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应力服从一类嵌套多元指数分布的结构可靠度估计 *

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摘　要

本文研究应力服从一类嵌套多元指数分布, 强度服从指数分布的应力—强度结构可靠度模型. 分别在强度参数未知、应力参数已知和强度参数已知、应力参数未知的情况下给出了结构可靠度P_A的估计$\widehat{P}_{A1}$和$\widehat{P}_{A2}$, 并讨论了它们的渐近性质,而且获得了P_A的近似置信区间. 最后对这两种情况下模型结构可靠度的估计$\widehat{P}_{A1}$和$\widehat{P}_{A2}$进行了随机模拟, 随机模拟结果令人满意.

关 键 词:　结构可靠度, 嵌套多元指数分布, 强相合性, 渐近正态性, 置信区间, 随机模拟.

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§1.　引　　言

假设某种产品的结构强度为Z, 应力为X (X, Z均为随机变量), 当Z大于X时产品正常工作, 反之产品失效. $\mathsf{P}(Z > X)$为产品正常工作的概率, 也称为应力—强度模型的结构可靠度. 许多文献[1–3, 7–9]对应力和强度都服从指数分布的情况作过讨论, 但是这些文章中的应力限于一维的情况, 即使是多维的情况, 也假定各个随机变量之间是独立的, 而对于应力为多维随机变量而各变量之间又是相关的情况讨论却很少. 可是, 在分析结构系统可靠性时, 必须考虑各组成部分之间的相关性, 它们对系统可靠性的影响是不容忽视的.

文献[4]考虑到暴风雨对不同地区的影响力不是相互独立的, 由于地理位置的不同存在或大或小的相关性, 所以提出了一类多元嵌套指数模型, 这类嵌套模型确定了一种有效的层次结构, 能够把不同地区的层次相关性表示出来. 此类模型中最简单的二级嵌套模型在研究英格兰东南海岸三个海洋监测站的年最高水位分布时有了初步的应用, 文献[5]给出了这个模型的相关参数的矩估计. 本文将研究应力为多维随机变量, 而这些随机变量又是复杂层次相关的应力—强度结构可靠度模型. 例如, 有四个防汛大堤, 它们的年最高水位每两个可用二元指数模型拟合, 但是由于这四个防汛大堤Q1, Q2, Q3, Q4的地理位置是Q1和Q2, Q3和Q4分别比较接近, 因此有理由认为它们之间的相关性比较强, 而这两组防汛大堤之间的相关性没有显著差别, 所以模型(1.1)是合适的.

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本文设上述四个防汛大堤串联而成的系统为A, 当暴风雨来临时, 各防汛大堤受到的影响力分别为X_1, X_2, X_3, X_4, 它们的联合可靠度函数为

$$\begin{aligned}
&\overline{F}(x_1,x_2,x_3,x_4)\\
=\ &\mathsf{P}(X_1>x_1,X_2>x_2,X_3>x_3,X_4>x_4)\\
=\ &\exp\Big[-\Big(\Big[\Big(\frac{x_1}{\theta_1}\Big)^{1/(\alpha\delta)}+\Big(\frac{x_2}{\theta_2}\Big)^{1/(\alpha\delta)}\Big]^{\alpha}+\Big[\Big(\frac{x_3}{\theta_3}\Big)^{1/(\beta\delta)}+\Big(\frac{x_4}{\theta_4}\Big)^{1/(\beta\delta)}\Big]^{\beta}\Big)^{\delta}\Big], \qquad (1.1)
\end{aligned}$$

其中$0<x_i<\infty$, $0<\alpha,\beta,\delta\le 1$, $0<\theta_i<\infty$, $i=1,2,3,4$. 各防汛大堤所能承受的强度服从期望为$1/\lambda$的指数分布, 其密度函数为

$$f(z)=\lambda \mathrm{e}^{-\lambda}, \qquad \lambda>0,\ z>0. \qquad (1.2)$$

在研究上述模型结构可靠度问题时我们假设应力和强度是相互独立的. 首先在强度参数未知, 应力参数已知的情况下给出了结构可靠度P_A的估计并讨论了它的渐近性质, 而且获得了P_A的置信度为$1-\alpha$的近似置信区间, 然后就强度参数已知, 应力参数未知的情况作了类似的讨论. 最后对这两种情况下模型结构可靠度的估计$\widehat{P}_{A1}$和$\widehat{P}_{A2}$进行了随机模拟, 给出了若干随机模拟结果.

§2. 结构可靠度表达式

设作用于串联系统四个防汛大堤的应力分别为X_1, X_2, X_3, X_4, 它们的联合可靠度函数为(1.1), 强度Z服从一维指数分布, 其密度函数为(1.2). 记系统的结构可靠度为P_A, 于是

