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带有不完全信息随机截尾试验下最大似然估计的重对数律 *

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摘　要

本文在条件(Φ)下, 证明了带有不完全信息随机截尾试验下最大似然估计的收敛速度符合重对数律, 并验证了Weibull分布、对数正态分布满足条件(Φ).

关 键 词:　带有不完全信息随机截尾试验, 最大似然估计, 重对数律.

学 科 分 类 号:　O213.2.

§1.　引　　言

设寿命变量$X_1, X_2, \cdots$是概率空间$(\Omega, \mathcal{F}, \mathsf{P}_\theta)$ $(\theta\in\Theta$, Θ是R^m空间上的开集)上的独立同分布随机变量序列, 其分布函数为$F(x,\theta)$, 密度函数为$f(x,\theta)$. 又设截尾变量$Y_1, Y_2, \cdots$是$(\Omega, \mathcal{F}, \mathsf{P}_\theta)$上相互独立的正值随机变量序列, 分布函数分别为$G_1(t), G_2(t), \cdots$, 并且假定$\{X_i\}$与$\{Y_i\}$相互独立.

为估计参数θ, 给出n个样品$\{X_i, 1\le i\le n\}$及观察数据$\{Z_i, 1\le i\le n\}$的取值情况如下:

(I) 当$X_i<Y_i$, 表示产品在截尾前失效. 在通常的截尾试验下有$Z_i=X_i$, 但在这里取值不同因为产品的失效状态还必须通过某种检测手段利用信号给予显示, 所以, 有两种可能情况发生, 失效状态以概率p $(0<p\le 1$, p是已知数, 与θ无关)被立即显示, 此时$Z_i=X_i$; 或以$1-p$未被立即显示, 直到截尾变量Y_i终止时才发现产品已失效, 此时获得了不完全信息, 即仅知道$X_i\le Y_i$而不知道寿命X_i的准确值, 故得$Z_i=Y_i$, 称p为失效显示概率.

(II) 当$X_i\ge Y_i$, 表示产品寿命已不小于截尾变量, 故得$Z_i=Y_i$.

若令$\alpha_i=\begin{cases}1 & 若\ X_i<Y_i\\ 0 & 若\ X_i\ge Y_i\end{cases}$, $\beta_i=\begin{cases}0 & 若\ X_i<Y_i, 且失效未被显示\\ 1 & 其它\end{cases}$, $(i=1,2,\cdots,n)$,
则$\{(Z_i,\alpha_i,\beta_i), 1\le i\le n\}$是带有不完全信息的随机截尾数据, 且对每个$i$ $(1\le i\le n)$有

$$Z_i=\begin{cases}X_i & \alpha_i=1,\ \beta_i=1,\\ Y_i & \alpha_i=1,\ \beta_i=0,\\ Y_i & \alpha_i=0\ (\beta_i=1),\end{cases}\qquad \mathsf{P}_\theta(\beta_i=1|\alpha_i=1)=p.$$

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由上易知, 随机截尾试验模型是带有不完全信息的随机截尾试验模型的特例. 文献[6]-[7]中, 分别给出了随机截尾情形下Weibull分布与分布较一般且截尾变量独立不同分布的分布参数最大似然估计的相合性及渐近正态性. 文献[8]验证了随机截尾情形下分布参数最大似然估计的收敛速度满足重对数律. 随着可靠性理论的深入发展, Elperin & Gertsbakh (1988)将随机截尾试验模型推广到带有不完全信息的随机截尾试验模型. 尔后, 文献[2]-[4]中, 分别证明了带有不完全信息随机截尾情形下指数分布、Weibull分布及分布较一般的分布参数最大似然估计的相合性及渐近正态性. 在文献[5]中, 验证了带有不完全信息随机截尾试验下Weibull分布的最大似然估计的收敛速度满足重对数律. 据作者所知, 对于分布较一般的带有不完全信息随机截尾试验下参数最大似然估计的收敛速度还没有一个较好的结果. 鉴于此, 本文在文献[4]的基础上进一步讨论了最大似然估计的收敛速度, 论证了收敛速度满足重对数律. 从而, 就分布类型和试验模型这两个方面推广了[5]和[8]的结果. 本文的结构如下, 除第一节引言外, 第二节为主要结果, 第三节为若干引理, 第四节为定理的证明.

§2. 主要结果

众所周知, 基于$\{(Z_i,\alpha_i,\beta_i),1\le i\le n\}$的似然函数为(见[7]):

$$L(\theta)=\prod_{i=1}^{n}f(Z_i,\theta)^{\alpha_i\beta_i}F(Z_i,\theta)^{\alpha_i(1-\beta_i)}\overline{F}(Z_i,\theta)^{1-\alpha_i}, \tag{2.1}$$

其中: $\overline{F}=1-F$. 似然方程组为

$$\frac{\partial\ln L}{\partial\theta}=0, \tag{2.2}$$

其中:

$$\frac{\partial\ln L}{\partial\theta}=\Big(\frac{\partial\ln L}{\partial\theta_1},\frac{\partial\ln L}{\partial\theta_2},\cdots,\frac{\partial\ln L}{\partial\theta_m}\Big)^T.$$

以下$\partial\ln f(x,\theta)/\partial\theta$, $\partial\ln F(x,\theta)/\partial\theta$, $\partial\ln\overline{F}(x,\theta)/\partial\theta$与之定义类同, 均为向量.

