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On Sufficient and Necessary Condition of a Strong Law of Large Numbers for Negatively Associated Random Variables *

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Abstract

In this paper, we obtain the sufficient and necessary condition of a strong law of large numbers for negatively associated random variables with different distributions and finite variances, the Egorov's results for independent random variables are generalized to the case of negatively associated random variables. We also establish a new strong laws of large numbers for negatively associated random variables.

Keywords: Negatively associated random variables, strong law of the large numbers, probability inequalities.

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§1. Introduction

The negatively associated random variables was introduced by Joag-Dev and Proschan [1].

Definition 1.1 A finite family of random variables $\{X_k, 1 \le k \le n\}$ is said to be negatively associated (NA) if

$$\mathsf{Cov}\,(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)) \le 0$$

holds for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \cdots, n\}$, and coordinatewise increasing functions f_1, f_2 such that above covariance exists. An infinite family is NA if each of its finite subfamily is negatively associated.

Clearly, a set of independent random variables is NA.

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As pointed out by Joag-Dev and Proschan[1], a lot of well-known multivariate distributions are NA. Negative association has been applied to reliability theory, multivariate statistical analysis and percolation theory, and attracted extensive attentions (see ref. [1]–[10]).

Let $\{Y_n, n \geq 1\}$ be independent random variables with $\mathsf{E}Y_n = 0$, $s_n^2 := \sum\limits_{k=1}^{n} \mathsf{E}Y_k^2 \uparrow \infty$. A well known law of large numbers is for all $\lambda > 1/2$,

$$\frac{\sum\limits_{k=1}^{n} Y_k}{s_n \log^{\lambda} s_n} \to 0 \qquad \text{a.s.} \qquad (n \to \infty) \tag{1.1}$$

(see Petrov[11]). Furthermore, Egorov[12] showed that for any $\lambda > 0$, (1.1) holds if and only if

$$\sum_{n=1}^{\infty} \mathsf{P}\big(|Y_n| > \varepsilon s_n \log^{\lambda} s_n\big) < \infty$$

for any $\varepsilon > 0$. The law of large numbers (1.1) has already been extended to the case that $\{Y_n, n \geq 1\}$ are NA random variables. For example, Su and Qin[7] obtained the following.

Theorem A Let $\{X_n, n \geq 1\}$ be NA random variables with $\mathsf{E}X_n = 0$. Suppose that there is a random variable X such that $\mathsf{P}(|X_n| > x) \leq \mathsf{P}(|X| > x)$ for any $x > 0$, and $\mathsf{E}X^2 < \infty$. Then

$$\lim_{n \to \infty} \frac{\sum\limits_{k=1}^{n} X_k}{(n \log^{\lambda} n)^{1/2}} = 0 \qquad \text{a.s.}$$

holds for all $\lambda > 0$.

In the present paper, we shall show a revision of the Egorov's theorem for the NA sequence. Our main result is given in Section 2 and the above Theorem A will be obtained as a corollary of Theorem 2.1. Theorem 3.1 in Section 3 will present a new law of large numbers for the NA sequence without finite variances.

We now quote two lemmas for use. The first one is an extension of probability inequalities for independent random variables in Fuk and Nagaev[13] and Borovkov[14], and its proof is obtained due to inequalities of Shao[5].

Lemma 1.1 (Liu and Wu[10]) Suppose that $X_1, X_2, \cdots, X_n$ are NA random variables with $\mathsf{E}X_k = 0$ for all $k = 1, \cdots, n$. If there exists real positive constants $\sigma_1, \sigma_2, \cdots, \sigma_n$ such that $\mathsf{E}X_k^2 \leq \sigma_k^2$, $k = 1, 2, \cdots, n$, then

$$\mathsf{P}\Big(\max_{1 \leq k \leq n} |S_k| \geq x\Big) \leq \sum_{k=1}^{n} \mathsf{P}(|X_k| > y_k) + 2\exp\Big\{1 + \frac{x}{y} - \Big(\frac{x}{y} + \frac{B_n}{y^2}\Big) \ln\Big(1 + \frac{xy}{B_n}\Big)\Big\} \tag{1.2}$$

holds for every $x > 0$, $y_1, \cdots, y_n > 0$, and $y \geq \max\{y_1, y_2, \cdots, y_n\}$, where $S_n := \sum\limits_{k=1}^{n} X_k$ and $B_n := \sum\limits_{k=1}^{n} \sigma_k^2 > 0$.