$$\begin{aligned}
P_A\ =\ &\mathsf{P}(X_1<Z,X_2<Z,X_3<Z,X_4<Z)\\
=\ &\int_0^{\infty}(X_1<z,X_2<z,X_3<z,X_4<z)\mathrm{d}F_z(z)\\
=\ &1-\sum_{1}^{4}\frac{\lambda\theta_i}{1+\lambda\theta_i}+\sum_{i=1}^{2}\sum_{j=3}^{4}\frac{\lambda\theta_i\theta_j}{\lambda\theta_i\theta_j+(\theta_i^{1/\delta}+\theta_j^{1/\delta})^{\delta}}\\
&+\frac{\lambda\theta_1\theta_2}{\lambda\theta_1\theta_2+(\theta_1^{1/(\alpha\delta)}+\theta_2^{1/(\alpha\delta)})^{\alpha\delta}}+\frac{\lambda\theta_3\theta_4}{\lambda\theta_3\theta_4+(\theta_3^{1/(\beta\delta)}+\theta_4^{1/(\beta\delta)})^{\beta\delta}}\\
&-\sum_{j=3}^{4}\frac{\lambda\theta_1\theta_2\theta_j}{\lambda\theta_1\theta_2\theta_j+[(\theta_1^{1/(\alpha\delta)}+\theta_2^{1/(\alpha\delta)})^{\alpha}\cdot\theta_j^{1/\delta}+\theta_1^{1/\delta}\theta_2^{1/\delta}]^{\delta}}\\
&-\sum_{i=1}^{2}\frac{\lambda\theta_3\theta_4\theta_i}{\lambda\theta_3\theta_4\theta_i+[(\theta_3^{1/(\beta\delta)}+\theta_4^{1/(\beta\delta)})^{\beta}\cdot\theta_i^{1/\delta}+\theta_3^{1/\delta}\theta_4^{1/\delta}]^{\delta}}\\
&+\frac{\lambda\theta_1\theta_2\theta_3\theta_4}{\lambda\theta_1\theta_2\theta_3\theta_4+[(\theta_1^{1/(\alpha\delta)}+\theta_2^{1/(\alpha\delta)})^{\alpha}\cdot\theta_3^{1/\delta}\theta_4^{1/\delta}+(\theta_3^{1/(\beta\delta)}+\theta_4^{1/(\beta\delta)})^{\beta}\cdot\theta_1^{1/\delta}\theta_2^{1/\delta}]^{\delta}}. \qquad (2.1)
\end{aligned}$$

在实际问题中, 我们经常会遇到边缘分布相同的情况, 因此, 我们考虑$\theta_1=\theta_2=\theta_3=\theta_4=\theta$的情况, 这时(2.1)简化为

$$\begin{aligned} P_A &= 1-\frac{4\lambda\theta}{1+\lambda\theta}+\frac{4\lambda\theta}{2^\delta+\lambda\theta}+\frac{\lambda\theta}{2^{\alpha\delta}+\lambda\theta}+\frac{\lambda\theta}{2^{\beta\delta}+\lambda\theta} \\ &\quad -\frac{2\lambda\theta}{(2^\alpha+1)^\delta+\lambda\theta}-\frac{2\lambda\theta}{(2^\beta+1)^\delta+\lambda\theta}+\frac{\lambda\theta}{(2^\alpha+2^\beta)^\delta+\lambda\theta}. \end{aligned} \tag{2.2}$$

§3. 应力参数已知强度参数未知的结构可靠度估计及其性质

当应力参数$\theta,\delta,\alpha,\beta$已知, 而强度参数$\lambda$未知时, 设$Z_1,Z_2,Z_3,Z_4$是来自$Z$的样本, 由指数分布的性质可知$\lambda$的矩估计和极大似然估计均为

$$\widehat{\lambda}=\frac{n}{\sum\limits_{i=1}^n Z_i}=\frac{1}{\overline{Z}}. \tag{3.1}$$

故我们可提出的P_A估计如下:

$$\begin{aligned} \widehat{P}_{A1} &= 1-\frac{4\widehat{\lambda}\theta}{1+\widehat{\lambda}\theta}+\frac{4\widehat{\lambda}\theta}{2^\delta+\widehat{\lambda}\theta}+\frac{\widehat{\lambda}\theta}{2^{\alpha\delta}+\widehat{\lambda}\theta}+\frac{\widehat{\lambda}\theta}{2^{\beta\delta}+\widehat{\lambda}\theta} \\ &\quad -\frac{2\widehat{\lambda}\theta}{(2^\alpha+1)^\delta+\widehat{\lambda}\theta}-\frac{2\widehat{\lambda}\theta}{(2^\beta+1)^\delta+\widehat{\lambda}\theta}+\frac{\widehat{\lambda}\theta}{(2^\alpha+2^\beta)^\delta+\widehat{\lambda}\theta}. \end{aligned} \tag{3.2}$$

定理 3.1　(1) $\widehat{P}_{A1}\to P_A\ (n\to\infty)$;

(2) $\sqrt{n}(\widehat{P}_{A1}-P_A)\to N(0,\sigma_1^2(\lambda))\ (n\to\infty)$, 其中

$$\begin{aligned} \sigma_1^2(\lambda) &= \frac{4\lambda\theta}{(1+\lambda\theta)^2}-\frac{4\lambda\theta 2^\delta}{(2^\delta+\lambda\theta)^2}-\frac{\lambda\theta}{(2^{\alpha\delta}+\lambda\theta)^2}-\frac{\lambda\theta}{(2^{\beta\delta}+\lambda\theta)^2} \\ &\quad +\frac{2\lambda\theta((2^\alpha+1)^\delta)}{[(2^\alpha+1)^\delta+\lambda\theta]^2}+\frac{2\lambda\theta(2^\beta+1)^\delta}{[(2^\beta+1)^\delta+\lambda\theta]^2}-\frac{\lambda\theta(2^\alpha+2^\beta)^\delta}{[(2^\alpha+2^\beta)^\delta+\lambda\theta]^2}; \end{aligned} \tag{3.3}$$

(3) P_A的置信度为$1-\alpha$的渐近置信区间为

$$\Big(\widehat{P}_{A1}-\frac{Z_{\alpha/2}\sigma_1(\widehat{\lambda})}{\sqrt{n}},\widehat{P}_{A1}+\frac{Z_{\alpha/2}\sigma_1(\widehat{\lambda})}{\sqrt{n}}\Big),$$

其中$Z_{\alpha/2}$是标准正态分布的上$\alpha/2$分位点.

