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Comparison of Simultaneous Intervals for the Mean of a Multivariate Normal Distribution

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Abstract

In this paper, we give an analytical comparison between Bonferroni and Scheffé simultaneous confidence intervals for the mean of a multivariate normal distribution, which concludes that the Bonferroni intervals are shorter than the Scheffé intervals when the dimension of the mean vector is between 2 and 12.

Keywords: Bonferroni simultaneous confidence interval, Scheffé simultaneous confidence interval, multiple comparisons.

AMS Subject Classification: 62J15.

§1. Introduction

Multiple comparisons arose in statistics because of the multiplicity impacts on statistical inferences, which increases the possibility of making type I error. For comprehensive treatments, see [1], [2], [3]. A commonly existing phenomenon in this field is that there usually exists a number of competitive procedures to tackle a given problem. Thus, many researchers have contributed their efforts to examine the merits among the simultaneous intervals obtained by various procedures, see for example [1], [4], [5], [6], [7] and [8]. Among all these multiple comparison procedures, the most commonly cited may be Bonferroni and Scheffé procedures. The comparison between them has been studied also by various authors including [4], [7] and [9]. While Alt and Spruill in [4] addressed their results based only on numerical outcomes or infinite degree of freedom, i.e. implicitly taking the assumption that the variances of the errors are known, and Nickerson in [7] declared a similar result that was substantiated by F-tables, Mi in [9] investigated the same problem under the framework of normally distributed linear models to infer the coefficients and analytically compared the simultaneous intervals obtained by these two multiple comparison procedures.

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However, all the results aforementioned involve only the univariate problems. It is well known that the theories in multivariate circumstance are usually not trivial extensions of those obtained in a univariate analysis due to the correlation between the component variables. Therefore, in this short note, we consider the problem of comparing the simultaneous confidence intervals produced by Bonferroni and Scheffé procedures for the mean of a multivariate normal distribution. To be specific, let $X \sim N_p(\mu, \Sigma)$, with unknown parameters $\mu = (\mu_1, \mu_2, \cdots, \mu_p)'$ and $\Sigma \geq 0$. We consider the problem of comparing the simultaneous confidence intervals of parameters $\mu_1, \mu_2, \cdots, \mu_p$: $\{(\widehat{\mu}_{iL}, \widehat{\mu}_{iU}), i = 1, 2, \cdots, p\}$ obtained by Bonferroni and Scheffé procedures. From the technical point of view, while it amounts to comparing the quantiles of two F-distributions sharing the same denominator degree of freedom in the univariate case (see e.g. [7], [9], [10]), we have to contrast those quantiles between two F-distributions that have the same sum of their numerator and denominator degrees of freedom in the present setting, see below for details. This difference raises great technical challenges and new skills should be employed so as to tackle it. In fact, it turns out to be quite difficult to obtain an analytical comparison between such F-distributions.

Here an analytical proof is given to show that Bonferroni interval is shorter than Scheffé interval under some conditions.

§2. Comparison of the Confidence Bounds

Let $X_1, X_2, \cdots, X_n$ be a random sample from an $N_p(\mu, \Sigma)$ population with corresponding sample mean $\overline{X} = (1/n) \cdot \sum_{j=1}^{n} X_j$ and sample covariance matrix $S = [1/(n-1)] \cdot \sum_{j=1}^{n} (X_i - \overline{X})(X_i - \overline{X})' = (s_{ij})_{1 \leq i,j \leq p}$. A number of ways have been developed to obtain simultaneous confidence intervals for the population mean μ, among which the most extensively used in practice may be Bonferroni and Scheffé intervals (the later is also called T^2-intervals, see e.g. [11]). Using the obvious inequality $\mathsf{P}(\bigcup_{j=1}^{n} A_j) \leq \sum_{j=1}^{n} \mathsf{P}(A_j)$, Bonferroni simultaneous $100(1-\alpha)\%$ confidence intervals of μ_i are written as

$$\mu_i \in \Big(\overline{X}_i \pm \sqrt{\frac{s_{ii}}{n}} t_{1-\alpha/(2p)}(n-1)\Big), \qquad i = 1, 2, \cdots, p, \tag{2.1}$$

where $t_{1-\alpha/(2p)}(n-1)$ is the upper $100(1-\alpha/(2p))\%$ quantile of t-distribution with degree of freedom $(n-1)$. On the other hand, since the joint Scheffé's simultaneous confidence intervals of $a'\mu$ for all $a \in R^p$ with confidence level $100(1-\alpha)\%$ are, see e.g. [11] again,

$$a'\mu \in \left(a'\overline{X} \pm \sqrt{F_{1-\alpha}(p, n-p)}\sqrt{\frac{p(n-1)a'Sa}{n(n-p)}}\right),$$

where $F_{1-\alpha}(p, n-p)$ is the upper $100(1-\alpha)\%$ quantile of $F(p, n-p)$ distribution. Taking especially $a_1' = (1, 0, \cdots, 0), a_2' = (0, 1, \cdots, 0), \cdots, a_p' = (0, 0, \cdots, 1)$, we have

$$\mu_i \in \left(\overline{X}_i \pm \sqrt{F_{1-\alpha}(p, n-p)}\sqrt{\frac{p(n-1)s_{ii}}{n(n-p)}}\right), \qquad i = 1, 2, \cdots, p, \tag{2.2}$$

with a probability of at least $(1-\alpha)$.

