\sum_{i=1}^{k}\eta_i^{(k)}\Big)\ &\geq\ \lim_{k\to\infty}k^{-1}L(k\mu_1,k\mu_2)\qquad \text{a.s.}\\ &=\ r(\mu_1,\mu_2).\end{aligned}\tag{3.5}$$

On the other hand, for every $m\geq 1$, Proposition 2.2 implies that

$$\begin{aligned}\limsup_{k\to\infty}\frac{1}{k}L\Big(\sum_{i=1}^{k}\xi_i^{(k)},\sum_{i=1}^{k}\eta_i^{(k)}\Big)\ &\leq\ \lim_{k\to\infty}\frac{1}{k}L\Big(\sum_{i=1}^{k}\xi_i^{(m)},\sum_{i=1}^{k}\eta_i^{(m)}\Big)\\ &=\ \lim_{k\to\infty}k^{-1}L(k\mu_1^{(m)},k\mu_2^{(m)})\qquad \text{a.s.}\\ &=\ r(\mu_1^{(m)},\mu_2^{(m)}).\end{aligned}$$

Let $m \to \infty$ and by (3.3) we deduce that

$$\begin{aligned}\limsup_{k\to\infty} \frac{1}{k} L\Big(\sum_{i=1}^{k} \xi_i^{(k)}, \sum_{i=1}^{k} \eta_i^{(k)}\Big) &\le \lim_{k\to\infty} k^{-1} L(k\mu_1, k\mu_2) \qquad \text{a.s.} \\ &= r(\mu_1, \mu_2). \end{aligned} \tag{3.6}$$

Then the result of the lemma follows from (3.5) and (3.6). #

Proposition 3.1 On the event $\{Z_n \to \infty\}$

$$\liminf_{n\to\infty} Z_n^{-1} Z_{n+1} > 1 \qquad \text{a.s..}$$

Proof By Lemma 3.1 we see that

$$r(\mu_1, \mu_2) = \lim_{k\to\infty} \frac{1}{k}\Big[L\Big(\sum_{i=1}^{k} \xi_{n,i}^{(k)}, \sum_{i=1}^{k} \eta_{n,i}^{(k)}\Big)\Big] = \lim_{k\to\infty} \frac{1}{k} L(k\mu_1, k\mu_2) \qquad \text{a.s..}$$

Let $r(x,y) = \lim\limits_{k\to\infty} (1/k)\cdot L(kx, ky)$. Since the function $r(x,y)$ is continuous in every non-negative valued (x,y), so if $r > 1$, i.e. $r(\mu_1,\mu_2) > 1$, then there exists $a, b \in R^+$ such that $\widetilde{r} = r(\mathsf{E}[\xi_{0,1}\wedge a], \mathsf{E}[\eta_{0,1}\wedge b]) > 1$. Now we define the sequence $\{\widetilde{Z}_n\}$ in terms of the given process $\{Z_n\}$ by

$$\widetilde{Z_0} = Z_0, \qquad \widetilde{Z}_{n+1} = L\Big(\sum_{i=1}^{Z_n} (\xi_{n,i}^{(Z_n)} \wedge a, \eta_{n,i}^{(Z_n)} \wedge b)\Big), \qquad n = 0, 1, \cdots.$$

Obviously, $Z_n \ge \widetilde{Z}_n$, for $n = 0, 1, \cdots$.

For $\forall \varepsilon > 0$, let $A_n = \{|Z_n^{-1}\widetilde{Z}_{n+1} - \widetilde{r}| < \varepsilon\}$, $n = 0, 1, \cdots$, then it suffice to show that

$$\mathsf{P}\Big(\liminf_{n\to\infty} A_n\Big) \ge \mathsf{P}(Z_n \to \infty) \qquad \text{for } 0 < \varepsilon < \widetilde{r} - 1. \tag{3.7}$$

But, by Lemma 3.1, and an analogous argument as Proposition 3.1 in [10]), shows (3.7), and the proof is complete. #