定义 称似然函数正规, 若对一切$n\ge 2$只要(Z_i,α_i,β_i), $i=1,2,\cdots,n$不全相等, 似然方程组(2.2)有唯一解$\widehat{\theta}^n=(\widehat{\theta}_1^n,\widehat{\theta}_2^n,\cdots,\widehat{\theta}_m^n)^T$, 其中$\widehat{\theta}_s^n=\widehat{\theta}_s^n(Z_1,Z_2,\cdots,Z_n,\alpha_1,\alpha_2,\cdots,\alpha_n,\beta_1,\beta_2,\cdots,\beta_n)$, $s=1,2,\cdots,m$.

当$(Z_1,\alpha_1,\beta_1)=(Z_2,\alpha_2,\beta_2)=\cdots=(Z_n,\alpha_n,\beta_n)$时, 令$\widehat{\theta}_i^n=\theta_i^0$, $i=1,2,\cdots,m$, 其中$(\theta_1^0,\theta_2^0,\cdots,\theta_m^0)$为$\Theta$中一固定点. 因此$\widehat{\theta}^n=(\widehat{\theta}_1^n,\widehat{\theta}_2^n,\cdots,\widehat{\theta}_m^n)^T$总有定义. 此时称$\widehat{\theta}^n$是$\theta$的最大似然估计.

其次, 我们对$f(x,\theta),F(x,\theta),G_i(x)$, $i\ge 1$施加如下条件, 合称为条件(Φ):

(1) $f(x,\theta)$为$[0,+\infty)\times\Theta$上定义的正值函数, $f(x,\theta)$关于x Borel可测且$\partial^3 f(x,\theta)/(\partial\theta_s\partial\theta_t\partial\theta_k)$, $\partial^2 f(x,\theta)/(\partial\theta_s\partial\theta_t)$, $\partial f(x,\theta)/\partial\theta_s$, $f(x,\theta)$ $(s,t,k=1,2,\cdots,m)$为θ的连续函数.

(2) 对于$\forall\,\theta^0\in\Theta$, 存在$\mu_{\theta^0}=\{\theta:\|\theta-\theta^0\|\le\eta_{\theta^0}\}\subset\Theta\ (\eta_{\theta^0}>0)$使得在$\mu_{\theta^0}$上, 下式成立,

$$\begin{aligned}
&\Big|\frac{\partial^3 f(x,\theta)}{\partial\theta_s\partial\theta_t\partial\theta_k}\Big|\le H^{(3)}_{\theta^0}(x), && \int_0^\infty [H^{(3)}_{\theta^0}(x)]^2\mathrm{d}x<+\infty,\\
&\Big|\frac{\partial^3 \ln f(x,\theta)}{\partial\theta_s\partial\theta_t\partial\theta_k}\Big|\le \widetilde{H}^{(3)}_{\theta^0}(x), && \int_0^\infty [\widetilde{H}^{(3)}_{\theta^0}(x)]^2 f(x,\theta^0)\mathrm{d}x<+\infty,\\
&\int_0^\infty\Big[\frac{\partial^2 f(x,\theta)}{\partial\theta_s\partial\theta_t}\Big]^2\mathrm{d}x<+\infty, && \int_0^\infty\Big[\frac{\partial^2 \ln f(x,\theta)}{\partial\theta_s\partial\theta_t}\Big]^2 f(x,\theta)\mathrm{d}x<+\infty,\\
&\Big|\frac{\partial^3 \ln \overline{F}(x,\theta)}{\partial\theta_s\partial\theta_t\partial\theta_k}\Big|\le \widehat{H}^{(3)}_{\theta^0}(x), && \sup_{x\ge 0}[\widehat{H}^{(3)}_{\theta^0}(x)]^2\overline{F}(x,\theta^0)\le M,\\
&\Big|\frac{\partial^2 \ln \overline{F}(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le \widehat{H}^{(2)}_{\theta^0}(x), && \sup_{x\ge 0}[\widehat{H}^{(2)}_{\theta^0}(x)]^2\overline{F}(x,\theta^0)\le M,\\
&\Big|\frac{\partial^3 \ln F(x,\theta)}{\partial\theta_s\partial\theta_t\partial\theta_k}\Big|\le \overline{H}^{(3)}_{\theta^0}(x), && \sup_{x\ge 0}[\overline{H}^{(3)}_{\theta^0}(x)]^2 F(x,\theta^0)\le M,\\
&\Big|\frac{\partial^2 \ln F(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le \overline{H}^{(2)}_{\theta^0}(x), && \sup_{x\ge 0}[\overline{H}^{(2)}_{\theta^0}(x)]^2 F(x,\theta^0)\le M,
\end{aligned}$$

其中: M与x无关, 与θ^0有关.

(3)

$$\begin{aligned}
&\int_0^\infty\Big(\frac{\partial\ln f(x,\theta)}{\partial\theta_s}\Big)^4 f(x,\theta)\mathrm{d}x<+\infty,\\
&\sup_{x\ge 0}\Big(\frac{\partial\ln \overline{F}(x,\theta)}{\partial\theta_s}\Big)^4\overline{F}(x,\theta)\le M,\\
&\sup_{x\ge 0}\Big(\frac{\partial\ln F(x,\theta)}{\partial\theta_s}\Big)^4 F(x,\theta)\le M.
\end{aligned}$$

(4) 似然函数正规(见定义).

(5) 对$\forall\,\theta\in\Theta$, $[\partial\ln f(x,\theta)/\partial\theta]\cdot[\partial\ln f(x,\theta)/\partial\theta]^T$或$[\partial\ln F(x,\theta)/\partial\theta]\cdot[\partial\ln F(x,\theta)/\partial\theta]^T$或$[\partial\ln\overline{F}f(x,\theta)/\partial\theta]\cdot[\partial\ln\overline{F}(x,\theta)/\partial\theta]^T$关于$x$正定.