By (2.1) and (2.2), we observe that Bonferroni and Scheffé simultaneous intervals for the parameters $\mu' = (\mu_1, \mu_2, \cdots, \mu_p)$ share the same form

$$\overline{X}_i \pm (\text{critical value})\sqrt{\frac{s_{ii}}{n}}, \qquad i = 1, 2, \cdots, p.$$

As a result, comparing the both types of confidence intervals amounts to examining the difference between the corresponding critical values $t_{1-\alpha/(2p)}(n-1) = \sqrt{F_{1-\alpha/p}(1, n-1)}$ and $\sqrt{[p(n-1)/(n-p)] \cdot F_{1-\alpha}(p, n-p)}$. For simplicity in representations, denote the critical values by

$$b(p, n, \alpha) = \sqrt{F_{1-\alpha/p}(1, n-1)} \tag{2.3}$$

and

$$s(p, n, \alpha) = \sqrt{\frac{p(n-1)}{n-p}F_{1-\alpha}(p, n-p)}. \tag{2.4}$$

Define for $z > 0$

$$f(z) = \frac{p\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}\frac{z^{-1/2}}{(1+z)^{n/2}} \tag{2.5}$$

and

$$g(z) = \frac{\Gamma(n/2)}{\Gamma(p/2)\Gamma((n-p)/2)}\frac{z^{p/2-1}}{(1+z)^{n/2}}. \tag{2.6}$$

It is easy to see that

$$\int_{\widetilde{b}(p,n,\alpha)}^{+\infty} f(z)\mathrm{d}z = \int_{\widetilde{s}(p,n,\alpha)}^{+\infty} g(z)\mathrm{d}z = \alpha, \tag{2.7}$$

where

$$\widetilde{b}(p, n, \alpha) = \frac{b^2(p, n, \alpha)}{n-1} \qquad \text{and} \qquad \widetilde{s}(p, n, \alpha) = \frac{s^2(p, n, \alpha)}{n-1}.$$

$b(p, n, \alpha)$ and $s(p, n, \alpha)$ can be compared by checking the order between $\widetilde{b}(p, n, \alpha)$ and $\widetilde{s}(p, n, \alpha)$.

First note that for $z>0$, the curve $f(z)$ crosses $g(z)$ only once at

$$z_0(p,n)=\Big(\frac{p\Gamma(p/2)\Gamma((n-p)/2)}{\sqrt{\pi}\Gamma((n-1)/2)}\Big)^{2/(p-1)} \tag{2.8}$$

and $f(z)>g(z)$ in the case of $0<z<z_0(p,n)$ and $f(z)<g(z)$ otherwise. We here give some lemmas that are helpful to prove the main theorem. The first lemma is a sufficient condition in terms of the relative locations of $\widetilde{b}(p,n,\alpha_0)$ and $\widetilde{s}(p,n,\alpha_0)$ regarding $z_0(p,n)$, which leads to the order relation between them.

Lemma 2.1 Given a fixed triplet (p,n,α_0), if either $\widetilde{b}(p,n,\alpha_0)$ or $\widetilde{s}(p,n,\alpha_0)$ is greater than $z_0(p,n)$, then $\widetilde{b}(p,n,\alpha_0)<\widetilde{s}(p,n,\alpha_0)$.

Proof First consider the case $\widetilde{b}(p,n,\alpha_0)>z_0(p,n)$. Since $f(z)<g(z)$ for $z>z_0(p,n)$, it follows that

$$\int_{\widetilde{b}(p,n,\alpha_0)}^{+\infty} f(z)\mathrm{d}z<\int_{\widetilde{b}(p,n,\alpha_0)}^{+\infty} g(z)\mathrm{d}z.$$

By (2.7),

$$\int_{\widetilde{s}(p,n,\alpha_0)}^{+\infty} g(z)\mathrm{d}z=\int_{\widetilde{b}(p,n,\alpha_0)}^{+\infty} f(z)\mathrm{d}z<\int_{\widetilde{b}(p,n,\alpha_0)}^{+\infty} g(z)\mathrm{d}z.$$

Therefore, $\widetilde{b}(p,n,\alpha_0)<\widetilde{s}(p,n,\alpha_0)$.

