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生长曲线模型中参数的Bayes线性无偏估计 *

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摘　要

本文在平方损失下导出了生长曲线模型中参数的Bayes线性无偏估计(LUE), 并在均方误差矩阵(MSEM)准则下研究了Bayes LUE相对于广义最小二乘估计(GLSE)的优良性. 对于非满秩情形, 获得了可估函数的Bayes LUE并讨论了其优良性问题.

关键词:　生长曲线模型, Bayes线性无偏估计, 广义最小二乘估计, 均方误差矩阵准则.

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§1.　引　　言

考虑下列生长曲线模型

$$\begin{cases} Y = X_1 B X_2 + \varepsilon, \\ \varepsilon \text{ 的行向量互不相关, 均值为0, 协方差阵为 } \Sigma, \end{cases} \tag{1.1}$$

其中Y为$n\times k$的随机观测阵, X_1和X_2分别为$n\times p$和$q\times k$的已知设计阵, B为$p\times q$未知的参数矩阵, ε为$n\times k$的随机误差阵; Σ为正定阵, 记为$\Sigma>0$.

利用矩阵向量化运算和Kronecker乘积, 将模型(1.1)表示为

$$\begin{cases} \mathrm{Vec}(Y) = (X_2'\otimes X_1)\mathrm{Vec}(B) + \mathrm{Vec}(\varepsilon), \\ \mathsf{E}(\mathrm{Vec}(\varepsilon)) = 0, \qquad \mathsf{Cov}\,(\mathrm{Vec}(\varepsilon)) = \Sigma\otimes I. \end{cases} \tag{1.2}$$

令$R(A)$表示矩阵A的秩, 设$R(X_1)=p$, $R(X_2)=q$. 由王松桂等[1]可知Vec(B)的广义最小二乘估计(GLSE)也是最佳线性无偏估计, 它的表达式为

$$\begin{aligned} \mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) &= [(X_2'\otimes X_1)'(\Sigma\otimes I)^{-1}(X_2'\otimes X_1)]^{-1}(X_2'\otimes X_1)'(\Sigma\otimes I)^{-1}\mathrm{Vec}(Y) \\ &= ((X_2\Sigma^{-1}X_2')^{-1}X_2\Sigma^{-1}\otimes(X_1'X_1)^{-1}X_1')\mathrm{Vec}(Y) \\ &= \mathrm{Vec}((X_1'X_1)^{-1}X_1'Y\Sigma^{-1}X_2'(X_2\Sigma^{-1}X_2')^{-1}). \end{aligned} \tag{1.3}$$

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它也可以表成下列等价的矩阵形式

$$\widehat{B}_{\rm LS}=(X_1'X_1)^{-1}X_1'Y\Sigma^{-1}X_2'(X_2\Sigma^{-1}X_2')^{-1}. \tag{1.4}$$

为求Vec(B)的Bayes估计, 假定Vec(B)的先验分布适合以下条件

$$\pi({\rm Vec}(B)):\ \mathsf{E}({\rm Vec}(B))={\rm Vec}(B_0),\ \mathsf{Cov}\,({\rm Vec}(B))=T\otimes V, \tag{1.5}$$

此处Vec(B_0)为已知常数矩阵, T和V为已知的正定阵, 即$T>0$, $V>0$.

线性模型中参数的Bayes估计方法除了多层先验方法外主要有两种方法: 一种是在正态线性模型下, 假定参数向量的先验分布为正态或者为无信息先验, 因此在二次损失下, Bayes估计由后验均值给出(见Box and Tiao[2], Berger[3], Wang and Chow[4]等). 另一种方法是在Gauss-Markov模型下, 假定先验分布的二阶矩存在, 并假设Bayes估计具有线性形式, 利用最优化方法使Bayes风险达到最小, 确定Bayes估计. 利用这种方法获得的估计通常称为线性Bayes估计(见Rao[5], Gruber[6], Trenkler and Wei[7], Zhang[8]等). 现有文献中对第一种方法研究较多, 对第二种方法研究的较少. 线性Bayes估计对样本模型和先验分布所加条件较弱, 适用范围广. 本文将运用第二种方法研究生长曲线模型中参数的Bayes估计. 生长曲线模型来自生物生长问题, 有广泛的应用背景, 它是多元线性模型(即矩阵线性模型)的一种. 我们将通过矩阵向量化方法和Kronecker乘积技巧将这一模型转化成一般向量形式的线性模型, 使问题得以简化, 并讨论其参数的Bayes估计及其优良性问题.