(6) 存在分布函数$G_0(x)$, 使得$\lim\limits_{n\to\infty}(1/n)\cdot\sum\limits_{i=1}^{n}G_i(x)=G_0(x)$, 且至少存在一点$x_0>0$, 使得$G_0(x_0)<1$.

同时, 我们由条件(1)–(3)可以证明, 存在函数$H^{(2)}_{\theta^0}(x)$, $H^{(1)}_{\theta^0}(x)$, $\widetilde{H}^{(2)}_{\theta^0}(x)$对一切$s$, t及$(x,\theta)\in[0,+\infty)\times\mu_{\theta^0}$满足

$$\begin{aligned}
&\Big|\frac{\partial^2 f(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le H^{(2)}_{\theta^0}(x), && \int_0^\infty [H^{(2)}_{\theta^0}(x)]^2\mathrm{d}x<+\infty,\\
&\Big|\frac{\partial f(x,\theta)}{\partial\theta_s}\Big|\le H^{(1)}_{\theta^0}(x), && \int_0^\infty H^{(1)}_{\theta^0}(x)\mathrm{d}x<+\infty,\\
&\Big|\frac{\partial^2 \ln f(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le \widetilde{H}^{(2)}_{\theta^0}(x), && \int_0^\infty [\widetilde{H}^{(2)}_{\theta^0}(x)]^2 f(x,\theta^0)\mathrm{d}x<+\infty.
\end{aligned}$$

易证Weibull分布、对数正态分布满足条件(Φ)中的(1)–(5).

我们的主要结论如下:

定理 设$\{Z_i, \alpha_i, \beta_i\}$, $i=1,2,\cdots,n$满足条件(Φ), $\widehat{\theta}^n$为θ的最大似然估计, 则有

(1)
$$\overline{\lim_{n\to\infty}} K(n)(\widehat{\theta}_k^n - \theta_k^0) = \varphi(k) \qquad \text{a.s.}\ \mathsf{P}_{\theta^0}, \tag{2.3}$$
$$\underline{\lim_{n\to\infty}} K(n)(\widehat{\theta}_k^n - \theta_k^0) = -\varphi(k) \qquad \text{a.s.}\ \mathsf{P}_{\theta^0}; \tag{2.4}$$

(2)
$$S\{K(n)(\widehat{\theta}_k^n - \theta_k^0)\} = [-\varphi(k), \varphi(k)] \qquad \text{a.s.}\ \mathsf{P}_{\theta^0}, \tag{2.5}$$

其中, $\theta^0 = (\theta_1^0, \theta_2^0, \cdots, \theta_m^0)^T$为参数真值, $k=1,2,\cdots,m$,

$$K(n) = (n/2\log\log n)^{1/2}, \qquad \varphi(k) > 0,$$

$$\begin{aligned}
\varphi^2(k) \;=\; & p\int_0^\infty \Big(\sum_{l=1}^m a_{kl}(\theta^0)\frac{\partial \ln f(x,\theta^0)}{\partial \theta_l}\Big)^2 \overline{G}_0(x) f(x,\theta^0)\mathrm{d}x \\
& + (1-p)\int_0^\infty \Big(\sum_{l=1}^m a_{kl}(\theta^0)\frac{\partial \ln F(x,\theta^0)}{\partial \theta_l}\Big)^2 F(x,\theta^0)\mathrm{d}G_0(x) \\
& + \int_0^\infty \Big(\sum_{l=1}^m a_{kl}(\theta^0)\frac{\partial \ln \overline{F}(x,\theta^0)}{\partial \theta_l}\Big)^2 \overline{F}(x,\theta^0)\mathrm{d}G_0(x),
\end{aligned} \tag{2.6}$$

$$(a_{kl}(\theta^0))_{m\times m}^{-1} = (\widehat{g}_{kl}(\theta^0))_{m\times m},$$

$$\begin{aligned}
\widehat{g}_{kl}(\theta) \;=\; & p\int_0^\infty \frac{\partial \ln f(x,\theta)}{\partial \theta_k}\frac{\partial \ln f(x,\theta)}{\partial \theta_l}\overline{G}_0(x) f(x,\theta^0)\mathrm{d}x \\
& + (1-p)\int_0^\infty \frac{\partial \ln F(x,\theta)}{\partial \theta_k}\frac{\partial \ln F(x,\theta)}{\partial \theta_l} F(x,\theta^0)\mathrm{d}G_0(x) \\
& + \int_0^\infty \frac{\partial \ln \overline{F}(x,\theta)}{\partial \theta_k}\frac{\partial \ln \overline{F}(x,\theta)}{\partial \theta_l} \overline{F}(x,\theta^0)\mathrm{d}G_0(x),
\end{aligned} \tag{2.7}$$

$S\{a_n\}$表示序列$\{a_n\}$的极限点的全体.

§3. 若干引理

为给出定理的证明, 我们需要如下一些引理.

引理 3.1 ([3]) 对任一给定的$\theta^0 \in \Theta$及任一Borel函数$T(x)$, 若$T(z_i)$关于概率测度P_{θ^0}可积, 则有,

$$\begin{aligned}
& \mathsf{E}_{\theta^0}[\alpha_i\beta_i T(z_i)] = p\int_0^\infty T(x)\overline{G}_i(x)\mathrm{d}F(x,\theta^0), \\
& \mathsf{E}_{\theta^0}[\alpha_i(1-\beta_i)T(z_i)] = (1-p)\int_0^\infty T(x)F(x,\theta^0)\mathrm{d}G_i(x), \\
& \mathsf{E}_{\theta^0}[(1-\alpha_i)T(z_i)] = \int_0^\infty T(x)\overline{F}(x,\theta^0)\mathrm{d}G_i(x).
\end{aligned} \tag{3.1}$$

其中：$\overline{G}_i(x) = 1 - G_i(x)$.