A similar argument results in $\widetilde{b}(p,n,\alpha_0)<\widetilde{s}(p,n,\alpha_0)$ in the case $\widetilde{s}(p,n,\alpha_0)>z_0(p,n)$.
□

The following lemma shows that we only need to compare $\widetilde{b}(p,n,\alpha)$ and $\widetilde{s}(p,n,\alpha)$ for some specially selected α_0.

Lemma 2.2 For any given p and n, $\widetilde{b}(p,n,\alpha)<\widetilde{s}(p,n,\alpha)$ for all $\alpha\le\alpha_0$ provided that there exists α_0 such that $\widetilde{b}(p,n,\alpha_0)<\widetilde{s}(p,n,\alpha_0)$.

Proof It is obvious by (2.7) that both $\widetilde{b}(p,n,\alpha)$ and $\widetilde{s}(p,n,\alpha)$ are decreasing in α. Thus $\widetilde{b}(p,n,\alpha)>\widetilde{b}(p,n,\alpha_0)$ and $\widetilde{s}(p,n,\alpha)>\widetilde{s}(p,n,\alpha_0)$ for all $\alpha\le\alpha_0$.

a) If either $\widetilde{b}(p,n,\alpha_0)$ or $\widetilde{s}(p,n,\alpha_0)$ is greater than $z_0(p,n)$, then either $\widetilde{b}(p,n,\alpha)$ or $\widetilde{s}(p,n,\alpha)$ is greater than $z_0(p,n)$. By Lemma 2.1, $\widetilde{b}(p,n,\alpha)<\widetilde{s}(p,n,\alpha)$ holds for all $\alpha\le\alpha_0$.

b) If both $\widetilde{b}(p,n,\alpha_0)$ and $\widetilde{s}(p,n,\alpha_0)$ are less than $z_0(p,n)$, then we consider both $\widetilde{b}(p,n,\alpha)$ and $\widetilde{s}(p,n,\alpha)$ are less than $z_0(p,n)$; otherwise, if one of $\widetilde{b}(p,n,\alpha)$ and $\widetilde{s}(p,n,\alpha)$ is greater than $z_0(p,n)$, then by Lemma 2.1, the result can be directly obtained.

From $\int_{\widetilde{b}(p,n,\alpha_0)}^{+\infty} f(z)\mathrm{d}z=\int_{\widetilde{s}(p,n,\alpha_0)}^{+\infty} g(z)\mathrm{d}z=\alpha_0$ and (2.7), we have

$$\int_{\widetilde{b}(p,n,\alpha_0)}^{\widetilde{b}(p,n,\alpha)} f(z)\mathrm{d}z=\int_{\widetilde{s}(p,n,\alpha_0)}^{\widetilde{s}(p,n,\alpha)} g(z)\mathrm{d}z=\alpha_0-\alpha.$$

Since $f(z) > g(z)$ when $0 < z < z_0(p,n)$,

$$\widetilde{b}(p,n,\alpha) - \widetilde{b}(p,n,\alpha_0) < \widetilde{s}(p,n,\alpha) - \widetilde{s}(p,n,\alpha_0).$$

By $\widetilde{b}(p,n,\alpha_0) < \widetilde{s}(p,n,\alpha_0)$, we obtain $\widetilde{b}(p,n,\alpha) < \widetilde{s}(p,n,\alpha)$ for all $\alpha \le \alpha_0$. $\square$

Lemma 2.3 For any given p, $(n-1)z_0(p,n)$ strictly decreases in n when $n \ge p+1$. And $\lim\limits_{n\to+\infty}(n-1)z_0(p,n) = 2[p\Gamma(p/2)/\sqrt{\pi}]^{2/(p-1)}$.

Proof By (2.8), write $(n-1)z_0(p,n)$ as

$$(n-1)z_0(p,n) = \left(\frac{p\Gamma(p/2)}{\sqrt{\pi}}\right)^{2/(p-1)} \exp\{h(n)\}, \tag{2.9}$$

where

$$h(n) = \ln(n-1) + \frac{2}{p-1}\left(\ln\Gamma\left(\frac{n-p}{2}\right) - \ln\Gamma\left(\frac{n-1}{2}\right)\right). \tag{2.10}$$

Note the formula

$$(\ln\Gamma(a))' = \frac{\Gamma'(a)}{\Gamma(a)} = C + \int_0^1 \frac{1-t^{a-1}}{1-t}\mathrm{d}t,$$

where C is Euler's constant. Notice that

$$\begin{aligned} h'(n) &= \frac{1}{n-1} - \frac{1}{p-1}\int_0^1 t^{(n-p)/2-1}\frac{(1-t^{(p-1)/2})}{1-t}\mathrm{d}t \\ &< \frac{1}{n-1} - \frac{1}{2(p-1)}\int_0^1 t^{(n-p)/2-1}(1+t^{1/2}+t+\cdots+t^{(p-2)/2})\mathrm{d}t \\ &< \frac{1}{n-1} - \frac{p-1}{2(p-1)}\int_0^1 t^{(n-2)/2-1}\mathrm{d}t \\ &= \frac{1}{n-1} - \frac{1}{n-2} \\ &< 0. \end{aligned}$$

Therefore, $h(n)$ and hence $(n-1)z_0(p,n)$ by (2.9), strictly decrease in n if $n \ge p+1$.

