应用概率统计　第二十四卷
第四期　2008年8月

Chinese Journal of Applied Probability
and Statistics Vol.24 No.4 Aug. 2008

Updating Equations of Linear Models with Dependent Errors *

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Abstract

Updating equations for linear models have been investigated for several years, however they have been restricted to the models with uncorrelated error structure, or considered for univariate linear models involving fixed parameters. This paper has considered updating equations for multivariate linear models with correlated error structure, and outlined updating equations of BLUE of unknown parameter matrix and residual when parameter, data or index is supplemented. The formulae are fitted for the cases of both fixed and random parameter matrices.

Keywords: Linear model, updating equation, best linear unbiased estimator (BLUE), minimum mean square linear estimator (MMSLE), residual.

AMS Subject Classification: Primary 62J05; secondary 62C05.

§1. Introduction

Consider the univariate linear model $\{y, X\beta, \sigma^2 V\}$, where the parameters β and σ^2 are unknown and the design matrix X is fixed. The statistical quantities we are interested in include: the best linear unbiased estimates (BLUEs) of the estimable parametric functions, variance-covariance matrices of such estimates, the residual sum of squares and the likelihood ratio tests for testable linear hypotheses. With the time going by, some observations and regression parameters sometimes would have some changes. So we are primarily concerned with the changes of these statistical quantities when some observations are appended or deleted, as well as when some regressors are added or dropped. This is the focus of updating equations for linear models. It may be categorized by additional

*This research is supported by National Social Science Foundation of China (07CTJ001), National Natural Science Foundation of China (10701021), Philosophy and Social Science Planning Project of Zhejiang Province of China (06CGYJ21YQB), Major Project of Zhejiang University of Finance and Economics (2008YJZ06).

Received March 22, 2007. Revised October 25, 2007.

data and/or parameters are added, by whether parameter vector are fixed, random, or mixed, and by whether the error process is uncorrelated or not.

Updating equations for parameter estimates when parameters are added were considered for uncorrelated errors and fixed parameter vectors by Cochran (1938) and Quenouille (1950). The case of additional data but no additional regressors for a fixed parameter vector and uncorrelated errors was analyzed by Plackett (1950) and Young (1984). McGilchrist and Sandland (1979) extended these results to correlated errors when additional data are added one at a time. Bhimasankaram and Jammalamadaka (1994) discussed deletion of a single observation of more than one correlated errors. Haslett (1985) considered simultaneous addition of more than one correlated datum.

When linear model parameters were random and varied with each update, Anderson and Moore (1979) studied parameter estimates where there is no correlation between the error structure from different updates. The methodology, again for uncorrelated errors, has been extended by Sallas and Harville (1981) to include mixed models, i.e. the linear models containing both fixed and random effects, all of which are to be estimated.

Haslett (1996) outlined updating equations of some statistical quantities in univariate linear models where adding parameters and data simultaneously. Jammalamadaka and D. Sengupta (1999) defined linear zero functions which provided an intuitive way of developing important results for both cases – nonsingular and singular variance matrices in connection with the general linear model, and then obtained updating equations when adding or deleting a set of observations, adding or dropping a group of parameters in the general linear model.

In this paper, we argue updating equations for multivariate linear model $Y = X\Theta + UE$ with correlated error structure. In § 2 some necessary preliminaries are listed. In § 3 we outline our main results: updating equations of unknown parameter matrices and residual matrices when adding parameters, observations or indexes.

§2. Preliminaries

Consider the multivariate linear model

$$\begin{cases} Y_{11} = X_{11}\Theta_{11} + U_1E_1, \\ \text{All lines of } E_1 \text{ are uncorrelated, with the same variance-covariance matrix} \\ \Sigma_{11} \text{ and the mean value is } 0, \end{cases} \tag{2.1}$$

where Y_{11} is an $n \times p$ matrix. It is observed matrix in the non-random case, and it is composed of the mean matrix of the random parameter Θ_{11} and observed matrix in the random case. Θ_{11} is a fixed or random parameter matrix to be estimated, X_{11} is a fixed design matrix, having full-column rank, U_1 has full-row rank, and $\Sigma_{11} > 0$.