本文第二节将导出未知参数的Bayes线性无偏估计; 第三节讨论其在MSEM准则下的优良性; 第四节讨论当设计阵为非列满秩情形时, 可估函数的Bayes估计及其在MSEM准则下的优良性.

§2. Bayes线性无偏估计

设Vec(B)的线性无偏估计为

$$\mathcal{F}=\{{\rm Vec}(\widehat{B})=A{\rm Vec}(Y)+b: A\text{ 为 }pq\times nk\text{ 矩阵},\ b\text{ 为 }pq\times 1\text{ 矩阵}\}. \tag{2.1}$$

为了导出Vec(B)的Bayes LUE, 取如下的二次损失函数

$$L(\widehat{B},B)=[{\rm Vec}(\widehat{B})-{\rm Vec}(B)]'D[{\rm Vec}(\widehat{B})-{\rm Vec}(B)], \tag{2.2}$$

此处D为正定阵, 即$D>0$.

Vec$(\widehat{B})$的Bayes风险为$R(\widehat{B},B)=\mathsf{E}[L(\widehat{B},B)]$, 此处均值$\mathsf{E}$关于Vec$(Y)$, $Vec(B)$的联合分布计算. 从而Vec(B)的Bayes线性无偏估计(LUE), Vec$(\widehat{B}_{\rm BE})=A{\rm Vec}(Y)+b$, 满足

$$R(\widehat{B}_{\rm BE},B)=\min_{A,b}R(\widehat{B},B)=\min_{A,b}\mathsf{E}[{\rm Vec}(\widehat{B})-{\rm Vec}(B)]'D[{\rm Vec}(\widehat{B})-{\rm Vec}(B)] \tag{2.3}$$

及无偏性条件$\mathsf{E}[\mathrm{Vec}(\widehat{B})-\mathrm{Vec}(B)]=0$. 由无偏性条件解得

$$b=[I_{pq}-A(X_2'\otimes X_1)]\mathrm{Vec}(B_0). \tag{2.4}$$

为获得矩阵A的最佳选择, 结合(2.4), 计算$\mathrm{Vec}(\widehat{B})$的Bayes风险如下:

$$\begin{aligned}
R(\widehat{B},B) &= \mathsf{E}\{[\mathrm{Vec}(\widehat{B})-\mathrm{Vec}(B)]'D[\mathrm{Vec}(\widehat{B})-\mathrm{Vec}(B)]\}\\
&= \mathsf{E}\{[A\mathrm{Vec}(Y)+b-\mathrm{Vec}(B)]'D[A\mathrm{Vec}(Y)+b-\mathrm{Vec}(B)]\}\\
&= \mathrm{tr}\{D\mathsf{E}[A(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))-(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))]\\
&\quad\cdot[A(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))-(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))]'\}\\
&= \mathrm{tr}\{DA\mathsf{E}[(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))']A'\}\\
&\quad+\mathrm{tr}\{D\mathsf{E}[(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))']\}\\
&\quad-\mathrm{tr}\{DA\mathsf{E}[(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))']\}\\
&\quad-\mathrm{tr}\{D\mathsf{E}[(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))']A'\}\\
&= \mathrm{tr}\{I_1+I_2-I_3-I_3'\}=\mathrm{tr}\{I_1+I_2-2I_3\}.
\end{aligned} \tag{2.5}$$

此处利用了trace运算的性质: $\mathrm{tr}(DAG)=\mathrm{tr}(G'A'D')=\mathrm{tr}(G'A'D)=\mathrm{tr}(DG'A')$, $\mathrm{tr}(I_3)=\mathrm{tr}(I_3')$. 显见

$$\begin{aligned}
&\mathsf{E}[(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))']\\
=\ &\mathsf{Cov}\,(\mathrm{Vec}(Y))=\mathsf{E}[\mathsf{Cov}\,(\mathrm{Vec}(Y))|B]+\mathsf{Cov}\,[\mathsf{E}(\mathrm{Vec}(Y)|B)]\\
=\ &\Sigma\otimes I+(X_2'\otimes X_1)(T\otimes V)(X_2'\otimes X_1)'\\
=\ &\Sigma\otimes I+(X_2'TX_2\otimes X_1VX_1'),
\end{aligned}$$