证明:　(1) $Z_i\ (i=1,2,\cdots,n)$独立同分布均服从指数分布, 故$\mathrm{E}(Z_i)=1/\lambda$, $(i=1,2,\cdots,n)$. 由强大数定理

$$\overline{Z}=\frac{1}{n}\sum_{i=1}^n Z_i\to\frac{1}{\lambda},\qquad (n\to\infty).$$

又P_A关于λ连续, 故$\widehat{P}_{A1}\to P_A$, $(n\to\infty)$.

(2) 由指数分布的性质可知$\mathsf{E}(Z_i)=1/\lambda$, $D(Z_i)=1/\lambda^2$, $(i=1,2,\cdots,n)$, 故由中心极限定理可知, $\sqrt{n}(\overline{Z}-1/\lambda)\to N(0,1/\lambda^2)$, $(n\to\infty)$. 又P_A关于λ连续, 由Delta方法[6]有

$$\sqrt{n}(\widehat{P}_{A1}-P_A)\to N(0,\sigma_1^2(\lambda)),\qquad (n\to\infty),$$

其中$\sigma_1^2(\lambda)=(1/\lambda)\cdot[\mathrm{d}P_A/\mathrm{d}(1/\lambda)]$如(3.3)式所示.

(3) 由(2)可得

$$\sqrt{n}\frac{(\widehat{P}_{A1}-P_A)}{\sigma_1(\lambda)}\to N(0,1),\qquad (n\to\infty).$$

故

$$\mathsf{P}\Big(\Big|\sqrt{n}\frac{(\widehat{P}_{A1}-P_A)}{\sigma_1(\lambda)}\Big|<Z_{\alpha/2}\Big)\approx 1-\alpha,$$

由此可得P_A的置信度为$1-\alpha$的近似置信区间. $\square$

§4. 强度参数已知应力参数未知的结构可靠度估计及其性质

本节讨论应力参数$\theta,\delta,\alpha,\beta$未知而强度参数$\lambda$已知时$P_A$的估计及其性质. 设$(X_{1i},X_{2i},X_{3i},X_{4i})$ $(i=1,2,\cdots,n)$是来自可靠度函数为(1.1)的四维总体(X_1,X_2,X_3,X_4)的样本. 由$\mathsf{E}(\ln X_j)=\ln\theta-\gamma$, $(j=1,2,3,4)$我们采用如下的θ的估计:

$$\widehat{\theta}=\frac{1}{4}[\exp(\overline{\ln X_1}+\gamma)+\exp(\overline{\ln X_2}+\gamma)+\exp(\overline{\ln X_3}+\gamma)+\exp(\overline{\ln X_4}+\gamma)].\tag{4.1}$$

经过计算, 我们得到下述相关矩[10],

$$\begin{aligned}\mathsf{Cov}\,(\ln X_i,\ln X_j)&=\frac{\pi^2}{6}(1-\delta^2)\qquad i=1,2;\ j=3,4;\\ \mathsf{Cov}\,(\ln X_1,\ln X_2)&=\frac{\pi^2}{6}(1-\alpha^2\delta^2);\\ \mathsf{Cov}\,(\ln X_3,\ln X_4)&=\frac{\pi^2}{6}(1-\beta^2\delta^2).\end{aligned}$$

所以我们采用了δ,α,β的估计如下:

$$\begin{aligned}\widehat{\delta}&=\frac{1}{4}\sum_{k=1}^{2}\sum_{j=3}^{4}\Big[1-\frac{\pi^2}{6}\Big(\frac{1}{n}\sum_{i=1}^{n}\ln X_{ki}\ln X_{ji}-\overline{\ln X_k}\cdot\overline{\ln X_j}\Big)\Big]^{1/2},\\ \widehat{\alpha}&=\frac{1}{\widehat{\delta}}\Big[1-\frac{\pi^2}{6}\Big(\frac{1}{n}\sum_{i=1}^{n}\ln X_{1i}\ln X_{2i}-\overline{\ln X_1}\cdot\overline{\ln X_2}\Big)\Big]^{1/2},\\ \widehat{\beta}&=\frac{1}{\widehat{\delta}}\Big[1-\frac{\pi^2}{6}\Big(\frac{1}{n}\sum_{i=1}^{n}\ln X_{3i}\ln X_{4i}-\overline{\ln X_3}\cdot\overline{\ln X_4}\Big)\Big]^{1/2}.\end{aligned}\tag{4.2}$$

因此我们提出的P_A估计如下:

$$
\begin{aligned}
\widehat{P}_{A2} \;=\;& 1-\frac{4\lambda\widehat{\theta}}{1+\lambda\widehat{\theta}}+\frac{4\lambda\widehat{\theta}}{2^{\widehat{\delta}}+\lambda\widehat{\theta}}+\frac{\lambda\widehat{\theta}}{2^{\widehat{\alpha}\widehat{\delta}}+\lambda\widehat{\theta}}+\frac{\lambda\widehat{\theta}}{2^{\widehat{\beta}\widehat{\delta}}+\lambda\widehat{\theta}} \\
&-\frac{2\lambda\widehat{\theta}}{(2^{\widehat{\alpha}}+1)^{\widehat{\delta}}+\lambda\widehat{\theta}}-\frac{2\lambda\widehat{\theta}}{(2^{\widehat{\beta}}+1)^{\widehat{\delta}}+\lambda\widehat{\theta}}+\frac{\lambda\widehat{\theta}}{(2^{\widehat{\alpha}}+2^{\widehat{\beta}})^{\widehat{\delta}}+\lambda\widehat{\theta}}.
\end{aligned}
\tag{4.3}
$$