引理 3.2 在条件(Φ)下，对每个i $(i = 1, 2, \cdots, n)$有

$$\mathsf{E}_{\theta^0}\left(\alpha_i\beta_i\frac{\partial \ln f(Z_i,\theta^0)}{\partial\theta_s} + \alpha_i(1-\beta_i)\frac{\partial \ln F(Z_i,\theta^0)}{\partial\theta_s} + (1-\alpha_i)\frac{\partial \ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s}\right) = 0$$

$$s = 1, 2, \cdots, m. \tag{3.2}$$

证明过程请见文献[4]中引理2的证明.

引理 3.3 (Petrov[9]) 设$\{X_i, i \geq 1\}$是独立随机变量序列，$\mathsf{E}X_i = 0$，$i \geq 1$，$S_n^2 = \sum\limits_{i=1}^n \mathsf{E}X_i^2$，又设$\varliminf\limits_{n\to\infty} S_n^2/n > 0$且存在$\delta > 0$，使得$\sup\limits_{i\geq 1}\mathsf{E}|X_i|^{2+\delta} < \infty$，则

$$\varlimsup_{n\to\infty}\frac{\sum\limits_{i=1}^n X_i}{\sqrt{2S_n^2\log\log S_n^2}} = 1 \qquad \text{a.s.}. \tag{3.3}$$

引理 3.4 在条件(Φ)下，则对任意$\delta \in [0, 2]$均有

$$\max_{1\leq s\leq m}\sup_{i\geq 1}\mathsf{E}_{\theta^0}\left|\alpha_i\beta_i\frac{\partial \ln f(Z_i,\theta^0)}{\partial\theta_s} + \alpha_i(1-\beta_i)\frac{\partial \ln F(Z_i,\theta^0)}{\partial\theta_s} + (1-\alpha_i)\frac{\partial \ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s}\right|^{2+\delta} \leq k, \tag{3.4}$$

其中：$k < +\infty$.

证明： 由矩不等式知

$$\begin{aligned}
&\mathsf{E}_{\theta^0}\left|\alpha_i\beta_i\frac{\partial \ln f(Z_i,\theta^0)}{\partial\theta_s} + \alpha_i(1-\beta_i)\frac{\partial \ln F(Z_i,\theta^0)}{\partial\theta_s} + (1-\alpha_i)\frac{\partial \ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s}\right|^{2+\delta} \\
\leq\ & \left\{\mathsf{E}_{\theta^0}\left|\alpha_i\beta_i\frac{\partial \ln f(Z_i,\theta^0)}{\partial\theta_s} + \alpha_i(1-\beta_i)\frac{\partial \ln F(Z_i,\theta^0)}{\partial\theta_s} + (1-\alpha_i)\frac{\partial \ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s}\right|^4\right\}^{(2+\delta)/4}.
\end{aligned}$$

为证明引理只需

$$\mathsf{E}_{\theta^0}\left|\alpha_i\beta_i\frac{\partial \ln f(Z_i,\theta^0)}{\partial\theta_s} + \alpha_i(1-\beta_i)\frac{\partial \ln F(Z_i,\theta^0)}{\partial\theta_s} + (1-\alpha_i)\frac{\partial \ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s}\right|^4$$

对一切i及s有界即可.

同时我们注意到$\alpha_i = 1$或0，$\beta_i = 1$或0，故由引理3.1及条件(Φ)中的(3)知，存在某常数$K_s > 0$，使

$$\begin{aligned}
&\mathsf{E}_{\theta^0}\left|\alpha_i\beta_i\frac{\partial \ln f(Z_i,\theta^0)}{\partial\theta_s} + \alpha_i(1-\beta_i)\frac{\partial \ln F(Z_i,\theta^0)}{\partial\theta_s} + (1-\alpha_i)\frac{\partial \ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s}\right|^4 \\
=\ & \mathsf{E}_{\theta^0}\left[\alpha_i\beta_i\left(\frac{\partial \ln f(Z_i,\theta^0)}{\partial\theta_s}\right)^4 + \alpha_i(1-\beta_i)\left(\frac{\partial \ln F(Z_i,\theta^0)}{\partial\theta_s}\right)^4 + (1-\alpha_i)\left(\frac{\partial \ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s}\right)^4\right]
\end{aligned}$$

$$
\begin{aligned}
&= p\int_0^\infty\Big(\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\Big)^4\overline{G}_i(x)\mathrm{d}F(x,\theta^0)+(1-p)\int_0^\infty\Big(\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big)^4F(x,\theta^0)\mathrm{d}G_i(x)\\
&\quad+\int_0^\infty\Big(\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\Big)^4\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\\
&\le p\int_0^\infty\Big(\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\Big)^4f(x,\theta^0)\mathrm{d}x+(1-p)\sup_{x\ge0}\Big(\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big)^4F(x,\theta^0)\\
&\quad+\sup_{x\ge0}\Big(\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\Big)^4\overline{F}(x,\theta^0)\\
&\le K_s<+\infty.
\end{aligned}
$$

于是令$K\triangleq\max\{K_1,K_2,\cdots,K_m\}$, 对任意$i\ge1$, $s=1,2,\cdots,m$有

$$
\begin{aligned}
&\mathsf{E}_{\theta^0}\Big|\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big|^4\\
\le\ & K<+\infty.
\end{aligned}\tag{3.5}
$$