Theorem 2.1 In the model (2.1), the minimum mean square linear estimator (MMSLE) of Θ is

$$\widehat{\Theta}_{11} = (X_{11}'(U_1U_1')^{-1}X_{11})^{-1}X_{11}'(U_1U_1')^{-1}Y_{11}, \tag{2.2}$$

and the residual is

$$\begin{aligned} &\mathrm{RSS}_1 = Y_{11}'(U_1U_1')^{-1}Y_{11} - \widehat{\Theta}_{11}'X_{11}'(U_1U_1')^{-1}Y_{11}, \\ \text{or}\quad &\mathrm{RSS}_1 = Y_{11}'(U_1U_1')^{-1/2}(I - P_{(U_1U_1')^{-1/2}X_{11}})(U_1U_1')^{-1/2}Y_{11}. \end{aligned} \tag{2.3}$$

Proof Herein we only prove the results in the non-random case, and the results in the random case can follow from Lemmas 2.4 and 2.6 (Duncan and Horn, 1972).

We can vectorize the model to be

$$\vec{Y}_{11} = (I \otimes X_{11})\vec{\Theta}_{11} + (I \otimes U_1)\vec{E}_1,$$

thus the original model is changed into

$$\begin{cases} \mathsf{E}\vec{Y}_{11} = (I \otimes X_{11})\vec{\Theta}_{11}, \\ \mathsf{Var}\,\vec{Y}_{11} = \Sigma_{11} \otimes U_1U_1'. \end{cases} \tag{2.4}$$

According to the classical theory of linear model (see Rao (1973)), Model (2.4) has MMSLE of $\vec{\Theta}$ given by

$$\begin{aligned} \widehat{\vec{\Theta}}_{11} &= [(I \otimes X_{11})'(\Sigma_{11} \otimes U_1U_1')^{-1}(I \otimes X_{11})]^{-1}(I \otimes X_{11})'(\Sigma_{11} \otimes U_1U_1')^{-1}\vec{Y}_{11} \\ &= \mathrm{Vec}((X_{11}'(U_1U_1')^{-1}X_{11})^{-1}X_{11}'(U_1U_1')^{-1}Y_{11}), \end{aligned}$$

thus

$$\widehat{\Theta}_{11} = (X_{11}'(U_1U_1')^{-1}X_{11})^{-1}X_{11}'(U_1U_1')^{-1}Y_{11}.$$

Use $(U_1U_1')^{-1/2}$ premultiply the model, and then

$$\overrightarrow{(U_1U_1')^{-1/2}Y_{11}} = (I \otimes (U_1U_1')^{-1/2}X_{11})\vec{\Theta}_{11} + (I \otimes (U_1U_1')^{-1/2}U_1)\vec{E}_1.$$

So

$$\begin{aligned} \mathsf{Var}\,(\overrightarrow{(U_1U_1')^{-1/2}Y_{11}}) &= (I \otimes (U_1U_1')^{-1/2}U_1)(\Sigma_{11} \otimes I)(I \otimes U_1'(U_1U_1')^{-1/2}) \\ &= \Sigma_{11} \otimes I, \end{aligned}$$

then

$$\begin{aligned}\mathrm{RSS}_1 &= [(U_1U_1')^{-1/2}(Y_{11}-X_{11}\widehat{\Theta}_{11})]'[(U_1U_1')^{-1/2}(Y_{11}-X_{11}\widehat{\Theta}_{11})]\\ &= Y_{11}'(U_1U_1')^{-1}Y_{11}-\widehat{\Theta}_{11}'X_{11}'(U_1U_1')^{-1}Y_{11},\end{aligned}$$

or

$$\mathrm{RSS}_1 = Y_{11}'(U_1U_1')^{-1/2}(I-P_{(U_1U_1')^{-1/2}X_{11}})(U_1U_1')^{-1/2}Y_{11}.$$

Suppose that $Y_{11}=(y_1,\cdots,y_p)$, $E_1=(\varepsilon_1,\cdots,\varepsilon_p)$, $(\Sigma_{11})_{ij}=\sigma_{ij}$, then

$$\begin{aligned}(\mathsf{E}(\mathrm{RSS}_1))_{ij} &= \mathsf{E}(y_i'(U_1U_1')^{-1/2}(I-P_{(U_1U_1')^{-1/2}X_{11}})(U_1U_1')^{-1/2}y_j)\\ &= \mathsf{E}\mathrm{tr}(\varepsilon_i'U_1'(U_1U_1')^{-1/2}(I-P_{(U_1U_1')^{-1/2}X_{11}})(U_1U_1')^{-1/2}U_1\varepsilon_j)\\ &= \mathrm{tr}(I-P_{(U_1U_1')^{-1/2}X_{11}})\mathsf{E}(U_1U_1')^{-1/2}U_1\varepsilon_j\varepsilon_i'U_1'(U_1U_1')^{-1/2}\\ &= \sigma_{ij}(n-rkX_{11}).\end{aligned}$$

So

$$\mathsf{E}(\mathrm{RSS}_1)=(n-rkX_{11})\Sigma_{11}.$$

$\square$

§3. Main Results

The original model can generate the following three models when adding parameters, observations or indexes. Now we discuss updating equations of the three models separately.