下面先讨论θ和δ,α,β估计的性质. 由矩估计的性质, 我们容易知道$\widehat{\theta},\widehat{\delta},\widehat{\alpha},\widehat{\beta}$分别是$\theta,\delta,\alpha,\beta$的强相合估计. 记协方差矩阵为

$$
\begin{aligned}
\Sigma_1 \;\widehat{=}\;& \mathsf{Cov}\,(\ln X_1\ln X_3,\ln X_1\ln X_4,\ln X_2\ln X_3,\ln X_2\ln X_4, \\
& \ln X_1\ln X_2,\ln X_3\ln X_4,\ln X_1,\ln X_2,\ln X_3,\ln X_4)',
\end{aligned}
$$

则

$$
\Sigma_1=\left(\begin{array}{cccccccccc}
 & & & & t'_5 & t'_6 & & & & \\
 & & A & & t'_5 & t'_6 & & C & & \\
 & & & & t'_5 & t'_6 & & & & \\
 & & & & t'_5 & t'_6 & & & & \\
t'_5 & t'_5 & t'_5 & t'_5 & t'_{13} & t'_{15} & t'_{19} & t'_{19} & t'_{20} & t'_{20} \\
t'_6 & t'_6 & t'_6 & t'_6 & t'_{15} & t'_{14} & t'_{20} & t'_{20} & t'_{21} & t'_{21} \\
 & & & & t'_{19} & t'_{20} & & & & \\
 & & C' & & t'_{19} & t'_{20} & & F & & \\
 & & & & t'_{20} & t'_{21} & & & & \\
 & & & & t'_{20} & t'_{21} & & & &
\end{array}\right),
$$

其中A,C,F都是4阶方阵,

$$
A=\begin{pmatrix} t'_1 & t'_2 & t'_3 & t'_4 \\ t'_2 & t'_1 & t'_4 & t'_3 \\ t'_3 & t'_4 & t'_1 & t'_2 \\ t'_4 & t'_3 & t'_2 & t'_1 \end{pmatrix},\quad
C=\begin{pmatrix} t'_{10} & t'_{11} & t'_{10} & t'_{12} \\ t'_{10} & t'_{11} & t'_{12} & t'_{10} \\ t'_{11} & t'_{10} & t'_{10} & t'_{12} \\ t'_{11} & t'_{10} & t'_{12} & t'_{10} \end{pmatrix},\quad
F=\begin{pmatrix} t'_{30} & t'_{32} & t'_{31} & t'_{31} \\ t'_{32} & t'_{30} & t'_{31} & t'_{31} \\ t'_{31} & t'_{31} & t'_{30} & t'_{33} \\ t'_{31} & t'_{31} & t'_{33} & t'_{30} \end{pmatrix},
$$

$$
\begin{aligned}
t'_1 &= D(\ln X_i\ln X_j)=t_1+4t_{10}\ln\theta+\frac{\pi^2}{3}(2-\delta^2)\ln\theta^2, && i=1,2;\ j=3,4, \\
t'_2 &= \mathsf{Cov}\,(\ln X_i\ln X_j,\ln X_i\ln X_l) \\
&= t_2+2\ln\theta(t_{10}+t_{12})+\frac{\pi^2}{6}(4-2\delta^2-\beta^2\delta^2)\ln\theta^2, && i=1,2;\ j,l=3,4,\ j\neq l, \\
t'_3 &= \mathsf{Cov}\,(\ln X_i\ln X_j,\ln X_k\ln X_j) \\
&= t_3+2\ln\theta(t_{10}+t_{11})+\frac{\pi^2}{6}(4-2\delta^2-\alpha^2\delta^2)\ln\theta^2, && i,k=1,2,\ i\neq k;\ j=3,4,
\end{aligned}
$$