故有

$$I_1=DA[\Sigma\otimes I+(X_2'TX_2\otimes X_1VX_1')]A'. \tag{2.6}$$

易见

$$I_2=D\mathsf{Cov}\,(\mathrm{Vec}(B))=D(T\otimes V). \tag{2.7}$$

而

$$\begin{aligned}
I_3 &= DA\mathsf{E}\{\mathsf{E}[(\mathrm{Vec}(Y)-\mathrm{Vec}(X_1B_0X_2))(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))'|B]\}\\
&= DA\mathsf{E}[(\mathrm{Vec}(X_1BX_2)-\mathrm{Vec}(X_1B_0X_2))(\mathrm{Vec}(B)-\mathrm{Vec}(B_0))']\\
&= DA(X_2'\otimes X_1)\mathsf{Cov}\,(\mathrm{Vec}(B))=DA(X_2'\otimes X_1)(T\otimes V).
\end{aligned} \tag{2.8}$$

将(2.6)-(2.8)代入(2.5)得

$$\begin{aligned}
R(\widehat{B},B) &= \mathrm{tr}\{DA[\Sigma\otimes I+(X_2'TX_2\otimes X_1VX_1')]A'\\
&\quad+D(T\otimes V)-2DA(X_2'\otimes X_1)(T\otimes V)\}\\
&\triangleq \varphi(A).
\end{aligned}$$

令$\partial\varphi(A)/\partial A=0$, 利用矩阵微商法则得

$$2DA[\Sigma\otimes I+(X_2'TX_2\otimes X_1VX_1')]-2D(T\otimes V)(X_2\otimes X_1')=0,$$

解此矩阵方程得

$$A=(TX_2\otimes VX_1')[(\Sigma\otimes I)+(X_2'TX_2\otimes X_1VX_1')]^{-1}. \tag{2.9}$$

注意到Σ,T,V可逆, 故由公式$(A+B'CB)^{-1}=A^{-1}-A^{-1}B'(C^{-1}+BA^{-1}B')^{-1}BA^{-1}$得

$$\begin{aligned}&[\Sigma\otimes I+(X_2'TX_2\otimes X_1VX_1')]^{-1}\\=&[\Sigma\otimes I+(X_2'\otimes X_1)(T\otimes V)(X_2\otimes X_1')]^{-1}\\=&\Sigma^{-1}\otimes I-(\Sigma^{-1}\otimes I)(X_2'\otimes X_1)[(T\otimes V)^{-1}\\&+(X_2\otimes X_1')(\Sigma^{-1}\otimes I)(X_2'\otimes X_1)]^{-1}(X_2\otimes X_1')(\Sigma^{-1}\otimes I)\\=&\Sigma^{-1}\otimes I-(\Sigma^{-1}X_2'\otimes X_1)[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(X_2\Sigma^{-1}\otimes X_1').\end{aligned} \tag{2.10}$$

又注意到

$$X_2\Sigma^{-1}\otimes X_1'=(X_2\Sigma^{-1}X_2'\otimes X_1'X_1)[(X_2\Sigma^{-1}X_2')^{-1}X_2\Sigma^{-1}\otimes(X_1'X_1)^{-1}X_1')], \tag{2.11}$$

将(2.10), (2.11)代入(2.9)得

$$\begin{aligned}A&=(TX_2\otimes VX_1')[(\Sigma\otimes I)+(X_2'TX_2\otimes X_1VX_1')]^{-1}\\&=(T\otimes V)(X_2\otimes X_1')\{(\Sigma^{-1}\otimes I)-(\Sigma^{-1}X_2'\otimes X_1)\\&\quad\cdot[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(X_2\Sigma^{-1}\otimes X_1')\}\\&=(T\otimes V)\{I_{pq}-(X_2\Sigma^{-1}X_2'\otimes X_1'X_1)[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}\}\\&\quad\cdot(X_2\Sigma^{-1}\otimes X_1')\\&=[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(X_2\Sigma^{-1}\otimes X_1')\\&=[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(X_2\Sigma^{-1}X_2'\otimes X_1'X_1)\\&\quad\cdot[(X_2\Sigma^{-1}X_2')^{-1}X_2\Sigma^{-1}\otimes(X_1'X_1)^{-1}X_1'].\end{aligned} \tag{2.12}$$

