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Valuation of Equity-Indexed Annuity under Jump Diffusion Process *

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Abstract

The Equity-Indexed Annuity (EIA) contract offers a proportional participation in the return on a specified equity index, in addition to a guaranteed return on the single premium. In general, valuation of Equity-Indexed Annuity is often assumed that the equity index is within the Black-Scholes framework. But some rare events (release of an unexpected economic figure, major political changes or even a natural disaster in a major economy) can lead to brusque variations in prices. So in the present work we study the equity index following a jump diffusion process. By Esscher transform, we obtain a closed form of the valuation of point-to-point EIA, which can be expressed as a function of some pricing factors. Finally, we conduct several numerical experiments in which, the break even participation rate α can be solved when the other factors are fixed. The relationship between α and the other factors are also discussed.

Keywords: Equity-indexed annuities, Esscher transform, participation rate, point to point, jump diffusion process.

AMS Subject Classification: 62E20.

§1. Introduction

Equity-Indexed Annuity (EIA) is essentially an equity-linked deferred annuity that earns a minimum rate of interest and offers a potential gain that is tied to the performance of a stock index or an equity mutual fund, typically of the Standard and Poor's 500 index, i.e. when a stock-index goes up, EIA provides policyholders with a rate of return connected to the index return; when the index goes down, EIA provides policyholders with a minimum guaranteed return.

The EIA product was originally introduced in 1995 by Keyport Life Insurance Co.. Since then, EIAs have enjoyed some popularity in both the United States and Canada. The sales of EIAs increased from \$1.5 billion in 1996 to \$6 billion 2001, and sales of EIAs

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in 2002, 2003, 2004, 2005 were at \$12 billion, \$14.4 billion, \$23.4 billion, \$26.7 billion, respectively. See the Advantage Group reports (Marrion 2003, 2004, 2005[15]) for more details.

There are several reasons for the increasing popularity of EIAs. First, EIAs offer returns that are linked to the performance of an equity index and never fall lower than a minimum guaranteed return (usually 3% per annum). Thus EIAs may be the most suitable products for those who are normally reluctant to buy traditional fixed annuities because of low returns and those who are reluctant to buy mutual funds and stocks for fear of the high volatility in the stock market. Second, EIAs are tax-deferred —— the customers pay no taxes on earnings until they make a withdrawal. Finally, EIAs appeal to insurance companies and agents because companies needn't register with the Securities and Exchange Commission (SEC) and agents do not need a special license to sell EIAs.

There are several indexing methods for EIAs. They are: point-to-point, annual reset, high water mark. The index growth with point-to-point indexing is based on the growth between two time point. The index growth with annual reset option is measured each year by comparing the index level at the beginning and the end of the year. The index growth with a high water mark feature is calculated to the highest index anniversary value over the entire term of the annuity.

Pricing EIAs is a challenging problem due to the complex payoff structure. How to price EIAs has been extensively studied. See Tiong (2000), Gerber and Shiu (2003), Lin and Tan (2003), Hardy (2003 and 2004), Lee (2003), Jaimungal (2004), Kijima and Wong (2006). In general, it is often assumed that the equity index is within the Black-Scholes framework. That is, the equity index follows a lognormal process. But some rare events (release of an unexpected economic figure, major political changes or even a natural disaster in a major economy) can lead to brusque variations in prices. In the present work we consider a more general economic model, by assuming that equity index follows a jump diffusion process. This model is an incomplete market model, so there are many equivalent martingale measures, four kinds of measures have been proposed, including the minimal martingale measure, the Esscher martingale measure, the minimal entropy martingale measure and the utility martingale measure. In this paper, we use the method of Esscher transform to price point-to-point EIA.

The paper is organized as follows. Esscher transform was introduced in Section 2. In section 3, we give the economic model and find Esscher measure under the given model. Finally, we price EIA under jump diffusion process in section 4. Some numerical experiments are also carried out in this section to show the relationship between the break even participation rate α and other parameters.

§2. The Risk-Neutral Esscher Transform

Definition 2.1 For a distribution function $F(x)$, let h be a real number such that

$$M(h)=\int_{-\infty}^{\infty}e^{hx}\mathrm{d}F(x) \tag{2.1}$$

exists. As a function in x,

$$F(x;h)=\int_{-\infty}^{x}\frac{e^{hy}}{M(h)}\mathrm{d}F(y) \tag{2.2}$$

is a distribution function, and it is called the Esscher transform (parameter h) of $F(x)$.