Theorem 3.2 On the event $\{Z_n \to \infty\}$

$$\lim_{n\to\infty} Z_n^{-1} F_{n+1} = \mu_1 \qquad \text{a.s..}$$

Proof First we define sequence $\{\overline{Z}_n\}_{n=0}^{\infty}$ via $\{Z_n\}_{n=0}^{\infty}$:

$$\overline{Z}_0 = N, \qquad \overline{Z}_{n+1} = L\Big(\sum_{i=1}^{Z_n} (\xi_{n,i}, \eta_{n,i})\Big), \qquad n = 0, 1, 2, \cdots,$$

where $(\xi_{n,i}, \eta_{n,i})$ $(i = 1, 2, \cdots;\ n = 0, 1, \cdots)$ are i.i.d. nonnegative, integer-valued random variables and have the same probability distribution as (ξ, η). By Proposition 2.2 we have that

$$\sum_{i=1}^{Z_n} \xi_{n,i}^{(Z_n)} \ge \sum_{i=1}^{Z_n} \xi_{n,i} \qquad \text{a.s..}$$

Then on $\{Z_n \to \infty\}$ we have that

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{Z_n} \xi_{n,i}^{(Z_n)}}{Z_n} \ge \lim_{n\to\infty} \frac{\sum_{i=1}^{Z_n} \xi_{n,i}}{Z_n} = \mu_1, \qquad \text{a.s.,} \tag{3.8}$$

where the last equality follows from Proposition 3.1 and Theorem 3.2 in [10] without immigration of mating units. #

Next, we define one more sequence $\{\widetilde{Z}_n\}_{n=0}^{\infty}$ in terms of the process $\{Z_n\}_{n=0}^{\infty}$.

For each $m \geq 1$, let

$$\widetilde{Z}_0 = N, \qquad \widetilde{Z}_{n+1} = L\Big(\sum_{i=1}^{Z_n} (\xi_{n,i}^{(m)}, \eta_{n,i}^{(m)})\Big), \qquad n = 0, 1, 2, \cdots,$$

where $(\xi_{n,i}^{(m)}, \eta_{n,i}^{(m)})$ $(i = 1, 2, \cdots;\ n = 0, 1, \cdots)$ are i.i.d. nonnegative, integer-valued random variables for fixed m. Then for $m \geq 1$, by Proposition 2.2 we see on $\{Z_n \to \infty\}$ that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{Z_n} \xi_{n,i}^{(Z_n)}}{Z_n} \leq \lim_{n\to\infty} \frac{\sum_{i=1}^{Z_n} \xi_{n,i}^{(m)}}{Z_n} = \mu_1^{(m)} \qquad \text{a.s.},$$

where the above equality is due to the same reason as in (3.8).

Let $m \to \infty$, we deduce that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{Z_n} \xi_{n,i}^{(Z_n)}}{Z_n} \leq \mu_1, \qquad \text{a.s..} \tag{3.9}$$

From (3.8) and (3.9), we obtain that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{Z_n} \xi_{n,i}^{(Z_n)}}{Z_n} = \mu_1, \qquad \text{a.s. on } \{Z_n \to \infty\}.$$

By a similar way, one can show that, on $\{Z_n \to \infty\}$, the sequence $\{Z_n^{-1} M_{n+1}\}_n$ converges a.s. to μ_2 as $n \to \infty$.

Corollary 3.1 On $\{Z_n \to \infty\}$

$$\lim_{n\to\infty} Z_n^{-1} Z_{n+1} = r \qquad \text{a.s..}$$

Proof Let $\{\overline{Z}_n\}_{n=0}^{\infty}$ and $\{\widetilde{Z}_n\}_{n=0}^{\infty}$ be defined as above. Theorem 3.2 and the continuity of the function $r(x, y)$ allow us to conclude that

$$\begin{aligned} Z_n^{-1}\overline{Z}_{n+1} &= Z_n^{-1} L(Z_n(Z_n^{-1}\overline{F}_{n+1}), Z_n(Z_n^{-1}\overline{M}_{n+1})) \\ &\to r(\mu_1, \mu_2) = r \qquad \text{a.s. on } \{Z_n \to \infty\}. \end{aligned}$$