由(1.4), (2.4)和(2.12)可知Vec(B)的Bayes LUE为

$$\begin{aligned}\mathrm{Vec}(\widehat{B}_{\mathrm{BE}})&=A\mathrm{Vec}(Y)+b\\&=[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(X_2\Sigma^{-1}X_2'\otimes X_1'X_1)\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})\\&\quad+\{I_{pq}-[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(X_2\Sigma^{-1}X_2'\otimes X_1'X_1)\}\mathrm{Vec}(B_0)\\&=\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})-[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(T\otimes V)^{-1}\\&\quad\cdot(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})-\mathrm{Vec}(B_0)).\end{aligned} \tag{2.13}$$

§3.　在MSEM准则下Bayes线性无偏估计的优良性

为了获得$\mathrm{Vec}(B_{\mathrm{BE}})$相对于$\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})$在MSEM准则下的优良性, 需要引入下列定义:

定义 3.1　设参数向量θ的估计量是$\widehat{\theta}$, $M(\widehat{\theta}) = \mathsf{E}[(\widehat{\theta}-\theta)(\widehat{\theta}-\theta)']$称为$\widehat{\theta}$的均方误差矩阵(MSEM), 而$\mathrm{MSE}(\widehat{\theta}) = \mathsf{E}[(\widehat{\theta}-\theta)'(\widehat{\theta}-\theta)]$称为$\widehat{\theta}$的均方误差(MSE). 设$\widehat{\theta}_1$和$\widehat{\theta}_2$为参数向量$\theta$的两个不同估计, 若$M(\widehat{\theta}_2) - M(\widehat{\theta}_1) \geq 0$ (或$\mathrm{MSE}(\widehat{\theta}_2) - \mathrm{MSE}(\widehat{\theta}_1) > 0$), 则称$\widehat{\theta}_1$在MSEM (或MSE)准则下优于$\widehat{\theta}_2$.

显见MSEM准则是比MSE更强的准则, 一个估计量在MSEM准则下优于另一个估计量, 则在MSE准则下也成立, 反之则未必.

注意本文以下证明中均值E都是关于$\mathrm{Vec}(Y)$和$\mathrm{Vec}(B)$的联合分布计算的.

定理 3.1　在生长曲线模型(1.2)和先验分布(1.5)下, 参数向量$\mathrm{Vec}(B)$的LS估计和Bayes LUE分别由(1.3)和(2.13)给出, 则有

$$M(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})) - M(\mathrm{Vec}(\widehat{B}_{\mathrm{BE}})) > 0.$$

证明:　记$G = [(T\otimes V)^{-1} + X_2\Sigma^{-1}X_2' \otimes X_1'X_1]^{-1}(T\otimes V)^{-1}$, 由(2.13)知

$$\begin{aligned}
M(\mathrm{Vec}(\widehat{B}_{\mathrm{BE}}) &= \mathsf{E}\{[(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B)) - G(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0))] \\
&\quad \cdot [(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B)) - G(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0))]'\} \\
&= M(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})) + G\mathsf{E}\{[\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0)][\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0)]'\}G' \\
&\quad - G\mathsf{E}\{[\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0)][\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B)]'\} \\
&\quad - \mathsf{E}\{[\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B)][\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0)]'\}G' \\
&= M(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})) + GJ_1G' - GJ_2 - J_2'G', \qquad (3.1)
\end{aligned}$$