从而引理得证. $\square$

引理 3.5 在条件(Φ)下, 对任意$t\in[0,1]$且$\theta^0\in\Theta$, 令$\theta(t)=(1-t)\theta^0+t\widehat{\theta}^n$, 其中$\widehat{\theta}^n$是$\theta$的最大似然估计, 则对一切$k,l$ $(k,l=1,2,\cdots,m)$有

$$
\sup_{0\le t\le1}|a_{kl}^{(n)}(\theta(t))+\widehat{g}_{kl}(\theta^0)|\longrightarrow0\qquad\text{a.s.}\ \ \mathsf{P}_{\theta^0}\ \ (n\to\infty),
$$

其中:

$$
a_{kl}^{(n)}(\theta)\triangleq\frac1n\sum_{i=1}^n\Big[\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_k\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_k\partial\theta_l}+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta)}{\partial\theta_k\partial\theta_l}\Big],
$$

$\widehat{g}_{kl}(\theta)$的定义见(2.7).

证明: 显然

$$
\sup_{0\le t\le1}|a_{kl}^{(n)}(\theta(t))+\widehat{g}_{kl}(\theta^0)|\le\sup_{0\le t\le1}|a_{kl}^{(n)}(\theta(t))-a_{kl}^{(n)}(\theta^0)|+|a_{kl}^{(n)}(\theta^0)+\widehat{g}_{kl}(\theta^0)|,\tag{3.6}
$$

所以证明引理分两步.

首先, 由引理3.1, 引理3.4[4]及强大数定律易知

$$
a_{kl}^{(n)}(\theta^0)\longrightarrow-\widehat{g}_{kl}(\theta^0)\qquad\text{a.s.}\ \ \mathsf{P}_{\theta^0}\ \ (n\to\infty).\tag{3.7}
$$

其次, 由于$\widehat{\theta}^n$是θ^0的强相合估计(见文献[4]), 故当n充分大时, $\widehat{\theta}^n\in\mu_{\theta^0},\theta(t)\in\mu_{\theta^0}$, 故

由多元函数的中值定理及条件(Φ)中的(2)可知, 当n充分大时, 有

$$\begin{aligned}
&|a_{kl}^{(n)}(\theta(t))-a_{kl}^{(n)}(\theta^0)|\\
=&\ \Big|\frac{1}{n}\sum_{i=1}^{n}\Big[\alpha_i\beta_i\Big(\frac{\partial^2\ln f(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}-\frac{\partial^2\ln f(Z_i,\theta^0)}{\partial\theta_k\partial\theta_l}\Big)\\
&+\alpha_i(1-\beta_i)\Big(\frac{\partial^2\ln F(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}-\frac{\partial^2\ln F(Z_i,\theta^0)}{\partial\theta_k\partial\theta_l}\Big)\\
&+(1-\alpha_i)\Big(\frac{\partial^2\ln \overline{F}(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}-\frac{\partial^2\ln \overline{F}(Z_i,\theta^0)}{\partial\theta_k\partial\theta_l}\Big)\Big]\Big|\\
=&\ \Big|\frac{1}{n}\sum_{i=1}^{n}\Big(\alpha_i\beta_i\frac{\partial^3\ln f(Z_i,\theta)}{\partial\theta_k\partial\theta_l\partial\theta}\Big|_{\theta=\theta_1(t)}+\alpha_i(1-\beta_i)\frac{\partial^3\ln F(Z_i,\theta)}{\partial\theta_k\partial\theta_l\partial\theta}\Big|_{\theta=\theta_2(t)}\\
&+(1-\alpha_i)\frac{\partial^3\ln \overline{F}(Z_i,\theta)}{\partial\theta_k\partial\theta_l\partial\theta}\Big|_{\theta=\theta_3(t)}\Big)\cdot t(\widehat{\theta}^n-\theta^0)\Big|\\
\le&\ \frac{1}{n}\sum_{i=1}^{n}[\alpha_i\beta_i\widetilde{H}_{\theta^0}^{(3)}(Z_i)+\alpha_i(1-\beta_i)\overline{H}_{\theta^0}^{(3)}(Z_i)+(1-\alpha_i)\widehat{H}_{\theta^0}^{(3)}(Z_i)]\cdot\sqrt{m}\|\widehat{\theta}^n-\theta^0\|,
\end{aligned}\tag{3.8}$$

其中$\theta_1(t),\theta_2(t),\theta_3(t)$是界于$\theta(t)$与$\theta^0$之间的向量. 同时由条件($\Phi$)中的(2)及引理3.1知, 对任意$i\ge 1$有

$$\begin{aligned}
&\mathsf{E}_{\theta^0}[\alpha_i\beta_i\widetilde{H}_{\theta^0}^{(3)}(Z_i)+\alpha_i(1-\beta_i)\overline{H}_{\theta^0}^{(3)}(Z_i)+(1-\alpha_i)\widehat{H}_{\theta^0}^{(3)}(Z_i)]^2\\
=&\ \mathsf{E}_{\theta^0}\big(\alpha_i\beta_i[\widetilde{H}_{\theta^0}^{(3)}(Z_i)]^2+\alpha_i(1-\beta_i)[\overline{H}_{\theta^0}^{(3)}(Z_i)]^2+(1-\alpha_i)[\widehat{H}_{\theta^0}^{(3)}(Z_i)]^2\big)\\
=&\ p\int_0^{\infty}[\widetilde{H}_{\theta^0}^{(3)}(x)]^2\overline{G}_i(x)f(x,\theta^0)\mathrm{d}x+(1-p)\int_0^{\infty}[\overline{H}_{\theta^0}^{(3)}(x)]^2F(x,\theta^0)\mathrm{d}G_i(x)\\
&+\int_0^{\infty}[\widehat{H}_{\theta^0}^{(3)}(x)]^2\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\\
\le&\ p\int_0^{\infty}[\widetilde{H}_{\theta^0}^{(3)}(x)]^2f(x,\theta^0)\mathrm{d}x+(1-p)\sup_{x\ge0}[\overline{H}_{\theta^0}^{(3)}(x)]^2F(x,\theta^0)\\
&+\sup_{x\ge0}[\widehat{H}_{\theta^0}^{(3)}(x)]^2\overline{F}(x,\theta^0)<+\infty.
\end{aligned}$$