In particular, suppose $F(x)$ has a density function $f(x)$, then

$$M(h)=\int_{-\infty}^{\infty}e^{hx}f(x)\mathrm{d}x, \tag{2.3}$$

and

$$f(x;h)=\frac{e^{hx}f(x)}{M(h)} \tag{2.4}$$

is also called the Esscher transform (parameter h) of the original distribution.

For $t\geq 0$, $S(t)$ denotes the price of a stock or security at time t. We assume that there is a stochastic process $\{X(t),t\geq 0\}$, with stationary and independent increments, $X(0)=0$, such that

$$S(t)=S(0)e^{X(t)},\qquad t\geq 0. \tag{2.5}$$

For each t, the random variable $X(t)$, which may be interpreted as the continuously compounded rate of return over the t periods, has an infinitely divisible distribution. Let

$$F(x,t)=\mathsf{P}[X(t)\leq x] \tag{2.6}$$

be its cumulative distribution function, and

$$M(z,t)=\mathsf{E}[e^{zX(t)}] \tag{2.7}$$

be its moment-generating function. By assuming that $M(z,t)$ is continuous at $t=0$, Feller (1971[3]) proved that

$$M(z,t)=[M(z,1)]^{t}. \tag{2.8}$$

For simplicity, we assume that the random variable $X(t)$ has a density

$$f(x,t)=\frac{\mathrm{d}}{\mathrm{d}x}F(x,t),\qquad t>0,$$

then

$$M(z,t)=\int_{-\infty}^{\infty}e^{zx}f(x,t)\mathrm{d}x.$$

Let h be a nonzero real number for which $M(h,t)$ exists (It follows form (2.8) that, if $M(h,t)$ exists for one positive number t, it exists for all positive t). We now introduce the Esscher transform (parameter h) of the process $\{X(t)\}$. The new probability density function of $X(t)$, for any $t>0$, is

$$f(x,t;h)=\frac{e^{hx}f(x,t)}{\displaystyle\int_{-\infty}^{\infty}e^{hy}f(y,t)\mathrm{d}y}=\frac{e^{hx}f(x,t)}{M(h,t)}. \tag{2.9}$$

That is, the modified distribution of $X(t)$ is the Esscher transform of the original distribution. The corresponding moment-generating function is

$$M(z,t;h)=\int_{-\infty}^{\infty}e^{zx}f(x,t;h)\mathrm{d}x=\frac{M(z+h,t)}{M(h,t)}. \tag{2.10}$$

By (2.8),

$$M(z,t;h)=[M(z,1;h)]^{t}. \tag{2.11}$$

Here, we consider the Esscher transform of a stochastic process. In other words, the probability measure of the process has been modified, which is specified in the following Lemma.

Lemma 2.1 (Lin 2006[14]) Let $\{\Lambda(t),0\le t\le T\}$ be a positive P-martingale defined on the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{T\ge t\ge 0},\mathsf{P})$ such that $\mathsf{E}[\Lambda(T)]=1$. Define Q by the relation $\mathsf{Q}(A)=\displaystyle\int_A\Lambda(T)\mathrm{d}\mathsf{P}$ (denoted as $\mathrm{d}\mathsf{Q}/\mathrm{d}\mathsf{P}=\Lambda(T)$). Then Q is a new probability measure absolutely continuous with respect to P and for any random variable X

$$\mathsf{E}_{\mathsf{Q}}[X]=\mathsf{E}_{\mathsf{P}}[\Lambda(T)X],\qquad \mathsf{E}_{\mathsf{Q}}[X|\mathcal{F}_t]=\mathsf{E}_{\mathsf{P}}\Big[\frac{\Lambda(T)}{\Lambda(t)}X\Big|\mathcal{F}_t\Big],$$

and if X is $\mathcal{F}_t$-measurable, then for $s\le t\le T$

$$\mathsf{E}_{\mathsf{Q}}[X|\mathcal{F}_s]=\mathsf{E}_{\mathsf{P}}\Big[\frac{\Lambda(t)}{\Lambda(s)}X\Big|\mathcal{F}_t\Big].$$

Lemma 2.2 (Tiong 2000[16]) Let U and V be two underlying random variable, suppose they are independent under the original measure. Then U and V remain independent under the Esscher measure Q_h for each parameter h.