其中

$$\begin{aligned}
J_1 &= \mathsf{Cov}\,(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})) \\
&= \mathsf{E}[\mathsf{Cov}\,(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})|\mathrm{Vec}(B))] + \mathsf{Cov}\,[\mathsf{E}(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})|\mathrm{Vec}(B))] \\
&= (X_2\Sigma^{-1}X_2')^{-1} \otimes (X_1'X_1)^{-1} + (T\otimes V), \qquad (3.2) \\
J_2 &= \mathsf{E}\{[\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0)][\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B)]'\} \\
&= \mathsf{E}\{(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0))[(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0)) - (\mathrm{Vec}(B) - \mathrm{Vec}(B_0))]'\} \\
&= \mathsf{Cov}\,(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})) - \mathsf{E}\{\mathsf{E}[(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) - \mathrm{Vec}(B_0))(\mathrm{Vec}(B) - \mathrm{Vec}(B_0))'|\mathrm{Vec}(B)]\} \\
&= \mathsf{Cov}\,(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})) - \mathsf{Cov}\,(\mathrm{Vec}(B)) \\
&= [(X_2\Sigma^{-1}X_2')^{-1} \otimes (X_1'X_1)^{-1} + (T\otimes V)] - (T\otimes V) \\
&= (X_2\Sigma^{-1}X_2')^{-1} \otimes (X_1'X_1)^{-1}. \qquad (3.3)
\end{aligned}$$

将(3.2)和(3.3)代入(3.1)得

$$\begin{aligned} & M(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}))-M(\mathrm{Vec}(\widehat{B}_{\mathrm{BE}})) \\ = \ & GJ_2+J_2'G'-GJ_1G' \\ = \ & G\{(X_2\Sigma^{-1}X_2')^{-1}\otimes(X_1'X_1)^{-1}G'^{-1}+G^{-1}(X_2\Sigma^{-1}X_2')^{-1}\otimes(X_1'X_1)^{-1} \\ & -[(X_2\Sigma^{-1}X_2')^{-1}\otimes(X_1'X_1)^{-1}+(T\otimes V)]\}G' \\ = \ & G\{(X_2\Sigma^{-1}X_2')^{-1}\otimes(X_1'X_1)^{-1}+(T\otimes V)\}G'>0, \end{aligned} \tag{3.4}$$

定理得证. □

利用事实$\mathrm{MSE}(\widehat{\theta})=\mathrm{tr}M(\widehat{\theta})$, 易证$\mathrm{Vec}(B)$的Bayes LUE在MSE准则下具有下列优良性:

推论 3.1 在定理3.1条件下, 有

$$\mathrm{MSE}(\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}))-\mathrm{MSE}(\mathrm{Vec}(\widehat{B}_{\mathrm{BE}}))>0.$$

§4. 可估函数的Bayes线性无偏估计及其优良性

若模型(1.2)中$R(X_1)<p$及$R(X_2)<q$至少有一个成立时, $\mathrm{Vec}(B)$就不可估了, 此时考虑可估函数

$$\varphi=\mathrm{tr}(P'B)=(\mathrm{Vec}(P))'\mathrm{Vec}(B).$$

此处P为$p\times q$的常数矩阵. 由文献[1]可知φ可估的充要条件是: 存在常数矩阵$U_{n\times k}$使得$P=X_1'UX_2'$, 即

$$\mathrm{Vec}(P)=(X_2\otimes X_1')\mathrm{Vec}(U),$$

故有

$$\varphi=[\mathrm{Vec}(U)]'(X_2'\otimes X_1)\mathrm{Vec}(B). \tag{4.1}$$

因此我们只要考虑可估函数$\mathrm{Vec}(\eta)=(X_2'\otimes X_1)\mathrm{Vec}(B)=\mathrm{Vec}(X_1BX_2)$即可, 因为任一可估函数$\varphi$可表为$\mathrm{Vec}(\eta)$的线性组合, 即$\varphi=[\mathrm{Vec}(U)]'\mathrm{Vec}(\eta)$. 下面讨论$\mathrm{Vec}(\eta)$的估计及其性质. 显见$\mathrm{Vec}(\eta)$的GLSE为

$$\begin{aligned} \mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}) &= \mathrm{Vec}(X_1\widehat{B}_{\mathrm{LS}}X_2)=(X_2'\otimes X_1)\mathrm{Vec}(\widehat{B}_{\mathrm{LS}}) \\ &= [X_2'(X_2\Sigma^{-1}X_2')^{-}X_2\Sigma^{-1}\otimes X_1(X_1'X_1)^{-}X_1']\mathrm{Vec}(Y). \end{aligned} \tag{4.2}$$

由于上述表达式与广义逆的选择无关, 故可用Moore-Penrose广义逆代替它, 得到

$$\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})=[X_2'(X_2\Sigma^{-1}X_2')^{+}X_2\Sigma^{-1}\otimes X_1(X_1'X_1)^{+}X_1']\mathrm{Vec}(Y). \tag{4.3}$$