故由控制收敛定理, Helly定理及强大数定律知

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}[\alpha_i\beta_i\widetilde{H}_{\theta^0}^{(3)}(Z_i)+\alpha_i(1-\beta_i)\overline{H}_{\theta^0}^{(3)}(Z_i)+(1-\alpha_i)\widehat{H}_{\theta^0}^{(3)}(Z_i)]\\
\longrightarrow&\ p\int_0^{\infty}\widetilde{H}_{\theta^0}^{(3)}(x)\overline{G}_0(x)f(x,\theta^0)\mathrm{d}x+(1-p)\int_0^{\infty}\overline{H}_{\theta^0}^{(3)}(x)F(x,\theta^0)\mathrm{d}G_0(x)\\
&+\int_0^{\infty}\widehat{H}_{\theta^0}^{(3)}(x)\overline{F}(x,\theta^0)\mathrm{d}G_0(x)\qquad \text{a.s. }\ \mathsf{P}_{\theta^0}\ \ (n\to\infty).
\end{aligned}\tag{3.9}$$

所以由(3.8), (3.9)及$\widehat{\theta}^n\to\theta^0$ a.s. P_{θ^0} $(n\to\infty)$ (见文献[4])可知

$$\sup_{0\le t\le 1}|a_{kl}^{(n)}(\theta(t))-a_{kl}^{(n)}(\theta^0)|\longrightarrow 0\qquad \text{a.s. }\ \mathsf{P}_{\theta^0}\ \ (n\to\infty).\tag{3.10}$$

最后, 由(3.6), (3.7), (3.10)可知引理成立. □

推论 3.1 在条件(Φ)下, 对任意$\theta^0\in\Theta$, 令$\theta(t)=(1-t)\theta^0+t\widehat{\theta}^n$, 其中$\widehat{\theta}^n$是$\theta$的最大似然估计, 则对$t\in[0,1]$一致地有

$$A_n(\theta(t))\longrightarrow -\widehat{G}(\theta^0)\qquad \text{a.s. } \mathsf{P}_{\theta^0}\ \ (n\to\infty), \tag{3.11}$$

其中$A_n(\theta(t))\triangleq\Big(a_{kl}^{(n)}(\theta(t))\Big)_{m\times m}$, $\widehat{G}(\theta^0)\triangleq\Big(\widehat{g}_{kl}(\theta^0)\Big)_{m\times m}$.

证明: 由引理3.5易知(3.11)成立. □

显然由条件(Φ)中的(5)易知$\widehat{G}(\theta^0)$正定(见文献[4]中的(4.6)式). 记

$$\Big(a_{kl}(\theta^0)\Big)_{m\times m}\triangleq\widehat{G}(\theta^0)^{-1}. \tag{3.12}$$

引理 3.6 在条件(Φ)下, 记$S_n^2(k)=\sum\limits_{i=1}^{n}\mathsf{E}_{\theta^0}\omega_i^2(k)$, 其中

$$\omega_i(k)=\sum_{l=1}^{m}a_{kl}(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big],$$

$a_{kl}(\theta^0)$见(3.12), 则有

$$\overline{\lim_{n\to\infty}}\frac{\sum\limits_{i=1}^{n}\omega_i(k)}{\sqrt{2S_n^2(k)\log\log S_n^2(k)}}=1\qquad\text{a.s. } \mathsf{P}_{\theta^0}. \tag{3.13}$$

证明: 首先, 由引理3.1及引理3.2知

$$\begin{aligned}\mathsf{E}_{\theta^0}\omega_i(k)&=\mathsf{E}_{\theta^0}\Big[\sum_{l=1}^{m}a_{kl}(\theta^0)\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big)\Big]\\&=\sum_{l=1}^{m}a_{kl}(\theta^0)\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big)\\&=0.\end{aligned}\tag{3.14}$$

其次, 对任意$i\geq 1$及$\varepsilon\in[0,1]$

$$\begin{aligned}\mathsf{E}_{\theta^0}|\omega_i(k)|^{2+\varepsilon}&=\mathsf{E}_{\theta^0}\Big|\sum_{l=1}^{m}a_{kl}(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]\Big|^{2+\varepsilon}\\&\leq\Big\{\mathsf{E}_{\theta^0}\Big|\sum_{l=1}^{m}a_{kl}(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]\Big|^4\Big\}^{(2+\varepsilon)/4}\end{aligned}$$

$$
\begin{aligned}
&\le \Big\{\mathsf{E}_{\theta^0}\Big|\sum_{l=1}^m a_{kl}(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]\Big|^4+1\Big\}^{(2+\varepsilon)/4}\\
&\le m^2\mathsf{E}_{\theta^0}\Big(\sum_{l=1}^m a_{kl}^2(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]^2\Big)^2+1\\
&\le m^3\mathsf{E}_{\theta^0}\Big(\sum_{l=1}^m a_{kl}^4(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]^4\Big)+1\\
&= m^3\sum_{l=1}^m a_{kl}^4(\theta^0)\mathsf{E}_{\theta^0}\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]^4+1.
\end{aligned}
$$