Lemma 2.3 (Lin 2006[14]) The process $\{X(t)\}$ is again a process with stationary and independent increments with respect to measure Q_h.

Definition 2.2 The Esscher measure of parameter $h=h^*$ is called risk-neutral Esscher measure, if the discounted stock process process $\{e^{-(r-\delta)t}S(t)\}$ is a martingale with respect to it, where r denotes the constant risk-free force of interest, δ be the constant nonnegative instanous dividend yield rates for asset, such that the assets pay out dividends $\delta S(t)dt$ between time t and time $t+dt$. The corresponding Esscher transform of parameter h^* is called risk-neutral Esscher transform.

Next we can get h^*, so that the Esscher measure of parameter h^* is a risk-neutral Esscher measure, from the following equation, for $\forall\, s<t$,

$$\mathsf{E}^*[e^{-(r-\delta)t}S(t)|\mathcal{F}_s]=e^{-(r-\delta)s}S(s),$$

where the expectation E^* is taken with respect to the Esscher measure of parameter h^*. Since

$$\begin{aligned}\mathsf{E}^*[e^{-(r-\delta)t}S(t)|\mathcal{F}_s] &= e^{-(r-\delta)t}S(0)\mathsf{E}\Big[e^{X(t)}\frac{e^{h^*X(t)}}{M(h^*,t)}\frac{M(h^*,s)}{e^{h^*X(s)}}\Big|\mathcal{F}_s\Big]\\ &= e^{-(r-\delta)t}S(s)\frac{M(h^*,s)}{M(h^*,t)}\mathsf{E}[e^{(h^*+1)(X(t)-X(s))}]\\ &= e^{-(r-\delta)t}S(s)\frac{\mathsf{E}[e^{(h^*+1)(X(t)-X(s))}]}{\mathsf{E}[e^{h^*(X(t)-X(s))}]}\\ &= e^{-(r-\delta)t}S(s)M(1,t-s;h^*)\\ &= e^{-(r-\delta)t}S(s)[M(1,1;h^*)]^{t-s},\end{aligned}$$

consequently

$$e^{-(r-\delta)t}S(s)[M(1,1;h^*)]^{t-s}=e^{-(r-\delta)s}S(s),$$

the parameter h^* is the solution of equation

$$e^{(r-\delta)(t-s)}=[M(1,1;h^*)]^{t-s}, \tag{2.12}$$

or equivalently

$$r-\delta=\ln[M(1,1;h^*)]. \tag{2.13}$$

Clearly, risk-neutral Esscher measure is an equivalent martingale measure. Although there may be other equivalent martingale measure, the risk-neutral Esscher measure is unique (See Gerber and Shiu 1994[5]).

§3. Economic Model

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a complete probability space. The following dynamic is proposed to model the asset price, $S(t)$, under the physical probability measure P:

$$\frac{\mathrm{d}S(t)}{S(t-)}=a\mathrm{d}t+\sigma\mathrm{d}W(t)+\mathrm{d}\Big(\sum_{i=1}^{N(t)}(V_i-1)\Big), \tag{3.1}$$

where $W(t)$ is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed nonnegative random variable such that $\ln V_i \sim N(\mu, \theta^2)$. In the model, all sources of randomness, $N(t)$, $W(t)$ and V_i, are assumed to be independent.

Solving the stochastic differential equation (3.1) gives the dynamics of the asset price:

$$S(t) = S(0)e^{X(t)+Y(t)}, \tag{3.2}$$

where $X(t) = (a - \sigma^2/2)t + \sigma W(t)$, $Y(t) = \sum\limits_{i=1}^{N(t)} \ln V_i$.

Put

$$\mathcal{F}_t = \sigma\{(W(s), N(s), V_i) : 0 \le s \le t, 1 \le i \le N(t)\}, \qquad t \ge 0.$$

Let r be the constant risk-free force of interest, δ be the constant nonnegative instanous dividend yield rates for asset. Now we want to find the risk-neutral Esscher measure Q such that $\mathsf{Q} \sim \mathsf{P}$ and $\{e^{-(r-\delta)t}S(t)\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \ge 0\}$ under the measure Q.