Lemma 1.2 (Wittmann[15], Lemma 3.3) Let $\{l_n\}$ be an increasing sequence of positive numbers with $\lim\limits_{n\to\infty} l_n=\infty$, then for any $M>1$, there exists a subsequence $\{n_k\}$ $\subset \mathbf{N}:=\{1,2,\cdots\}$ such that

$$Ml_{n_k}\le l_{n_{k+1}}\le M^3 l_{n_k+1}.$$

§2. The Main Results

In the following, suppose that $\{X_n,n\ge1\}$ are NA random variables with $\mathsf{E}X_n=0$ and finite variances, and $\{\sigma_n,n\ge1\}$ are real numbers such that $\mathsf{E}X_n^2\le\sigma_n^2$ for each $n\ge1$. Denote $S_n=\sum\limits_{k=1}^n X_k$, $B_n=\sum\limits_{k=1}^n\sigma_k^2$, and $\log x=\ln\max(e,x)$ for every $x>0$. Write K for a positive constant whose value may be changed in different places. The Egorov's results are extended as follows

Theorem 2.1 If $B_n\uparrow\infty$ as $n\to\infty$, then for any $\lambda>0$

$$\lim_{n\to\infty}\frac{S_n}{(B_n\log^\lambda B_n)^{1/2}}=0\qquad\text{a.s.}\tag{2.1}$$

if and only if

$$\sum_{n=1}^\infty\mathsf{P}\big(|X_n|>\varepsilon(B_n\log^\lambda B_n)^{1/2}\big)<\infty\tag{2.2}$$

holds for any $\varepsilon>0$.

Proof The necessity of the theorem follows from Lemma 3 of Matula[3]. We need only to show the sufficiency. Assume that (2.2) holds for any $\varepsilon>0$. Denoting $a_n=\varepsilon(B_n\log^\lambda B_n)^{1/2}$,

$$Y_n=X_nI_{\{|X_n|\le a_n\}}+a_nI_{\{X_n>a_n\}}-a_nI_{\{X_n<-a_n\}}\qquad\text{and}\qquad Z_n=X_n-Y_n$$

for each $n\ge1$, we have

$$S_n=\sum_{k=1}^n(Y_k-\mathsf{E}Y_k)+\sum_{k=1}^n(Z_k-\mathsf{E}Z_k).\tag{2.3}$$

At first, we show that

$$(B_n\log^\lambda B_n)^{-1/2}\sum_{k=1}^n(Z_k-\mathsf{E}Z_k)\to0\qquad\text{a.s.}\qquad(n\to\infty).\tag{2.4}$$

It is easily seen that

$$\begin{aligned}(B_n\log^\lambda B_n)^{-1/2}\sum_{k=1}^n\mathsf{E}|X_k|I_{\{|X_k|>a_n\}}&\le(B_n\log^\lambda B_n)^{-1/2}a_n^{-1}\sum_{k=1}^n\mathsf{E}|X_k|^2\\&=(\varepsilon B_n\log^\lambda B_n)^{-1}\sum_{k=1}^n\mathsf{E}|X_k|^2\\&\le(\varepsilon\log^\lambda B_n)^{-1}\to0\end{aligned}\tag{2.5}$$

as $n \to \infty$. Note that for any $m \le n$,

$$\begin{aligned}
& (B_n \log^\lambda B_n)^{-1/2} \sum_{k=1}^n \mathsf{E}|X_k| I_{\{a_k < |X_k| \le a_n\}} \\
= \ & (B_n \log^\lambda B_n)^{-1/2} \Big(\sum_{k=1}^m + \sum_{k=m+1}^n \Big) \mathsf{E}|X_k| I_{\{a_k < |X_k| \le a_n\}} \\
\le \ & (B_n \log^\lambda B_n)^{-1/2} \sum_{k=1}^m \mathsf{E}|X_k| + \varepsilon \sum_{k=m+1}^\infty \mathsf{P}(|X_k| > a_k).
\end{aligned}$$