$$\begin{aligned}
t_4' &= \mathsf{Cov}\,(\ln X_i \ln X_j, \ln X_k \ln X_l) \\
&= t_4 + 2\ln\theta(t_{11}+t_{12}) + \frac{\pi^2}{6}(4-2\delta^2-\beta^2\delta^2-\alpha^2\delta^2)\ln\theta^2, \\
&\qquad\qquad\qquad i,k=1,2,\ i\neq k;\ j,l=3,4,\ j\neq l, \\
t_5' &= \mathsf{Cov}\,(\ln X_1 \ln X_2, \ln X_i \ln X_j) \\
&= t_5 + \ln\theta(t_{10}+t_{11}+t_{19}+t_{20}) + \frac{\pi^2}{6}(4-2\delta^2-\alpha^2\delta^2)\ln\theta^2, \qquad i,k=1,2;\ j=3,4, \\
t_6' &= \mathsf{Cov}\,(\ln X_3 \ln X_4, \ln X_i \ln X_j) \\
&= t_6 + \ln\theta(t_{10}+t_{12}+t_{20}+t_{21}) + \frac{\pi^2}{6}(4-2\delta^2-\beta^2\delta^2)\ln\theta^2, \qquad i,k=1,2;\ j=3,4, \\
t_{10}' &= \mathsf{Cov}\,(\ln X_i \ln X_j, \ln X_i) = t_{10} + \frac{\pi^2}{6}(2-\delta^2)\ln\theta, \qquad i=1,2;\ j=3,4, \\
t_{11}' &= \mathsf{Cov}\,(\ln X_i \ln X_j, \ln X_k) \\
&= t_{11} + \frac{\pi^2}{6}(2-\delta^2-\alpha^2\delta^2)\ln\theta, \qquad i,k=1,2,\ i\neq k;\ j=3,4, \\
t_{12}' &= \mathsf{Cov}\,(\ln X_i \ln X_j, \ln X_l) \\
&= t_{12} + \frac{\pi^2}{6}(2-\delta^2-\beta^2\delta^2)\ln\theta, \qquad i=1,2;\ j,l=3,4,\ j\neq l, \\
t_{13}' &= D(\ln X_1 \ln X_2) = t_{13} + 4t_{19}\ln\theta + \frac{\pi^2}{3}(2-\alpha^2\delta^2)\ln\theta^2, \\
t_{14}' &= D(\ln X_3 \ln X_4) = t_{14} + 4t_{21}\ln\theta + \frac{\pi^2}{3}(2-\beta^2\delta^2)\ln\theta^2, \\
t_{15}' &= \mathsf{Cov}\,(\ln X_1 \ln X_2, \ln X_3 \ln X_4) = t_{15} + 4t_{20}\ln\theta + \frac{2\pi^2}{3}(1-\delta^2)\ln\theta^2, \\
t_{19}' &= \mathsf{Cov}\,(\ln X_1 \ln X_2, \ln X_i) = t_{19} + \frac{\pi^2}{6}(2-\alpha^2\delta^2)\ln\theta, \qquad i=1,2, \\
t_{20}' &= \mathsf{Cov}\,(\ln X_1 \ln X_2, \ln X_j) \\
&= \mathsf{Cov}\,(\ln X_3 \ln X_4, \ln X_i) = t_{20} + \frac{\pi^2}{6}(2-\delta^2)\ln\theta, \qquad i=1,2;\ j=3,4, \\
t_{21}' &= \mathsf{Cov}\,(\ln X_3 \ln X_4, \ln X_j) = t_{21} + \frac{\pi^2}{6}(2-\beta^2\delta^2)\ln\theta, \qquad j=3,4, \\
t_{30}' &= D(\ln X_i) = \frac{\pi^2}{6}, \qquad i=1,2,3,4, \\
t_{31}' &= \mathsf{Cov}\,(\ln X_i, \ln X_j) = \frac{\pi^2}{6}(1-\delta^2), \qquad i=1,2;\ j=3,4, \\
t_{32}' &= \mathsf{Cov}\,(\ln X_1, \ln X_2) = \frac{\pi^2}{6}(1-\alpha^2\delta^2), \\
t_{33}' &= \mathsf{Cov}\,(\ln X_3, \ln X_4) = \frac{\pi^2}{6}(1-\beta^2\delta^2).
\end{aligned}$$

上述各式中t_i, $i=1,2,\cdots,21$ (计算可参见文献[10])分别为

$$t_1 = 6\zeta(4) + 8\zeta(3)\gamma + \frac{\pi^4}{18} + \frac{2\pi^2\gamma^2}{3} - \delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big) - 8\gamma\zeta(3)\delta^3 + \delta^4\Big(\frac{\pi^4}{36} - 6\zeta(4)\Big),$$

$$
\begin{aligned}
t_2 &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big)-8\gamma\zeta(3)\delta^3+\delta^4\Big(\frac{\pi^4}{36}-6\zeta(4)\Big)\\
&\quad -\beta^2\delta^2\frac{\pi^2}{6}\Big(\frac{\pi^2}{6}+\gamma^2\Big),\\
t_3 &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big)-8\gamma\zeta(3)\delta^3+\delta^4\Big(\frac{\pi^4}{36}-6\zeta(4)\Big)\\
&\quad -\alpha^2\delta^2\frac{\pi^2}{6}\Big(\frac{\pi^2}{6}+\gamma^2\Big),\\
t_4 &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big)-8\gamma\zeta(3)\delta^3+\delta^4\Big(\frac{\pi^4}{36}-6\zeta(4)\Big)\\
&\quad -\frac{\pi^2}{6}\Big(\frac{\pi^2}{6}+\gamma^2\Big)(\alpha^2+\beta^2)\delta^2+\frac{\pi^4}{36}\alpha^2\beta^2\delta^4,\\
t_5 &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big)-6\gamma\zeta(3)\delta^3-2\gamma\zeta(3)\alpha^3\delta^3\\
&\quad -\alpha^2\delta^2\Big(\frac{\pi^4}{36}+\frac{\pi^2\gamma^2}{6}\Big)+\delta^4\Big(\frac{\pi^4}{36}\alpha^2-6\zeta(4)\Big),\\
t_6 &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big)-6\gamma\zeta(3)\delta^3-2\gamma\zeta(3)\beta^3\delta^3\\
&\quad -\beta^2\delta^2\Big(\frac{\pi^4}{36}+\frac{\pi^2\gamma^2}{6}\Big)+\delta^4\Big(\frac{\pi^4}{36}\beta^2-6\zeta(4)\Big),\\
t_{10} &= -2\zeta(3)-\frac{\pi^2\gamma}{3}+\frac{\pi^2\delta^2\gamma}{3}+2\zeta(3)\delta^3,\\
t_{11} &= -2\zeta(3)-\frac{\pi^2\gamma}{3}+\frac{\pi^2\delta^2\gamma}{6}+2\zeta(3)\delta^3+\frac{\pi^2\alpha^2\delta^2\gamma}{6},\\
t_{12} &= -2\zeta(3)-\frac{\pi^2\gamma}{3}+\frac{\pi^2\delta^2\gamma}{6}+2\zeta(3)\delta^3+\frac{\pi^2\beta^2\delta^2\gamma}{6},\\
t_{13} &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\alpha^2\delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big)-8\gamma\zeta(3)\alpha^3\delta^3\\
&\quad +\alpha^4\delta^4\Big(\frac{\pi^4}{36}-6\zeta(4)\Big),\\
t_{14} &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\beta^2\delta^2\Big(\frac{\pi^4}{18}+\frac{\pi^2\gamma^2}{3}\Big)-8\gamma\zeta(3)\beta^3\delta^3\\
&\quad +\beta^4\delta^4\Big(\frac{\pi^4}{36}-6\zeta(4)\Big),\\
t_{15} &= 6\zeta(4)+8\zeta(3)\gamma+\frac{\pi^4}{18}+\frac{2\pi^2\gamma^2}{3}-\delta^2\Big(\frac{\pi^4}{9}+\frac{2\pi^2\gamma^2}{3}\Big)-8\gamma\zeta(3)\delta^3+\delta^4\Big(\frac{\pi^4}{18}-6\zeta(4)\Big),\\
t_{19} &= -2\zeta(3)-\frac{\pi^2\gamma}{3}+2\zeta(3)\alpha^3\delta^3+\frac{\pi^2\alpha^2\delta^2\gamma}{6},\\
t_{20} &= -2\zeta(3)-\frac{\pi^2\gamma}{3}+2\zeta(3)\delta^3+\frac{\pi^2\delta^2\gamma}{6},\\
t_{21} &= -2\zeta(3)-\frac{\pi^2\gamma}{3}+2\zeta(3)\beta^3\delta^3+\frac{\pi^2\beta^2\delta^2\gamma}{6}.
\end{aligned}
$$