(I) Model 1 (additional parameters)

$$Y_{11}=(X_{11}:X_{12})\begin{pmatrix}\Theta_{11}\\ \Theta_{21}\end{pmatrix}+U_1E_1, \tag{3.1}$$

where $\mathsf{Var}\,E_1=\Sigma_{11}>0$, $X_{12}\neq 0$.

Define that

$$\begin{aligned}&R_{11}=(U_1U_1')^{-1}-(U_1U_1')^{-1}X_{11}(X_{11}'(U_1U_1')^{-1}X_{11})^{-1}X_{11}'(U_1U_1')^{-1},\\ &L_{21}=X_{12}'(U_1U_1')^{-1}X_{11}(X_{11}'(U_1U_1')^{-1}X_{11})^{-1},\\ &M_{22}=X_{12}'R_{11}X_{12}.\end{aligned} \tag{3.2}$$

Theorem 3.1 In the model (3.1), if X_{12} is of full-column rank, and $R(X_{11}) \cap R(X_{12}) = \{0\}$, then the MMSLE $\widetilde{\Theta} = (\widetilde{\Theta}'_{11} : \widetilde{\Theta}'_{21})'$ is given by

$$\begin{pmatrix} \widetilde{\Theta}_{11} \\ \widetilde{\Theta}_{21} \end{pmatrix} = \begin{pmatrix} \widehat{\Theta}_{11} \\ 0 \end{pmatrix} - \begin{pmatrix} L'_{21} \\ -I \end{pmatrix} M_{22}^{-1} X'_{12} R_{11} Y_{11}, \tag{3.3}$$

and the residual RSS_2 is

$$\text{RSS}_2 = \text{RSS}_1 - \widetilde{\Theta}'_{21} X'_{12} R_{11} Y_{11}. \tag{3.4}$$

Proof The proof method is similar to Theorem 3.1 of Haslett (1985), so omitted. □

For $\text{RSS}_2 = \text{RSS}_1 - \widetilde{\Theta}'_{21} X'_{12} R_{11} Y_{11}$, and $\widetilde{\Theta}'_{21} X'_{12} R_{11} Y_{11} = Y'_{11} R_{11} X_{12} M_{22}^{-1} X'_{12} R_{11} Y_{11} \geq 0$, so $\text{RSS}_2 \leq \text{RSS}_1$. This result shows that the addition of paramors improves the precision of MMSLE of Θ.

When $(X_{11} : X_{12})$ has not full-column rank, we can't use this formula directly. However because X_{11} has full-column rank, we can combine the columns of X_{12} with those of X_{11}. The columns which are linearly independent with columns of X_{11} are left, and the others are replaced with zero vectors. Then $(X_{11} : X_{12})$ is changed into $(X_{11} : X^*_{12} : 0)$. At this time $(X_{11} : X^*_{12})$ has full-column rank, and the columns which are linearly dependent with them can be linearly expressed by them. So these columns are unnecessary, and they can be deleted. Thus $(X_{11} : X_{12})$ turns to $(X_{11} : X^*_{12})$, and $\begin{pmatrix} \Theta_{11} \\ \Theta_{21} \end{pmatrix}$ turns to $\begin{pmatrix} \Theta_{11} \\ \Theta^*_{21} \end{pmatrix}$. In this way we don't lose information, and $(X_{11} : X^*_{12})$ has full-column rank, so we can get the updating equations according to the way of this part.

(II) Model 2 (additional data)

$$\begin{pmatrix} Y_{11} \\ Y_{21} \end{pmatrix} = \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix} \Theta_{11} + \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} E_1, \tag{3.5}$$

where $\mathsf{Var}\, E_1 = \Sigma_{11} > 0$, $U_2 \neq 0$.

Define that

$$\begin{aligned} R_A &= Y_{21} - G'_A Y_{11}, & R_B &= G'_B X'_{11} (U_1 U'_1)^{-1} Y_{11}, \\ C_A &= U_2 U'_2 - U_2 U'_1 G_A, & C_B &= G'_B Z_A, \\ G_A &= (U_1 U'_1)^{-1} U_1 U'_2, & G_B &= (X'_{11} (U_1 U'_1)^{-1} X_{11})^{-1} Z_A, \\ Z_A &= X'_{21} - X'_{11} G_A. \end{aligned} \tag{3.6}$$