Theorem 3.1 On the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{T\ge t\ge 0}, \mathsf{P})$, let $Z(t) = [e^{h_1^* X(t)}/M_{X(t)}(h_1^*)] \cdot [e^{h_2^* Y(t)}/M_{Y(t)}(h_2^*)]$, where $h_1^* = (r - a - \delta)/\sigma^2$, $h_2{}^* = -\mu/\theta^2 - 1/2$. Then we can define the new probability Q by the relation $\mathrm{d}\mathsf{Q}/\mathrm{d}\mathsf{P}|_{\mathcal{F}_t} = Z(t)$, and $\{e^{-(r-\delta)t}e^{X(t)+Y(t)}\}$ is a martingale under Q.

Proof Without much difficulty, we have

$$\begin{aligned}
\mathsf{E}[Z(t)|\mathcal{F}_s] &= \mathsf{E}\Big[\frac{e^{h_1^*X(t)}}{M_{X(t)}(h_1^*)}\frac{e^{h_2^*Y(t)}}{M_{Y(t)}(h_2^*)}\Big|\mathcal{F}_s\Big] \\
&= \frac{e^{h_1^*X(s)+h_2^*Y(s)}}{M_{X(t)}(h_1^*)M_{Y(t)}(h_2^*)}\mathsf{E}\big[e^{h_1^*(X(t)-X(s))}e^{h_2^*(Y(t)-Y(s))}|\mathcal{F}_s\big] \\
&= e^{h_1^*X(s)}e^{h_2^*Y(s)}\frac{\mathsf{E}e^{h_1^*(X(t)-X(s))}}{\mathsf{E}e^{h_1^*X(t)}}\frac{\mathsf{E}e^{h_2^*(Y(t)-Y(s))}}{\mathsf{E}e^{h_2^*Y(t)}} \\
&= \frac{e^{h_1^*X(s)}}{M_{X(s)}(h_1^*)}\frac{e^{h_2^*Y(s)}}{M_{Y(s)}(h_2^*)} \\
&= Z(s).
\end{aligned}$$

Therefore, $\{Z(t), t \ge 0\}$ is a martingale and $Z(0) = 1$. By Lemma 2.1 we can define measure Q, such that

$$\frac{\mathrm{d}\mathsf{Q}}{\mathrm{d}\mathsf{P}}\Big|_{\mathcal{F}_t} = Z(t).$$

For each $t > s \geq 0$, we have

$$\begin{aligned}
\mathsf{E}_{\mathsf{Q}}\big[e^{-(r-\delta)t}e^{X(t)+Y(t)}|\mathcal{F}_s\big] &= \mathsf{E}\Big[e^{-(r-\delta)t}e^{X(t)+Y(t)}\frac{Z(t)}{Z(s)}\Big|\mathcal{F}_s\Big] \\
&= \frac{e^{-(r-\delta)t}M_{X(s)}(h_1^*)M_{Y(s)}(h_2^*)}{e^{h_1^*X(s)+h_2^*Y(s)}}\frac{\mathsf{E}\big[e^{X(t)(h_1^*+1)}e^{Y(t)(h_2^*+1)}|\mathcal{F}_s\big]}{M_{X(t)}(h_1^*)M_{Y(t)}(h_2^*)} \\
&= e^{-(r-\delta)t}e^{X(s)+Y(s)}\frac{M_{X(t-s)}(h_1^*+1)}{M_{X(t-s)}(h_1^*)}\frac{M_{Y(t-s)}(h_2^*+1)}{M_{Y(t-s)}(h_2^*)} \\
&= e^{-(r-\delta)t}e^{X(s)+Y(s)}e^{(t-s)(r-\delta)} \\
&= e^{-(r-\delta)s}e^{X(s)+Y(s)},
\end{aligned}$$

so $\{e^{-(r-\delta)t}e^{X(t)+Y(t)}\}$ is a martingale under Q. $\square$

Theorem 3.2 Under the measure Q, we have the following (1) $\{X(t), t \geq 0\}$ is still a Brownian motion with drift $r-\delta-\sigma^2/2$ and volatility σ; (2) $\{Y(t), t \geq 0\}$ is still a compound Poisson process with parameter $\lambda e^{\theta^2/8-\mu^2/2\theta^2}$, and $\ln V_i \sim N(-\theta^2/2, \theta^2)$.