By letting $n \to \infty$ and $m \to \infty$ in turn, we get further

$$(B_n \log^\lambda B_n)^{-1/2} \sum_{k=1}^n \mathsf{E}|X_k| I_{\{a_k < |X_k| \le a_n\}} \to 0 \tag{2.6}$$

as $n \to \infty$. Since $Z_k = (X_k - a_k) I_{\{X_k > a_k\}} + (X_k + a_k) I_{\{X_k < -a_k\}}$, it follows from (2.5) and (2.6) that

$$\begin{aligned}
& (B_n \log^\lambda B_n)^{-1/2} \sum_{k=1}^n \mathsf{E}|Z_k| \\
\le \ & (B_n \log^\lambda B_n)^{-1/2} \sum_{k=1}^n \mathsf{E}|X_k| I_{\{|X_k| > a_k\}} \\
\le \ & (B_n \log^\lambda B_n)^{-1/2} \Big(\sum_{k=1}^n \mathsf{E}|X_k| I_{\{|X_k| > a_n\}} + \sum_{k=1}^n \mathsf{E}|X_k| I_{\{a_k < |X_k| \le a_n\}} \Big) \to 0.
\end{aligned} \tag{2.7}$$

Therefore (2.4) is obtained by (2.2) and the Borel-Cantelli Lemma.

Secondly, note that $\{Y_n - \mathsf{E}Y_n, n \ge 1\}$ are also NA random variables (see Joag-Dev and Proschan[1]), and

$$\mathsf{E}(Y_n - \mathsf{E}Y_n)^2 \le \mathsf{E}Y_n^2 = \mathsf{E}X_n^2 I_{\{|X_n| \le a_n\}} + a_n^2 I_{\{|X_n| > a_n\}} \le \mathsf{E}X_n^2 \le \sigma_n^2$$

for each $n \ge 1$. Applying Lemma 1.1 to $\{Y_n - \mathsf{E}Y_n, n \ge 1\}$, set $d > 2/\lambda$, $x = da_n$, $y = 2a_n$ and $y_k = 2a_k$ for each $1 \le k \le n$, we get

$$\begin{aligned}
\mathsf{P}\Big(\max_{1 \le k \le n} \Big| \sum_{j=1}^k (Y_j - \mathsf{E}Y_j) \Big| \ge da_n \Big) & \le 2e \exp\Big\{ 1 + \frac{x}{y} - \Big(\frac{x}{y} + \frac{B_n}{y^2} \Big) \ln \Big(1 + \frac{xy}{B_n} \Big) \Big\} \\
& \le K \exp\Big\{ -\frac{d}{2} \ln(1 + 2d\varepsilon^2 \log^\lambda B_n) \Big\} \\
& \le K (\log B_n)^{-d\lambda/2}.
\end{aligned} \tag{2.8}$$

Moreover, for any $M > 1$, it follows from Lemma 1.2 that there exists a subsequence $\{n_k\} \subset \mathbf{N}$ such that

$$M a_{n_k} \le a_{n_{k+1}} \le M^3 a_{n_k + 1}.$$

Since $a_{n+1}/a_n \le B_{n+1}/B_n$ holds for large n, the following inequality

$$M \le a_{n_{k+1}}/a_{n_k} \le B_{n_{k+1}}/B_{n_k}$$

is obtained for large k. Therefore we have form (2.8) that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathsf{P}\Big(\max_{n_k<n\le n_{k+1}}\Big(\Big|\sum_{j=1}^{n}(Y_j-\mathsf{E}Y_j)\Big|/(M^3a_n)\Big)>d\Big)\\
\le\ &\sum_{k=1}^{\infty} \mathsf{P}\Big(\max_{n_k<n\le n_{k+1}}\Big|\sum_{j=1}^{n}(Y_j-\mathsf{E}Y_j)\Big|>da_{n_{k+1}}\Big)\\
\le\ &\sum_{k=1}^{\infty} \mathsf{P}\Big(\max_{1<n\le n_{k+1}}\Big|\sum_{j=1}^{n}(Y_j-\mathsf{E}Y_j)\Big|>da_{n_{k+1}}\Big)\\
\le\ &K\sum_{k=1}^{\infty}(\log B_{n_{k+1}})^{-d\lambda/2}\le K\sum_{k=1}^{\infty}(k\log M+\log B_{n_1})^{-d\lambda/2}<\infty.
\end{aligned}$$

By the Borel-Cantelli Lemma, it holds that

$$\limsup_{n\to\infty}\Big|\sum_{k=1}^{n}(Y_k-\mathsf{E}Y_k)\Big|/a_n\le dM^3 \qquad \text{a.s..}$$

Since $M>1$ and $d>2/\lambda$ are arbitrary, we get further

$$\limsup_{n\to\infty}\Big|\sum_{k=1}^{n}(Y_k-\mathsf{E}Y_k)\Big|/(B_n\log^{\lambda}B_n)^{1/2}\le 2\varepsilon/\lambda \qquad \text{a.s..} \tag{2.9}$$