故由(3.5)式, 有

$$
\sup_{i\ge1}\mathsf{E}_{\theta^0}|\omega_i(k)|^{2+\varepsilon}<+\infty. \tag{3.15}
$$

最后, 再根据引理3.1可知

$$
\begin{aligned}
S_n^2(k) &= \sum_{i=1}^n\mathsf{E}_{\theta^0}\omega_i^2(k)\\
&= \sum_{i=1}^n\mathsf{E}_{\theta^0}\Big|\sum_{l=1}^m a_{kl}(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]\Big|^2\\
&= \sum_{s,l=1}^m a_{ks}(\theta^0)a_{kl}(\theta^0)\cdot\sum_{i=1}^n\mathsf{E}_{\theta^0}\Big\{\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big]\cdot\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]\Big\}\\
&= \sum_{s,l=1}^m a_{ks}(\theta^0)a_{kl}(\theta^0)\cdot\Big\{p\int_0^\infty\frac{\partial\ln f(x,\theta)}{\partial\theta_s}\frac{\partial\ln f(x,\theta)}{\partial\theta_l}f(x,\theta^0)\Big(n-\sum_{i=1}^n G_i(x)\Big)\mathrm{d}x\\
&\quad+(1-p)\int_0^\infty\frac{\partial\ln F(x,\theta)}{\partial\theta_s}\frac{\partial\ln F(x,\theta)}{\partial\theta_l}F(x,\theta^0)\mathrm{d}\sum_{i=1}^n G_i(x)\\
&\quad+\int_0^\infty\frac{\partial\ln\overline{F}(x,\theta)}{\partial\theta_s}\frac{\partial\ln\overline{F}(x,\theta)}{\partial\theta_l}\overline{F}(x,\theta^0)\mathrm{d}\sum_{i=1}^n G_i(x)\Big\}.
\end{aligned}
$$

故由条件(Φ)中的(2), (6)及控制收敛定理、Helly定理可知

$$\lim_{n\to\infty}\frac{S_n^2(k)}{n}=\sum_{s,l=0}^{m}a_{ks}(\theta^0)a_{kl}(\theta^0)\widehat{g}_{sl}(\theta^0)<\infty, \tag{3.16}$$

其中$\widehat{g}_{sl}(\theta^0)$定义见(2.7). 显然我们知道$\widehat{G}(\theta^0)$正定(见文献[4]中的(4.6)式), 所以

$$\lim_{n\to\infty}\frac{S_n^2(k)}{n}>0. \tag{3.17}$$

综合考虑(3.14), (3.15), (3.17)及引理3.3知, (3.13)成立. 证毕. □

§4. 定理的证明

定理的证明: (1) 令

$$\theta(t)=(1-t)\theta^0+t\widehat{\theta}^n,\qquad t\in[0,1], \tag{4.1}$$

$$u_n(\theta)=\frac{\partial\ln L(\theta)}{\partial\theta},\qquad g(t)=u_n(\theta(t)), \tag{4.2}$$

则$g(1)-g(0)=\int_0^1 g'(t)\mathrm{d}t$, 其中$g(1)=u_n(\widehat{\theta}^n)=0$ (由似然方程组(2.2)式知),

$$\begin{aligned}
g(0)&=u_n(\theta^0)=\frac{\partial\ln L(\theta^0)}{\partial\theta}\\
&=\Big(\cdots,\sum_{i=1}^{n}\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big],\cdots\Big)^T,\\
g'(t)&=\Big(\sum_{i=1}^{n}\Big[\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}\Big]\Big)_{m\times m}\cdot(\widehat{\theta}^n-\theta^0)\\
&=\Big(\frac{1}{n}\sum_{i=1}^{n}\Big[\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}\\
&\quad+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta(t))}{\partial\theta_k\partial\theta_l}\Big]\Big)_{m\times m}\cdot n(\widehat{\theta}^n-\theta^0)\\
&=A_n(\theta(t))\cdot n(\widehat{\theta}^n-\theta^0).
\end{aligned}\tag{4.3}$$

其中$A_n(\theta(t))$参见推论3.1, 于是

$$u_n(\theta^0)=-\int_0^1 A_n(\theta(t))\mathrm{d}t\cdot n(\widehat{\theta}^n-\theta^0). \tag{4.4}$$

为了证明定理的(1)成立, 我们分两步.

首先, 我们确定$\varphi(k)$的表达式, 记

$$\varphi^2(k)=\lim_{n\to\infty}\frac{S_n^2(k)\log\log S_n^2(k)}{n\log\log n}. \tag{4.5}$$

由(3.16)式可知,

$$\begin{aligned}\varphi^2(k) &= \lim_{n\to\infty}\frac{S_n^2(k)\log\log S_n^2(k)}{n\log\log n}\\ &= \lim_{n\to\infty}\frac{S_n^2(k)}{n}\lim_{n\to\infty}\Big(1+\frac{\log(1+\log(S_n^2(k)/n)/\log n)}{\log\log n}\Big)=\lim_{n\to\infty}\frac{S_n^2(k)}{n}\\ &= p\int_0^\infty\Big(\sum_{l=1}^m a_{kl}(\theta^0)\frac{\partial\ln f(x,\theta^0)}{\partial\theta_l}\Big)^2\overline{G}_0(x)f(x,\theta^0)\mathrm{d}x\\ &\quad+(1-p)\int_0^\infty\Big(\sum_{l=1}^m a_{kl}(\theta^0)\frac{\partial\ln F(x,\theta^0)}{\partial\theta_l}\Big)^2F(x,\theta^0)\mathrm{d}G_0(x)\\ &\quad+\int_0^\infty\Big(\sum_{l=1}^m a_{kl}(\theta^0)\frac{\partial\ln \overline{F}(x,\theta^0)}{\partial\theta_l}\Big)^2\overline{F}(x,\theta^0)\mathrm{d}G_0(x).\end{aligned} \tag{4.6}$$