Proof Firstly we prove (1).

$$\begin{aligned}
\mathsf{E}_{\mathsf{Q}}e^{bX(t)} &= \mathsf{E}\Big[e^{bX(t)}\frac{e^{h_1^*X(t)}}{M_{X(t)}(h_1^*)}\frac{e^{h_2^*Y(t)}}{M_{Y(t)}(h_2^*)}\Big] = \frac{\mathsf{E}e^{(b+h_1^*)X(t)}}{\mathsf{E}e^{h_1^*X(t)}}\frac{\mathsf{E}e^{h_2^*Y(t)}}{\mathsf{E}e^{h_2^*Y(t)}} \\
&= e^{(a-\sigma^2/2)(b+h_1^*)t+(\sigma^2/2)t(b+h_1^*)^2-(a-\sigma^2/2)h_1^*t-(\sigma^2/2)th_1^{*2}} \\
&= e^{(a-\sigma^2/2)bt+(\sigma^2/2)tb(2h_1^*+b)} \\
&= e^{(a-\sigma^2/2+\sigma^2h_1^*)bt+(\sigma^2/2)b^2t} \\
&= e^{(r-\delta-\sigma^2/2)bt+(1/2)\sigma^2b^2t},
\end{aligned} \tag{3.3}$$

thus $\{X(t), t \geq 0\}$ is also a Brownian motion with drift $r-\delta-\sigma^2/2$ and volatility σ under the measure Q.

Now we prove (2).

$$\begin{aligned}
\mathsf{E}_{\mathsf{Q}}[e^{bY(t)}] &= \mathsf{E}\Big[e^{bY(t)}\frac{e^{h_1^*X(t)}}{M_{X(t)}(h_1^*)}\frac{e^{h_2^*Y(t)}}{M_{Y(t)}(h_2^*)}\Big] = \frac{\mathsf{E}e^{(b+h_2^*)Y(t)}}{\mathsf{E}e^{h_2^*Y(t)}} \\
&= \exp\big\{\lambda t\big[e^{\mu(b+h_2^*)+(\theta^2/2)(b+h_2^*)^2} - e^{\mu h_2^*+(\theta^2/2)h_2^{*2}}\big]\big\} \\
&= \exp\big\{\lambda t e^{\mu h_2^*+(\theta^2/2)h_2^{*2}}\big[e^{(\mu+\theta^2h_2^*)b+(\theta^2/2)b^2} - 1\big]\big\},
\end{aligned} \tag{3.4}$$

where

$$\mu h_2^* + \frac{\theta^2}{2}h_2^{*2} = \mu\Big(-\frac{\mu}{\theta^2}-\frac{1}{2}\Big) + \frac{\theta^2}{2}\Big(\frac{\mu}{\theta^2}+\frac{1}{2}\Big)^2 = \frac{\theta^2}{8} - \frac{\mu^2}{2\theta^2},$$

$$\mu + \theta^2 h_2^* = \mu - \mu - \frac{\theta^2}{2} = -\frac{\theta^2}{2}.$$

This indicates that $\{Y(t), t \geq 0\}$ is still a compound Poisson process under the measure Q with parameter $\lambda e^{\theta^2/8-\mu^2/(2\theta^2)}$, and $\ln V_i \sim N(-\theta^2/2, \theta^2)$. $\square$

§4. Pricing EIA

Now we price EIA of point-to-point design.

The model of the value of equity index is (3.2). Let α be the participation rate, which is usually less than or equal to 1 in practice. Suppose that at time T, $T > 0$, given an initial premium of \$1, we have a policy that pays $e^{\alpha(X(T)+Y(T))}$ or e^{gT}, whichever is higher. Therefore, at maturity, this policy earns a percentage of the realized return on the risky asset over T periods, with the provision of a minimum guaranteed return of g compounded continuously over time.