Combining (2.4), (2.9) with (2.3), we obtain

$$\limsup_{n\to\infty}\frac{|S_n|}{(B_n\log^{\lambda}B_n)^{1/2}}\le\frac{2\varepsilon}{\lambda} \qquad \text{a.s..}$$

This proves (2.2) by letting $\varepsilon\to 0$, the proof of the theorem is completed. □

Corollary 2.1 Let $\{X_n, n\ge 1\}$ satisfy the conditions of Theorem 2.1. If $\lambda>1$, then (2.1) holds.

Proof By Markov's inequality and Lemma 6.18 in Petrov[11], it is easily seen that

$$\sum_{n=1}^{\infty}\mathsf{P}(|X_n|>\varepsilon(B_n\log^{\lambda}B_n)^{1/2})\le\frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{\mathsf{E}X_n^2}{B_n\log^{\lambda}B_n}<\infty$$

for $\lambda>1$ and any $\varepsilon>0$. Hence the assertion of the corollary follows from Theorem 2.1. □

Remark 2.1 Consider the case of $0<\lambda\le 1$. We have from the proof of Theorem 2.1 that, if there exist $c>0$ and $0<\lambda\le 1$ such that

$$\sum_{n=1}^{\infty}\mathsf{P}(|X_n|>c(B_n\log^{\lambda}B_n)^{1/2})<\infty,$$

then

$$\limsup_{n\to\infty}\frac{S_n}{(B_n\log^{\lambda}B_n)^{1/2}}\le\frac{2c}{\lambda} \qquad \text{a.s..} \tag{2.10}$$

Remark 2.2 Theorem A of Su and Qin[7] is a corollary of Theorem 2.1. In fact, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\mathsf{P}(|X_n|>\varepsilon(n\log^{\lambda}n)^{1/2}) &\le \sum_{n=1}^{\infty}\mathsf{P}(|X|>\varepsilon(n\log^{\lambda}n)^{1/2})\\ &\le \sum_{n=1}^{\infty}\mathsf{P}(X^2>\varepsilon^2 n)\le\frac{\mathsf{E}X^2}{\varepsilon^2}<\infty\end{aligned}$$

for any $\varepsilon>0$. Hence the conditions of Theorem 2.1 are satisfied.

Remark 2.3 In particular, if these exist $p>2$ such that

$$\sum_{n=1}^{\infty}(B_n\log^{\lambda}B_n)^{-p/2}\mathsf{E}|X_n|^p<\infty$$

for some $0<\lambda\le 1$. Then (2.1) holds.

§3. Applications

We now use Theorem 2.1 to get a new strong laws of large numbers similar to (1.1).

Theorem 3.1 Suppose that there is a random variable X, real numbers $a>0$, $b>0$ and $n_0\in\mathbf{N}$ such that $\sup\limits_{n\ge n_0}\mathsf{P}(|X_n|>x)\le a\mathsf{P}(|X|>bx)$ for all $x>0$. If

$$\mathsf{E}(|X|^t\log^{-\beta}|X|)<\infty \tag{3.1}$$

holds for some $\beta\ge 0$ and either $0<t\le 1$ or $1<t\le 2$ with $\mathsf{E}X_n=0$ for all $n\ge n_0$, then

$$\lim_{n\to\infty}\frac{S_n}{n^{1/t}\log^{\alpha}n}=0\qquad \text{a.s.} \tag{3.2}$$

for all $\alpha>\beta/t$.

Proof Without loss generality we may assume $n_0=1$, and denote $a_n=\varepsilon n^{1/t}\log^{\alpha}n$ for given $0<\varepsilon\le 1$ and $\alpha>\beta/t$. Let Y_n and Z_n be the same as in the proof of Theorem 2.1. It is easily seen form (3.1) that

$$\begin{aligned}\sum_{n=1}^{\infty}\mathsf{P}(|X_n|>a_n) &\le a\sum_{n=1}^{\infty}\mathsf{P}(|X|>b\varepsilon n^{1/t}\log^{\alpha}n)\\ &\le a\sum_{n=1}^{\infty}\mathsf{P}(|X|^t\log^{-\beta}|X|>Kn\log^{\alpha t-\beta}n)\\ &\le K\mathsf{E}(|X|^t\log^{-\beta}|X|)<\infty.\end{aligned}\tag{3.3}$$