Theorem 3.2 In Model (3.5), when $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ is of full-row rank,

$$\begin{aligned} &\text{(i)} \quad (UU')^{-1} = \begin{pmatrix} (U_1U_1')^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} G_A \\ -I \end{pmatrix} C_A^{-1}(G_A' : -I), \\ &\text{(ii)} \quad (X'(UU')^{-1}X)^{-1} = (X_{11}'(U_1U_1')^{-1}X_{11})^{-1} - G_B(C_A+C_B)^{-1}G_B', \\ &\text{(iii)} \quad X'(UU')^{-1}Y = X_{11}'(U_1U_1')^{-1}Y_{11} + Z_AC_A^{-1}R_A, \end{aligned} \tag{3.7}$$

where $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$, $X = \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix}$, and $Y = \begin{pmatrix} Y_{11} \\ Y_{21} \end{pmatrix}$.

Proof It can be proved with a series of inverse operations of matrix. □

Theorem 3.3 In Model (3.5), if $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ is of full-row rank, then the MMSLE of Θ_{11} is given by

$$\widetilde{\Theta}_{11} = \widehat{\Theta}_{11} + G_B(C_A+C_B)^{-1}(R_A - R_B), \tag{3.8}$$

and the residual RSS_2 is given by

$$\mathrm{RSS}_2 = \mathrm{RSS}_1 + (R_A-R_B)'(C_A+C_B)^{-1}(R_A-R_B). \tag{3.9}$$

Proof It's easy to prove with Theorem 3.2. □

Notice that the equations (3.8) and (3.9) have strong statistics meaning. When $\Sigma_{11} = I$, $\mathsf{Var}\,(R_A - R_B) = C_A + C_B$. When $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ hasn't full-row rank, we can't use Theorem 3.3 directly for the conditions of this Theorem are not satisfied. But we can do as just now combining rows of U_2 with them of U_1. Those rows, linearly independent with them of U_1 are left, and the others, linearly dependent with rows of U_1, are replaced with zero vectors. Then the model will turn to

$$\begin{pmatrix} Y_{11} \\ Y_{21}^* \\ 0 \end{pmatrix} = \begin{pmatrix} X_{11} \\ X_{21}^* \\ 0 \end{pmatrix} \Theta_{11} + \begin{pmatrix} U_1 \\ U_2^* \\ 0 \end{pmatrix} E_1,$$

where $\begin{pmatrix} U_1 \\ U_2^* \end{pmatrix}$ has full-row rank, and the rows linearly dependent with them can be expressed by them. And then the model is changed into

$$\begin{pmatrix} Y_{11} \\ Y_{21}^* \end{pmatrix} = \begin{pmatrix} X_{11} \\ X_{21}^* \end{pmatrix} \Theta_{11} + \begin{pmatrix} U_1 \\ U_2^* \end{pmatrix} E_1,$$

with full-row rank $\begin{pmatrix} U_1 \\ U_2^* \end{pmatrix}$. So we can obtain the updating equations using the method of this part.

(III) Model 3 (additional index)

$$(Y_{11} : Y_{12}) = X_{11}(\Theta_{11} : \Theta_{12}) + U_1(E_1 : E_2), \tag{3.10}$$

where $\mathsf{Var}\,(E_1 : E_2) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} > 0$.

Theorem 3.4 In Model (3.10), the MMSLE $\widetilde{\Theta} = (\widetilde{\Theta}_{11} : \widetilde{\Theta}_{12})$ is given by

$$\widetilde{\Theta} = (\widehat{\Theta}_{11} : (X'_{11}(U_1U'_1)^{-1}X_{11})^{-1}X'_{11}(U_1U'_1)^{-1}Y_{12}), \tag{3.11}$$

and the residual RSS_2 is

$$\mathrm{RSS}_2 = \begin{pmatrix} \mathrm{RSS}_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & Y'_{11}R_{11}Y_{12} \\ Y'_{12}R_{11}Y_{11} & Y'_{12}R_{11}Y_{12} \end{pmatrix}, \tag{3.12}$$

where R_{11} is defined as (3.2).

Proof Because $X = X_{11}$ and $U = U_1$ in Model (3.10) are completely as same as them in Model (2.1), it's easy to get the result according to Theorem 2.1. □

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具有相关误差的多元线性模型的更新方程

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已知的线性模型的更新方程是在对模型加了不相关误差结构的约束, 或只对带有固定参数的一元线性模型考虑的. 本文考虑具有相关误差的多元线性模型下的更新方程, 给出了在补充参数, 数据或指标时, 未知参数阵的最佳线性无偏估计及残积阵的更新方程. 公式适用于固定参数与随机参数两种情形.

关键词: 线性模型, 更新方程, 最佳线性无偏估计, 最小均方线性估计, 残积阵.

学科分类号: O212.1.