Theorem 4.1 The present value of the EIA P_{pp} is defined by

$$\begin{aligned} P_{pp} &= \sum_{n=1}^{\infty}\frac{e^{-\lambda^*T}(\lambda^*T)^n}{n!}e^{-rT}\Big[e^{gT}\Phi\Big(\frac{gT/\alpha-(r-\delta-\sigma^2/2)T+n\theta^2/2}{\sqrt{\sigma^2T+n\theta^2}}\Big)\\ &\quad + e^{\alpha[(r-\delta-\sigma^2/2)T-n\theta^2/2]+(\alpha^2/2)(\sigma^2T+n\theta^2)}\\ &\quad \cdot\Phi\Big(\frac{(r-\delta-\sigma^2/2)T-n\theta^2/2+\alpha(\sigma^2T+n\theta^2)-gT/\alpha}{\sqrt{\sigma^2T+n\theta^2}}\Big)\Big], \end{aligned} \tag{4.1}$$

where $\lambda^* = \lambda e^{\theta^2/8-\mu^2/(2\theta^2)}$.

Proof Under the risk-neutral Esscher measure Q, we can write the value of this policy, as

$$P_{pp} = \mathsf{E}_{\mathsf{Q}}\Big[e^{-rT}\max\Big(e^{\alpha(X(T)+\sum\limits_{i=1}^{N(T)}\ln V_i)}, e^{gT}\Big)\Big]. \tag{4.2}$$

We can know from theorem 3.2

$$\begin{aligned} &X(T)\sim N\Big(\Big(r-\delta-\frac{\sigma^2}{2}\Big)T, \sigma^2T\Big),\\ &N(T)\sim \mathrm{Poi}(\lambda e^{\theta^2/8-\mu^2/(2\theta^2)}T),\\ &\ln V_i\sim N\Big(-\frac{\theta^2}{2}, \theta^2\Big), \end{aligned}$$

under measure Q. Let $\lambda^* = \lambda e^{\theta^2/8-\mu^2/(2\theta^2)}$, then

$$\begin{aligned} P_{pp} &= \mathsf{E}_{\mathsf{Q}}\mathsf{E}_{\mathsf{Q}}\Big[e^{-rT}\max\Big(e^{\alpha(X(T)+\sum\limits_{i=1}^{N(T)}\ln V_i)}, e^{gT}\Big)\Big|N(T)\Big]\\ &= \sum_{n=1}^{\infty}\frac{e^{-\lambda^*T}(\lambda^*T)^n}{n!}\mathsf{E}_{\mathsf{Q}}\Big[e^{-rT}\max\Big(e^{\alpha(X(T)+\sum\limits_{i=1}^{n}\ln V_i)}, e^{gT}\Big)\Big|N(T)=n\Big]. \end{aligned}$$

$N(t)$ is independent of $X(T)$ and V_i under measure Q by Lemma 2.2.

So

$$P_{pp} = \sum_{n=1}^{\infty}\frac{e^{-\lambda^*T}(\lambda^*T)^n}{n!}\mathsf{E}_{\mathsf{Q}}\Big[e^{-rT}\max\Big(e^{\alpha(X(T)+\sum\limits_{i=1}^{n}\ln V_i)}, e^{gT}\Big)\Big].$$

Let

$$K(T)=X(T)+\sum_{i=1}^{n}\ln V_i,$$

then

$$K(T)\sim N\Big(\Big(r-\delta-\frac{\sigma^2}{2}\Big)T-\frac{n\theta^2}{2},\sigma^2T+n\theta^2\Big).$$