Since $Z_{n+1} \geq \overline{Z}_{n+1}$, $n = 0, 1, \cdots$, then we have that, on $\{Z_n \to \infty\}$,

$$\liminf_{n\to\infty} \frac{Z_{n+1}}{Z_n} \geq \lim_{n\to\infty} \frac{\overline{Z}_{n+1}}{Z_n} = r(\mu_1, \mu_2) = r \qquad \text{a.s..}$$

Similarly, we have

$$\begin{aligned} Z_n^{-1}\widetilde{Z}_{n+1} &= Z_n^{-1} L(Z_n(Z_n^{-1}\widetilde{F}_{n+1}), Z_n(Z_n^{-1}\widetilde{M}_{n+1})) \\ &\to r(\mu_1^{(m)}, \mu_2^{(m)}) \qquad \text{a.s. on } \{Z_n \to \infty\}. \end{aligned}$$

Note that $Z_{n+1} \le \widetilde{Z}_{n+1}$, $n = 0, 1, \cdots$, then we have that, on $\{Z_n \to \infty\}$,

$$\limsup_{n\to\infty} \frac{Z_{n+1}}{Z_n} \le \lim_{n\to\infty} \frac{\widetilde{Z}_{n+1}}{Z_n} = r(\mu_1^{(m)}, \mu_2^{(m)}) \qquad \text{a.s..}$$

Hence, on $\{Z_n \to \infty\}$,

$$\limsup_{n\to\infty} \frac{Z_{n+1}}{Z_n} \le \lim_{m\to\infty} r(\mu_1^{(m)}, \mu_2^{(m)}) = r(\mu_1, \mu_2) = r \qquad \text{a.s..}$$

So the proof is completed. #

Corollary 3.2 On $\{Z_n \to \infty\}$, both $\{F_n^{-1}F_{n+1}\}_{n=0}^{\infty}$ and $\{M_n^{-1}M_{n+1}\}_{n=0}^{\infty}$ are a.s. convergent to r.

Proof Note that for $n = 1, 2, \cdots$

$$F_n^{-1}F_{n+1} = Z_n^{-1}F_{n+1}Z_{n-1}^{-1}Z_nF_n^{-1}Z_{n-1},$$

and

$$M_n^{-1}M_{n+1} = Z_n^{-1}M_{n+1}Z_{n-1}^{-1}Z_nM_n^{-1}Z_{n-1}.$$

Then the conclusions follow from Theorem 3.2 and Corollary 3.1. #

Proposition 3.2 On $\{Z_n \to \infty\}$ the following assertions are equivalent:

(1) $\{r^{-n}Z_n\}_n$ converges a.s. to W;

(2) $\{r^{-n}F_n\}_n$ converges a.s. to $r^{-1}\mu_1 W$;

(3) $\{r^{-n}M_n\}_n$ converges a.s. $r^{-1}\mu_2 W$.

Proof It is enough to show that (1) and (2) are equivalent.

Suppose that $\{r^{-n}Z_n\}_{n=0}^{\infty}$ converges a.s. to W. We are to prove that $\{r^{-n}F_n\}_{n=0}^{\infty}$ converges a.s. to $r^{-1}\mu_1 W$ as $n \to \infty$:

By Theorem 3.2, since $\{r^{-n}Z_n\}_{n=0}^{\infty}$ converges a.s. to W, we have, on $\{Z_n \to \infty\}$,

$$r^{-(n+1)}F_{n+1} = r^{-1}Z_n^{-1}F_{n+1}r^{-n}Z_n \to r^{-1}\mu_1 W \quad \text{a.s.} \qquad \text{as } n \to \infty.$$

Thus (1) implies (2). Analogously, one can show (2) implies (1). #

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人口数相依的两性分支过程的极限性质

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本文研究了后代分布依赖于人口数的两性 Galton-Watson 分支过程，在对后代分布的适当假设下，对于上临界的情况，我们研究了有关过程的几乎处处收敛的极限性质.

关 键 词: 两性的 Galton-Watson 分支过程，人口数相依的分支过程，几乎处处收敛.

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