在上述各式的计算过程中用到了以下函数: γ是Euler常数, $\gamma=0.5722\cdots$; $\zeta(\cdot)$表示zeta函

数, $\zeta(\alpha)=\sum\limits_{1}^{\infty}1/k^{\alpha}$, $(\alpha>1)$. 那么, 由多维中心极限定理

$$\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n}[\ln X_{1i}\ln X_{3i}-\mathsf{E}(\ln X_{1i}\ln X_{3i})],\cdots,\frac{1}{n}\sum_{i=1}^{n}[\ln X_{3i}\ln X_{4i}-\mathsf{E}(\ln X_{3i}\ln X_{4i})],$$
$$\frac{1}{n}\sum_{i=1}^{n}[\ln X_{1i}-\mathsf{E}(\ln X_{1i})],\cdots,\frac{1}{n}\sum_{i=1}^{n}[\ln X_{4i}-\mathsf{E}(\ln X_{4i})]\Big)'\rightarrow N(\underline{0},\Sigma_1),\qquad n\rightarrow\infty,$$

其中$\underline{0}=(0,0,\cdots,0)'$. 记

$$\begin{aligned}
a_{kj}&=\mathsf{E}(\ln X_{ki}\ln X_{ji})=\frac{\pi^2}{6}(1-\delta^2)+(\ln\theta-\gamma)^2,\qquad k=1,2;\ j=3,4;\\
c_j&=\mathsf{E}(\ln X_{ji})=\ln\theta-\gamma,\qquad j=1,\cdots,4;\\
d_0&=\mathsf{E}(\ln X_{1i}\ln X_{2i})=\frac{\pi^2}{6}(1-\alpha^2\delta^2)+(\ln\theta-\gamma)^2;\\
d_1&=\mathsf{E}(\ln X_{3i}\ln X_{4i})=\frac{\pi^2}{6}(1-\beta^2\delta^2)+(\ln\theta-\gamma)^2,\qquad i=1,\cdots,n.
\end{aligned}$$

所以

$$\begin{aligned}
\delta&=\frac{1}{4}\sum_{k=1}^{2}\sum_{j=3}^{4}\Big[1-\frac{6}{\pi^2}(a_{kj}-c_kc_j)\Big]^{1/2},\\
\alpha&=\frac{1}{\delta}\Big[1-\frac{6}{\pi^2}(a_{12}-c_1c_2)\Big]^{1/2},\\
\beta&=\frac{1}{\delta}\Big[1-\frac{6}{\pi^2}(a_{34}-c_3c_4)\Big]^{1/2},\\
\theta&=\frac{1}{4}[\exp(c_1+\gamma)+\exp(c_2+\gamma)+\exp(c_3+\gamma)+\exp(c_4+\gamma)].
\end{aligned}$$

$$\frac{\partial(\delta,\alpha,\beta,\theta)}{\partial(a_{13},a_{14},a_{23},a_{24},d_0,d_1,c_1,c_2,c_3,c_4)}\triangleq\frac{3}{2\pi^2}\cdot M_1=\frac{3}{2\pi^2}\cdot(m_{ij})_{10\times4},$$