Consequently

$$\begin{aligned}\mathsf{E}_{\mathsf{Q}}[\max(e^{\alpha K(T)},e^{gT})] &= \mathsf{E}_{\mathsf{Q}}[e^{\alpha K(T)}I(\alpha K(T)>gT)]+\mathsf{E}_{\mathsf{Q}}[e^{gT}I(\alpha K(T)<gT)]\\ &= \int_{gT/\alpha}^{\infty}e^{\alpha x}f(x)\mathrm{d}x+e^{gT}\mathsf{P}\Big(K(T)<\frac{gT}{\alpha}\Big)\\ &= e^{\alpha[(r-\delta-\sigma^2/2)T-n\theta^2/2]+(\alpha^2/2)(\sigma^2T+n\theta^2)}\\ &\quad\cdot\Phi\Big(\frac{(r-\delta-\sigma^2/2)T-n\theta^2/2+\alpha(\sigma^2T+n\theta^2)-gT/\alpha}{\sqrt{\sigma^2T+n\theta^2}}\Big)\\ &\quad+e^{gT}\Phi\Big(\frac{gT/\alpha-(r-\delta-\sigma^2/2)T+n\theta^2/2}{\sqrt{\sigma^2T+n\theta^2}}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}P_{pp} &= \sum_{n=1}^{\infty}\frac{e^{-\lambda^*T}(\lambda^*T)^n}{n!}e^{-rT}\Big[e^{gT}\Phi\Big(\frac{gT/\alpha-(r-\delta-\sigma^2/2)T+n\theta^2/2}{\sqrt{\sigma^2T+n\theta^2}}\Big)\\ &\quad+e^{\alpha[(r-\delta-\sigma^2/2)T-n\theta^2/2]+(\alpha^2/2)(\sigma^2T+n\theta^2)}\\ &\quad\cdot\Phi\Big(\frac{(r-\delta-\sigma^2/2)T-n\theta^2/2+\alpha(\sigma^2T+n\theta^2)-gT/\alpha}{\sqrt{\sigma^2T+n\theta^2}}\Big)\Big],\end{aligned}$$

where $\lambda^*=\lambda e^{\theta^2/8-\mu^2/(2\theta^2)}$. $\square$

Consequently, the present value of the EIA P_{pp} is a function of the participation rate α. Assuming an initial premium of \$1, we have $P_{pp}=1$ and it follows from equation (4.1) that the participation rate α is a solution to the following pricing equation:

$$\begin{aligned}1 &= \sum_{n=1}^{\infty}\frac{e^{-\lambda^*T}(\lambda^*T)^n}{n!}e^{-rT}\Big[e^{gT}\Phi\Big(\frac{gT/\alpha-(r-\delta-\sigma^2/2)T+(n\theta^2)/2}{\sqrt{\sigma^2T+n\theta^2}}\Big)\\ &\quad+e^{\alpha[(r-\delta-\sigma^2/2)T-n\theta^2/2]+(\alpha^2/2)(\sigma^2T+n\theta^2)}\\ &\quad\cdot\Phi\Big(\frac{(r-\delta-\sigma^2/2)T-n\theta^2/2+\alpha(\sigma^2T+n\theta^2)-gT/\alpha}{\sqrt{\sigma^2T+n\theta^2}}\Big)\Big].\end{aligned}\tag{4.3}$$

We call the implied participation rate satisfying the above equation break even participation rate. By holding all other parameter values constant, we can get break even participation rate α.

Some Monte Carlo simulations are conducted, and the results are summarized in Table 1, which lists the break even participation rates for α under different scenarios.

Table 1　Breakeven Participation Rates α for point-to-point EIA

$g=0.03,\ \mu=1,\ \theta=1$								
			$\sigma=0.2$			$\sigma=0.3$		
λ	δ	T	$r=0.04$	$r=0.05$	$r=0.06$	$r=0.04$	$r=0.05$	$r=0.06$
0	1%	3	0.4277	0.5753	0.6833	0.3270	0.4570	0.5555
		5	0.5273	0.6943	0.7998	0.4160	0.5742	0.6738
		7	0.6004	0.7667	0.8682	0.4931	0.6501	0.7529
	2%	3	0.4570	0.6149	0.7317	0.3431	0.4816	0.5860
		5	0.5801	0.7525	0.8671	0.4460	0.6094	0.7148
		7	0.6672	0.8555	0.9543	0.5266	0.7017	0.8101
1	1%	3	0.2873	0.4087	0.4977	0.2595	0.3752	0.4627
		5	0.4002	0.5447	0.6460	0.3633	0.5009	0.5999
		7	0.4923	0.6442	0.7502	0.4451	0.5975	0.6995
	2%	3	0.2976	0.4274	0.5246	0.2695	0.3827	0.4753
		5	0.4219	0.5771	0.6863	0.3809	0.5251	0.6330
		7	0.5273	0.6925	0.7852	0.4713	0.6270	0.7347

Remark　When $\lambda=0$, the situation is the same as the equity index following the Black-Scholes framework. Hence, our numerical results also provide a comparison between the valuation under tradition model and the valuation under jump diffusion model.