类似于第二节的方法可得$\mathrm{Vec}(\eta)$的Bayes LUE为

$$\begin{aligned}\mathrm{Vec}(\widehat{\eta}_{\mathrm{BE}}) &= (X_2'\otimes X_1)\mathrm{Vec}(\widehat{B}_{\mathrm{BE}})\\ &= (X_2'\otimes X_1)[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}\\ &\quad\cdot(X_2\Sigma^{-1}\otimes X_1')(X_2'\otimes X_1)\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})\\ &\quad+(X_2'\otimes X_1)\{I_{pq}-[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}\\ &\quad\cdot(X_2\Sigma^{-1}X_2'\otimes X_1'X_1)\}\mathrm{Vec}(B_0)\\ &= C(X_2'\otimes X_1)\mathrm{Vec}(\widehat{B}_{\mathrm{LS}})+(I-C)(X_2'\otimes X_1)\mathrm{Vec}(B_0)\\ &= C\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})+(I-C)\mathrm{Vec}(\eta_0), \end{aligned}\tag{4.4}$$

其中$\mathrm{Vec}(\eta_0)=(X_2'\otimes X_1)\mathrm{Vec}(B_0)=\mathrm{Vec}(X_1B_0X_2)$, $\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})$由(4.2)给出, 而

$$C=(X_2'\otimes X_1)[(T\otimes V)^{-1}+X_2\Sigma^{-1}X_2'\otimes X_1'X_1]^{-1}(X_2\Sigma^{-1}\otimes X_1').$$

关于$\mathrm{Vec}(\eta)$的Bayes LUE在MSEM准则下的优良性有如下结果:

定理 4.1 在生长曲线模型(1.2)和先验(1.5)下, 假定$R(X_1)<p$和$R(X_2)<q$中至少有一个成立, 参数向量$\mathrm{Vec}(\eta)$的LS估计和Bayes LUE分别由(4.3)和(4.4)给出, 则有

$$M(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}))-M(\mathrm{Vec}(\widehat{\eta}_{\mathrm{BE}})\geq 0.$$

证明:

$$\begin{aligned}M(\mathrm{Vec}(\widehat{\eta}_{\mathrm{BE}})) &= \mathsf{E}\{[\mathrm{Vec}(\widehat{\eta}_{\mathrm{BE}})-\mathrm{Vec}(\eta)][\mathrm{Vec}(\widehat{\eta}_{\mathrm{BE}})-\mathrm{Vec}(\eta)]'\}\\ &= \mathsf{E}\{[(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta))-(I-C)(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0))]\\ &\quad\cdot[(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta))-(I-C)(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0))]'\}\\ &= M(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}))+(I-C)\mathsf{E}\{[\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)]\\ &\quad\cdot[\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)]'\}(I-C)'\\ &\quad-(I-C)\mathsf{E}\{[\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)][\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta)]'\}\\ &\quad-\mathsf{E}\{[\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta)][\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)]'\}(I-C)'\\ &\triangleq M(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}))+(I-C)J_1(I-C)'-(I-C)J_2-J_2'(I-C)', \end{aligned}\tag{4.5}$$

其中

$$\begin{aligned}J_1 &= \mathsf{E}\{[\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)][\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)]'\}\\ &= \mathsf{Cov}\,\{\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})\}\\ &= \mathsf{Cov}\,\{[\mathsf{E}(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})|\mathrm{Vec}(B))]\}+\mathsf{E}[\mathsf{Cov}\,(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})|\mathrm{Vec}(B))]\\ &= X_2'TX_2\otimes X_1VX_1'+X_2'(X_2\Sigma^{-1}X_2')^{+}X_2\otimes P_{X_1}, \end{aligned}\tag{4.6}$$