其中

$$\sigma_2^2(\delta,\alpha,\beta,\theta)=\Big(\frac{\partial P_A}{\partial\delta},\frac{\partial P_A}{\partial\alpha},\frac{\partial P_A}{\partial\beta},\frac{\partial P_A}{\partial\theta}\Big)\sum_2\Big(\frac{\partial P_A}{\partial\delta},\frac{\partial P_A}{\partial\alpha},\frac{\partial P_A}{\partial\beta},\frac{\partial P_A}{\partial\theta}\Big)',$$

$$\begin{aligned}
\frac{\partial P_A}{\partial\delta}&=-\frac{4\lambda\theta2^{\delta}}{(\lambda\theta+2^{\delta})^2}\cdot\ln2-\frac{\lambda\theta\alpha2^{\alpha}\delta}{(\lambda\theta+2^{\alpha}\delta)^2}\cdot\ln2-\frac{\lambda\theta\beta2^{\beta}\delta}{(\lambda\theta+2^{\beta}\delta)^2}\cdot\ln2\\
&\quad+\frac{2\lambda\theta(2^{\alpha}+1)^{\delta}}{[\lambda\theta+(2^{\alpha}+1)^{\delta}]^2}\cdot\ln(2^{\alpha}+1)+\frac{2\lambda\theta(2^{\beta}+1)^{\delta}}{[\lambda\theta+(2^{\beta}+1)^{\delta}]^2}\cdot\ln(2^{\beta}+1)\\
&\quad-\frac{\lambda\theta(2^{\alpha}+2^{\beta})^{\delta}}{[\lambda\theta+(2^{\alpha}+2^{\beta})^{\delta}]^2}\cdot\ln(2^{\alpha}+2^{\beta}),\\
\frac{\partial P_A}{\partial\alpha}&=-\frac{\lambda\theta\delta2^{\alpha}\delta}{(\lambda\theta+2^{\alpha}\delta)^2}\cdot\ln2+\frac{2\lambda\theta\delta(2^{\alpha}+1)^{\delta-1}}{[\lambda\theta+(2^{\alpha}+1)^{\delta}]^2}\cdot\ln2-\frac{\lambda\theta(2^{\alpha}+2^{\beta})^{\delta-1}\cdot2^{\alpha}}{[\lambda\theta+(2^{\alpha}+2^{\beta})^{\delta}]^2}\cdot\ln2,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial P_A}{\partial \beta} &= -\frac{\lambda\theta\delta 2^{\beta}\delta}{(\lambda\theta+2^{\beta}\delta)^2}\cdot\ln 2+\frac{2\lambda\theta\delta(2^{\beta}+1)^{\delta-1}}{[\lambda\theta+(2^{\beta}+1)^{\delta}]^2}\cdot\ln 2-\frac{\lambda\theta(2^{\alpha}+2^{\beta})^{\delta-1}\cdot 2^{\beta}}{[\lambda\theta+(2^{\alpha}+2^{\beta})^{\delta}]^2}\cdot\ln 2,\\
\frac{\partial P_A}{\partial \theta} &= \frac{4\lambda}{(1+\lambda\theta)^2}+\frac{4\lambda 2^{\delta}}{(2^{\delta}+\lambda\theta)^2}+\frac{\lambda 2^{\alpha\delta}}{(2^{\alpha\delta}+\lambda\theta)^2}+\frac{\lambda 2^{\beta\delta}}{(2^{\beta\delta}+\lambda\theta)^2}-\frac{2\lambda(2^{\alpha}+1)^{\delta}}{[(2^{\alpha}+1)^{\delta}+\lambda\theta]^2}\\
&\quad -\frac{2\lambda(2^{\beta}+1)^{\delta}}{[(2^{\beta}+1)^{\delta}+\lambda\theta]^2}+\frac{\lambda(2^{\alpha}+2^{\beta})^{\delta}}{[(2^{\alpha}+2^{\beta})^{\delta}+\lambda\theta]^2}.
\end{aligned}$$

P_A的置信度为$1-\alpha$的渐近置信区间为

$$\Big(\widehat{P}_{A2}-\frac{Z_{\alpha/2}\sigma_2(\widehat{\delta},\widehat{\alpha},\widehat{\beta},\widehat{\theta})}{\sqrt{n}},\widehat{P}_{A2}+\frac{Z_{\alpha/2}\sigma_2(\widehat{\delta},\widehat{\alpha},\widehat{\beta},\widehat{\theta})}{\sqrt{n}}\Big).$$

证明从略.

§5. 随机模拟

5.1 强度参数未知的模型结构可靠度的随机模拟

指数分布的随机模拟可通过逆变换法实现, 若$X\sim U(0,1)$分布, 令

$$Y=-\frac{1}{\lambda}\cdot\ln X,$$

则Y服从期望为$1/\lambda$的指数分布. 在本文的模拟过程中, 采用了以下的具体数值: λ分别取0.25, 0.5, 1, 3, 5; $\theta=1.2$; $\delta=0.5$; $\alpha=0.6$, $\beta=0.4$, 样本容量$n=100$, 循环100次, 第l次获得的估计记为$\widehat{P}_{A1l}$, 记

$$\text{均值为}\ \widehat{P}_{A1}=\sum_{l=1}^{100}\frac{\widehat{P}_{A1l}}{100},\qquad \text{样本方差为}\ D(\widehat{P}_{A1})=\sum_{l=1}^{100}\frac{(\widehat{P}_{A1l}-\widehat{P}_{A1})^2}{100},$$

所获得的100组模拟结果的样本均值和方差如表1. 从表1中数据可以看到, $\widehat{P}_{A1}$的渐近方差$\sigma_1^2(\lambda)$和$D(\widehat{P}_{A1})$呈现出随着λ的增大而先增大后减小的趋势.