Moreover, applying Stirling's formula

$$\ln\Gamma(a) = \ln\sqrt{2\pi} + \left(a - \frac{1}{2}\right)\ln a - a + \frac{\theta}{12a}, \qquad 0 < \theta < 1, \tag{2.11}$$

into (2.10) yields

$$\begin{aligned} \lim_{n\to+\infty} h(n) &= \lim_{n\to+\infty}\left\{\ln(n-1) + \frac{2}{p-1}\left(\ln\Gamma\left(\frac{n-p}{2}\right) - \ln\Gamma\left(\frac{n-1}{2}\right)\right)\right\} \\ &= \lim_{n\to+\infty}\left(\frac{n-p-1}{p-1}\ln\left(\frac{n-p}{n-1}\right) + \ln 2 + 1\right) \\ &= \ln 2. \end{aligned}$$

Therefore, for fixed p, this equality along with (2.9) gives

$$\lim_{n\to+\infty}(n-1)z_0(p,n) = 2\Big(\frac{p\Gamma(p/2)}{\sqrt{\pi}}\Big)^{2/(p-1)}.$$

□

We now state and prove the main theorem on the comparison between $b(p,n,\alpha)$ and $s(p,n,\alpha)$.

Theorem 2.1 For $2\leq p\leq 12$ and $n\geq p+1$, $b(p,n,\alpha)<s(p,n,\alpha)$ holds for all $\alpha\leq 0.10$.

Proof First take $\alpha=0.10$. Because for any given $p\geq 2$, $t_{1-0.05/p}(n-1)$ is positive and strictly decreases in n (see e.g. [10]), $b^2(p,n,0.10)$ decreasingly tends to $(U_{(1-0.05/p)})^2$, where $U_{(1-0.05/p)}$ is $100(1-0.05/p)\%$ percentile of the standard normal distribution $N(0,1)$.

Moreover, it is easy to numerically check that $2[p\Gamma(p/2)/\sqrt{\pi}]^{2/(p-1)}<(U_{(1-0.05/p)})^2$ for $2\leq p\leq 12$. Then by Lemma 2.3 and the monotonicity of $(n-1)z_0(p,n)$ in n, there exists an $n_0(p)$ such that $(n-1)z_0(p,n)<(U_{(1-0.05/p)})^2$ as $n>n_0(p)$. The following table lists the values of $n_0(p)$ for $2\leq p\leq 12$.

Table 1 $n_0(p)$'s Values

p	2	3	4	5	6	7	8	9	10	11	12
$n_0(p)$	5	6	9	11	15	20	26	37	55	93	229

Therefore, $(n-1)z_0(p,n)<b^2(p,n,0.10)$ holds for $2\leq p\leq 12$ and $n>n_0(p)$. By Lemma 2.1 and Lemma 2.2, this leads to the conclusion that $(n-1)z_0(p,n)<b^2(p,n,\alpha)$ for $2\leq p\leq 12$, and thereby $b(p,n,\alpha)<s(p,n,\alpha)$ for all $\alpha\leq 0.10$, if $n>n_0(p)$.

For $(p+1)\leq n\leq n_0(p)$, computing all $b(p,n,0.10)$ and $s(p,n,0.10)$, we find that $b(p,n,0.10)<s(p,n,0.10)$ holds also. So by Lemma 2.2, we conclude that $b(p,n,\alpha)<s(p,n,\alpha)$ for all $\alpha\leq 0.10$. □

To conclude this note we remark that although we only analytically proved that Bonferroni simultaneous confidence intervals are shorter than Scheffé ones for $2\leq p\leq 12$, considering the fact that when p is large a Bonferroni interval may be too long to present any meaningful guide for the practicers, our result presents a useful guide for the selection between these two procedures for mild large p.

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单个多元正态总体均值向量联合置信区间的比较

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单个多元正态总体均值向量的Bonferroni和Scheffé联合置信区间在实际中经常用到, 本文主要采用解析的办法比较这两个置信区间的长短, 证明了当均值向量的维数$2 \leq p \leq 12$时, Bonferroni联合置信区间比Scheffé联合置信区间短.

关键词: Bonferroni联合置信区间, Scheffé联合置信区间, 多重比较.

学科分类号: O212.4.