Furthermore, next we consider the effect of parameters on α. We can see from Figures 1 and 2 that the break even participation rates for α is an increasing function with respect to parameters T and r. From Figures 3 and 4, we can see that the break even participation rates for α is decreasing in λ and σ.

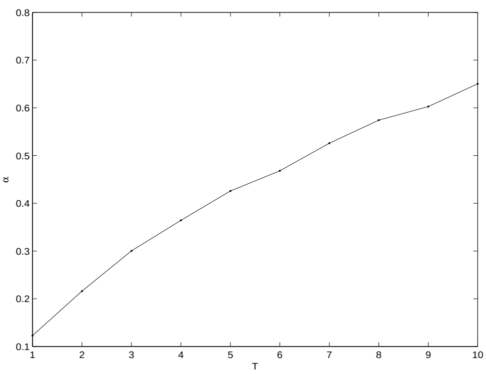

Figure 1　The effect of T on α

$\lambda=1,\ \mu=1,\ \theta=1,\ r=0.04,$

$g=0.03,\ \delta=0.02,\ \sigma=0.2$

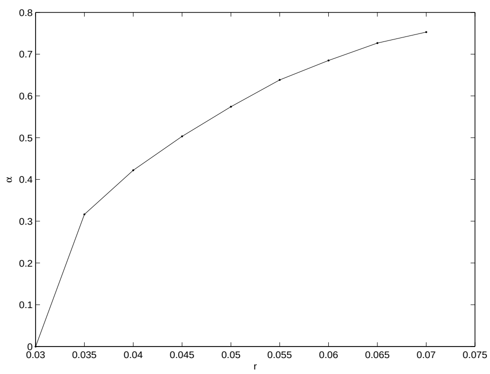

Figure 2　The effect of r on α

$\lambda=1,\ \mu=1,\ \theta=1,\ T=5,$

$g=0.03,\ \delta=0.02,\ \sigma=0.2$

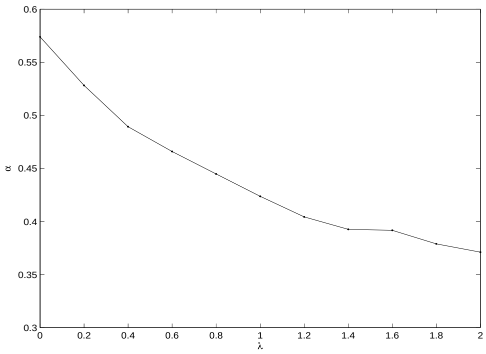

Figure 3 The effect of λ on α

$T=5$, $\mu=1$, $\theta=1$, $r=0.04$,
$g=0.03$, $\delta=0.02$, $\sigma=0.2$

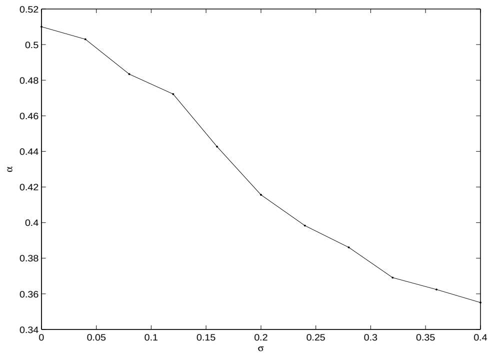

Figure 4 The effect of σ on α

$\lambda=1$, $\mu=1$, $\theta=1$, $T=5$,
$g=0.03$, $\delta=0.02$, $r=0.04$

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跳扩散模型下权益指数年金的定价

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权益指数年金收益在最小保证基础上, 能参与特定权益的收益. 通常权益指数年金定价是在假设权益指数遵从Black-Scholes模式下进行的, 但是一些例外事件(比如, 重大的政治事件)的发生, 会导致价格的巨幅波动, 这个假设并不合理. 因此本文研究了权益指数在跳扩散模型下权益指数年金的定价问题. 运用Esscher变换方法得到了点对点指数收益方法下权益指数年金定价的显示解, 并对结果作了敏感性分析.

关键词: 权益指数年金, Esscher变换, 参与率, 点对点, 跳扩散过程.

学科分类号: O211.4.