表1 $\theta=1.2$, $\alpha=0.6$, $\beta=0.4$, $\delta=0.5$

λ	P_A	$\widehat{P}_{A1}$	$D(\widehat{P}_{A1})$	$\sigma_1^2(\lambda)$
0.25	0.8352	0.8348	0.0183	0.0225
0.50	0.7736	0.7733	0.0195	0.0264
1.00	0.9139	0.9117	0.0246	0.0134
3.00	0.8136	0.8134	0.0137	0.0127
5.00	0.8622	0.8623	0.0092	0.0078

5.2 应力参数未知的模型结构可靠度的随机模拟

1) 先产生一组服从(1.1)分布的样本, 由$(x_{1i},x_{2i},x_{3i},x_{4i})$ $(i=1,2,\cdots,n)$可得到$\theta,\delta,\alpha,\beta$的估计量$\widehat{\theta},\widehat{\delta},\widehat{\alpha},\widehat{\beta}$.

2) 根据所得的估计量$\widehat{\theta},\widehat{\delta},\widehat{\alpha},\widehat{\beta}$, 由(4.3)式即可得$P_A$的估计量$\widehat{P}_{A2}$. 在模拟过程中, 采用了以下几组具体的数值: $\lambda=3$, $\theta=1.2$, $\alpha=0.6$, $\beta=0.4$, $\delta=0.1(0.1)0.9$; $\lambda=3$, $\theta=1.2$, $\delta=0.5$, $\beta=0.4$, $\alpha=0.1(0.2)0.9$; $\lambda=3$, $\theta=1.2$, $\delta=0.5$, $\alpha=0.6$, $\beta=0.2,0.4,0.6,0.8,0.9$; $\lambda=3$, $\theta=1.2$, $\delta=0.5$, $\alpha=0.6$, $\beta=0.4$, $\theta=0.1,0.3,0.4,0.6,0.8,1,1.6,2,3,5$.

同样, 取样本容量$n=100$, 循环100次, 第l次获得的估计记为$\widehat{P}_{A2l}$, 记

$$\text{均值为 } \widehat{P}_{A2}=\sum_{l=1}^{100}\frac{\widehat{P}_{A2l}}{100},\qquad \text{样本方差为 } D(\widehat{P}_{A2})=\sum_{l=1}^{100}\frac{(\widehat{P}_{A2l}-\widehat{P}_{A2})^2}{100},$$

所获得的模拟结果如表2–表5. 从模拟结果来看, 本文对结构可靠度P_A所作的估计$\widehat{P}_{A2}$还是很理想的, δ值的变化对$\widehat{P}_{A2}$的影响相对于θ,α,β 值的变化对$\widehat{P}_{A2}$的影响要显著. 随着δ,α,β数值的增大, P_A的值都呈现减小的趋势; 随着θ的增大, P_A的值都呈现先减小后增大的趋势.

表2 $\lambda=3$, $\theta=1.2$, $\alpha=0.6$, $\beta=0.4$

δ	P_A	$\widehat{P}_{A2}$	$D(\widehat{P}_{A2})$
0.1	0.9650	0.9669	0.0459
0.2	0.9288	0.9257	0.0348
0.3	0.8915	0.8934	0.0256
0.4	0.8531	0.8496	0.0538
0.5	0.8136	0.8154	0.0428
0.6	0.7730	0.7718	0.0279
0.7	0.7415	0.7339	0.0212
0.8	0.6890	0.6871	0.0179
0.9	0.6457	0.6483	0.0132

表3 $\lambda=3$, $\theta=1.2$, $\delta=0.5$, $\beta=0.4$

α	P_A	$\widehat{P}_{A2}$	$D(\widehat{P}_{A2})$
0.1	0.8410	0.8424	0.0087
0.3	0.8304	0.8295	0.0069
0.5	0.8193	0.8179	0.0037
0.7	0.8078	0.8063	0.0045
0.9	0.7956	0.7942	0.0032

表4 $\lambda=3,\ \theta=1.2,\ \delta=0.5,\ \alpha=0.6$

β	P_A	$\widehat{P}_{A2}$	$D(\widehat{P}_{A2})$
0.2	0.8236	0.8227	0.0032
0.4	0.8136	0.8130	0.0024
0.6	0.8031	0.8043	0.0041
0.8	0.7922	0.7916	0.0053
0.9	0.7866	0.7848	0.0027

表5 $\lambda=3,\ \delta=0.5,\ \alpha=0.6,\ \beta=0.4$

θ	P_A	$\widehat{P}_{A2}$	$D(\widehat{P}_{A2})$
0.1	0.8351	0.8367	0.0448
0.3	0.7522	0.7508	0.0321
0.4	0.7482	0.7491	0.0145
0.6	0.7597	0.7610	0.0118
0.8	0.7783	0.7795	0.0076
1.0	0.7968	0.7932	0.0059
1.6	0.8412	0.8441	0.0037
2.0	0.8623	0.8635	0.0026
3.0	0.8972	0.8991	0.0023
5.0	0.9321	0.9346	0.0117

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Estimation of Structural Reliability for a Nested Multivariate Exponential Distributed Stress

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We study a kind of stress-strength structural reliability model with stress being nested multivariate exponential distributed and strength being one-dimensional exponential distributed in this paper. On condition that strength parameter unknown and stress parameters known as well as contrary situations, the estimators $\widehat{P}_{A1}$ and $\widehat{P}_{A2}$ of the structural reliability P_A are proposed and their consistency and asymptotic normality are discussed, meanwhile, the estimators' confidence intervals are also shown. Finally, we randomly simulate the estimators $\widehat{P}_{A1}$ and $\widehat{P}_{A2}$ of P_A with the satisfactory results.

Keywords: Structural reliability, nested multivariate exponential distribution, strong consistency, asymptotic property, confidence interval, random simulation.

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