此处$P_{X_1}=X_1(X_1'X_1)^+X_1'$, 而

$$\begin{aligned} J_2 &= \mathsf{E}\{[\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)][\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta)]'\} \\ &= \mathsf{E}\{[\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0)][(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0))+(\mathrm{Vec}(\eta_0)-\mathrm{Vec}(\eta))]'\} \\ &= \mathsf{Cov}\,(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}))-\mathsf{E}\{\mathsf{E}[(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}})-\mathrm{Vec}(\eta_0))(\mathrm{Vec}(\eta)-\mathrm{Vec}(\eta_0))'|\mathrm{Vec}(B)]\} \\ &= \mathsf{Cov}\,(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}))-\mathsf{Cov}\,(\mathrm{Vec}(\eta)) \\ &= [X_2'TX_2\otimes X_1VX_1'+X_2'(X_2\Sigma^{-1}X_2')^+X_2\otimes P_{X_1}]-(X_2'TX_2\otimes X_1VX_1') \\ &= X_2'(X_2\Sigma^{-1}X_2')^+X_2\otimes P_{X_1}. \end{aligned} \tag{4.7}$$

由(2.10)式可知:

$$(\Sigma^{-1}\otimes I)(I-C)=[(\Sigma\otimes I)+(X_2'TX_2\otimes X_1VX_1')]^{-1},$$

故有

$$(I-C)=(\Sigma\otimes I)[(\Sigma\otimes I)+(X_2'TX_2\otimes X_1VX_1')]^{-1}.$$

因此有

$$(I-C)^{-1}=[(\Sigma\otimes I)+(X_2'TX_2\otimes X_1VX_1')](\Sigma\otimes I)^{-1}. \tag{4.8}$$

将(4.6)和(4.7)代入(4.5), 并利用(4.8)可知

$$\begin{aligned} &M(\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}))-M(\mathrm{Vec}(\widehat{\eta}_{\mathrm{BE}})) \\ =\ &(I-C)J_2+J_2'(I-C)'-(I-C)J_1(I-C)' \\ =\ &(I-C)\{(X_2'(X_2\Sigma^{-1}X_2')^+X_2\otimes P_{X_1})(I-C)'^{-1} \\ &+(I-C)^{-1}(X_2'(X_2\Sigma^{-1}X_2')^+X_2\otimes P_{X_1}) \\ &-[X_2'TX_2\otimes X_1VX_1'+X_2'(X_2\Sigma^{-1}X_2')^+X_2\otimes P_{X_1}]\}(I-C)' \\ =\ &(I-C)[(X_2'TX_2\otimes X_1VX_1')+(X_2'(X_2\Sigma^{-1}X_2')^+X_2\otimes P_{X_1})](I-C)'\geq 0, \end{aligned} \tag{4.9}$$

命题得证. □

由定理4.1的结果, 易知对一般可估函数$\varphi=[\mathrm{Vec}(U)]'\mathrm{Vec}(\eta)$, 其LS估计和Bayes LUE分别为:

$$\widehat{\varphi}_{\mathrm{LS}}=[\mathrm{Vec}(U)]'\mathrm{Vec}(\widehat{\eta}_{\mathrm{LS}}), \tag{4.10}$$

$$\widehat{\varphi}_{\mathrm{BE}}=[\mathrm{Vec}(U)]'\mathrm{Vec}(\widehat{\eta}_{\mathrm{BE}}). \tag{4.11}$$

关于可估函数$\varphi=[\mathrm{Vec}(U)]'\mathrm{Vec}(\eta)$的Bayes LUE在MSE准则下的优良性有下列结果:

推论 4.1 在生长曲线模型(1.2)和先验(1.5)下, 对任意可估函数$\varphi=\mathrm{tr}(P'B)=[\mathrm{Vec}(U)]'\mathrm{Vec}(\eta)$, 假定$\varphi$的LS估计和BLU估计由(4.10)和(4.11)给出, 则有

$$\mathrm{MSE}(\widehat{\varphi}_{\mathrm{LS}})-\mathrm{MSE}(\widehat{\varphi}_{\mathrm{BE}})>0.$$

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The Superiority of Bayes Linear Unbiased Estimation in the Growth Curve Model

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Under quadratic loss function, the Bayes linear unbiased estimator (LUE) is derived for the growth curve model. The superiority of Bayes LUE over the generalized least square estimator (GLSE) is studied in terms of the mean square error matrix (MSEM) criterion. Finally, the superiority of the Bayes LUE of estimable functions for non-full rank case is considered further.

Keywords: The growth curve model, Bayes linear unbiased estimator, generalized least square estimator, mean square error matrix criterion.

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