其次, 由推论3.1可知

$$-\int_0^1 A_n(\theta(t))\mathrm{d}t\longrightarrow\widehat{G}(\theta^0)\qquad \text{a.s. } \mathsf{P}_{\theta^0}\ \ (n\to\infty). \tag{4.7}$$

又由$\widehat{G}(\theta^0)$的正定性及(4.7)式易知, 存在$N>0$, 使得$-\int_0^1 A_n(\theta(t))\mathrm{d}t$可逆, 且

$$\sup_{n>N}\Big\|\Big(-\int_0^1 A_n(\theta(t))\mathrm{d}t\Big)^{-1}\Big\|<\infty,$$

记

$$\nabla_n\triangleq\Big(-\int_0^1 A_n(\theta(t))\mathrm{d}t\Big)^{-1}-\widehat{G}^{-1}(\theta^0), \tag{4.8}$$

于是

$$\|\nabla_n\|\le\Big\|\Big(-\int_0^1 A_n(\theta(t))\mathrm{d}t\Big)^{-1}\Big\|\cdot\Big\|\widehat{G}(\theta^0)+\int_0^1 A_n(\theta(t))\mathrm{d}t\Big\|\cdot\|\widehat{G}^{-1}(\theta^0)\|\longrightarrow 0,\quad n\to\infty.$$

由上式易知, 当$n\to\infty$时, $\nabla_n\widehat{G}(\theta^0)+I\to I$, 所以由(4.4), (4.8)知,

$$\begin{aligned}\sqrt{\frac{n}{2\log\log n}}(\widehat{\theta}^n-\theta^0) &= \frac{1}{\sqrt{2n\log\log n}}\Big(-\int_0^1 A_n(\theta(t))\mathrm{d}t\Big)^{-1}u_n(\theta^0)\\ &= \frac{1}{\sqrt{2n\log\log n}}(\widehat{G}^{-1}(\theta^0)+\nabla_n)u_n(\theta^0)\\ &= \frac{1}{\sqrt{2n\log\log n}}(I+\nabla_n\widehat{G}(\theta^0))\widehat{G}^{-1}(\theta^0)u_n(\theta^0)\\ &= (1+o(1))\frac{1}{\sqrt{2n\log\log n}}\widehat{G}^{-1}(\theta^0)u_n(\theta^0).\end{aligned} \tag{4.9}$$

最后由引理3.6和(4.6), (4.9), (4.3), (3.12)可以得到, 对任意的k ($k=1,2,\cdots,m$)有

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\sqrt{\frac{n}{2\log\log n}}(\widehat{\theta}_k^n-\theta_k^0)\\
=&\overline{\lim_{n\to\infty}}\frac{1}{\sqrt{2n\log\log n}}\sum_{i=1}^{n}\Big\{\sum_{l=1}^{m}a_{kl}(\theta^0)\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}\\
&+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]\Big\}\\
=&\overline{\lim_{n\to\infty}}\sqrt{\frac{S_n^2(k)\log\log S_n^2(k)}{n\log\log n}}\frac{1}{\sqrt{2S_n^2(k)\log\log S_n^2(k)}}\sum_{i=1}^{n}\omega_i(k)\\
=&\varphi(k).
\end{aligned}$$

即(2.3)成立, 同理可证(2.4)成立.

(2) 显然, 对任意的k ($k=1,2,\cdots,m$), 当$n\geq 2$时, 都有

$$n(\widehat{\theta}_k^n-\widehat{\theta}_k^{n-1})+(\widehat{\theta}_k^{n-1}-\theta_k^0)=n(\widehat{\theta}_k^n-\theta_k^0)-(n-1)(\widehat{\theta}_k^{n-1}-\theta_k^0). \tag{4.10}$$

根据上述所证定理的(1)的结论及(4.10)易知

$$\lim_{n\to\infty}\frac{n}{\sqrt{2n\log\log n}}(\widehat{\theta}_k^n-\widehat{\theta}_k^{n-1})+\lim_{n\to\infty}\frac{1}{\sqrt{2n\log\log n}}(\widehat{\theta}_k^{n-1}-\theta_k^0)=0,$$

即

$$\lim_{n\to\infty}K(n)(\widehat{\theta}_k^n-\widehat{\theta}_k^{n-1})=0.$$

又由引理3.7[8]知,

$$S\{K(n)(\widehat{\theta}_k^n-\theta_k^0)\}=[-\varphi(k),\varphi(k)]\qquad \text{a.s.}\ \ \mathsf{P}_{\theta^0}.$$

即(2.5)式成立. 证毕. □

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A Law of the Iterated Logarithm for MLE Based on Random Censoring Model with Incomplete Information

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In this paper, we prove that MLE for life distributed parameter converges to the true parameter at the rate of the law of iterated logarithm. At the same time, we verify that Weibull distribution and lognormal distribution are stratified with conditions (Φ) proposed here.

Keywords: Random censoring model with incomplete information, MLE, the law of Iterated Logarithm.

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