Note that $\{Y_n-\mathsf{E}Y_n,n\ge 1\}$ are NA random variables such that

$$\begin{aligned}\sum_{k=1}^{n}\mathsf{E}(Y_k-\mathsf{E}Y_k)^2 &\le \sum_{k=1}^{n}(\mathsf{E}X_k^2 I_{\{|X_k|\le a_k\}}+a_k^2\mathsf{P}(|X_k|>a_k))\\ &\le Kna_n^{2-t}\log^{\beta}a_n\mathsf{E}(|X|^t\log^{-\beta}|X|)\\ &\le Mn^{2/t}(\log n)^{\alpha(2-t)+\beta}\end{aligned}$$

holds for some $M > 4$ and large n. Letting $B_n = Mn^{2/t}(\log n)^{\alpha(2-t)+\beta}$ and $\lambda = \alpha t - \beta$, we have

$$\lim_{n\to\infty} B_n \log^\lambda B_n (n^{1/t}\log^\alpha n)^{-2} = M > 4.$$

It follows from (2.10) that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(Y_k - \mathsf{E}Y_k)}{n^{1/t}\log^\alpha n} = M^{1/2} \limsup_{n\to\infty} \frac{\sum_{k=1}^{n}(Y_k - \mathsf{E}Y_k)}{(B_n \log^\lambda B_n)^{1/2}} \le \frac{2\varepsilon M^{1/2}}{\alpha t - \beta} \qquad \text{a.s..} \tag{3.4}$$

Consider the case that $1 < t \le 2$ and $\mathsf{E}X_n = 0$ for all $n \ge 1$. It follows from (3.1) that

$$\begin{aligned}
\frac{1}{n^{1/t}\log^\alpha n}\sum_{k=1}^{n}\mathsf{E}|X_k|I_{\{|X_k|>a_n\}} &\le K\frac{n}{n^{1/t}\log^\alpha n}\mathsf{E}|X|I_{\{|X|>ba_n\}} \\
&\le K\frac{n}{n^{1/t}\log^\alpha n}\frac{\log^\beta a_n}{a_n^{t-1}}\mathsf{E}(X^t\log^{-\beta}|X|) \\
&\le K(\log n)^{\beta-\alpha t} \to 0 \qquad (n\to\infty).
\end{aligned}$$

By similar argument as in the proof of Theorem 2.1, we may show

$$\frac{1}{n^{1/t}\log^\alpha n}\Big|\sum_{k=1}^{n}\mathsf{E}Y_k\Big| = \frac{1}{n^{1/t}\log^\alpha n}\Big|\sum_{k=1}^{n}\mathsf{E}Z_k\Big| \to 0 \qquad \text{a.s.}$$

as $n \to \infty$. Suppose $0 < t \le 1$, we conclude that

$$\begin{aligned}
\frac{1}{n^{1/t}\log^\alpha n}\Big|\sum_{k=1}^{n}\mathsf{E}Y_k\Big| &\le \frac{1}{n^{1/t}\log^\alpha n}\sum_{k=1}^{n}(\mathsf{E}|X_k|I_{\{|X_k|\le a_k\}} + a_k\mathsf{P}(|X_k|>a_k)) \\
&\le K\frac{n}{n^{1/t}\log^\alpha n}a_n^{1-t}\log^\beta a_n\mathsf{E}(|X|^t\log^{-\beta}|X|) \\
&\le K(\log n)^{\beta-\alpha t} \to 0
\end{aligned}$$

as $n \to \infty$. This implies by (3.4), (3.3) and the Borel-Cantelli Lemma that

$$\limsup_{n\to\infty}\frac{S_n}{n^{1/t}\log^\alpha n} \le \frac{2\varepsilon M^{1/2}}{\alpha t - \beta} \qquad \text{a.s..}$$

Therefore (3.2) is proved by letting $\varepsilon \to 0$. The proof of Theorem 3.1 is completed. □

Remark 3.1 As a particular case, if $t = 2$, $0 \le \beta < 1$, then result of Liang and Su[8] follows from the Theorem 3.1 and can be improved.

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NA随机变量列一类强大数律的充分必要条件

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本文给出了具有不同分布NA随机变量列满足一类强大数律的充分必要条件, 从而将Egorov对独立随机变量列建立的结果推广到NA随机变量情形; 作为应用, 我们还建立了一个新的强大数律.

关键词: NA随机变量, 强大数律, 概率不等式.

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