

The Random Attractors of Stochastic Duffing-Van Der Pol Equations with Jumps *

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Abstract

In this paper, the theory of random dynamical systems and stochastic analysis are used to research the existence of random attractors and also stochastic bifurcation behavior for stochastic Duffing-van der Pol equation with jumps under some assumptions.

Keywords: Random attractors, random dynamical system, stochastic bifurcation, stochastic Duffing-van der Pol oscillator with jumps.

AMS Subject Classification: 60H10.

§1. Introduction

The deterministic nonlinear Duffing-van der Pol equation

$$\ddot{x} = \alpha x + \beta \dot{x} + \gamma x^3 + \delta x^2 \dot{x} + \varepsilon \dot{x}^3, \quad \alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R} \quad (1.1)$$

has become a paradigm for mathematicians, physicists and engineers. There are numerous physical and engineering problems whose dynamics are described by (1.1) for some parameter values.

The properties of random Duffing-van der Pol oscillator which is perturbed by white noise or real noise have been researched by Arnold^[1], Arnold, Sri Namachchivaya and Schenk-Hoppé^[2] and Schenk-Hoppé^[10–13]. They used Lyapunov's direct method to prove the existence of random attractors and the stochastic bifurcation theory for random Duffing-van der Pol equation. In this paper, we will use the theory of random dynamical systems in Arnold^[1] and Schenk-Hoppé^[10] and stochastic analysis method to research the existence of random attractors of stochastic Duffing-van der Pol equations with jumps and study their bifurcation behavior under some assumptions.

*Project is supported by the National Natural Science Foundation of China (10671168, 10971180) and the Natural Science Foundation of the Jiangsu Province (07-333).

Received October 28, 2005. Revised January 13, 2009.

Random dynamical system is one of important branches of probability theory. For continuous case, they have been studied by many authors, their main results can be found in Arnold's book^[1] and the references therein. However, the world is more complicated and models which are allowed to have jumps — both big and small — are desirable. Hence, it is necessary to study stochastic dynamical systems with jumps.

In [7], Kunita indicated that Itô's equation on \mathbb{R}^d driven by a Lévy process or a semimartingale with jumps $Z_t = (Z_t^1, \dots, Z_t^m)$ can be expressed in the form:

$$dX_t = \sum_{i=1}^m a_i(X_t) dZ_t^i. \quad (1.2)$$

Assuming that a_1, \dots, a_m are Lipschitz continuous maps from \mathbb{R}^d into itself, then the solution of (1.2) admits a version of a stochastic flow $\varphi_s(t, \omega)x$ which is continuous in $x \in \mathbb{R}^d$ and càdlàg in $t \in [s, \infty)$. However, the maps $\varphi_s(t, \omega)x$ are not homeomorphisms in general, since the maps $f: x \rightarrow x + \sum_{i=1}^m \Delta Z_t^i a_i(x)$ caused by the jumps $\Delta Z_t^i = Z_t^i - Z_{t-}^i$ of the driving process may not be homeomorphisms. But, if we give appropriate conditions on a_1, \dots, a_m , we can get that $\varphi_s(t, \omega)x$ is homeomorphisms in $x \in \mathbb{R}^d$. For example, the random dynamical system $\varphi(t, \omega)x$ generated by that one-dimension linear stochastic differential equation

$$dX_t = aX_t + \sigma X_t \circ dW_t + cX_{t-} dN_t$$

is homeomorphisms in $x \in \mathbb{R}$ with $c \neq -1$, since $\varphi(t, \omega, x) = xe^{at} \prod_{0 < s \leq t} (1 + c\Delta N_s)$ is homeomorphisms in $x \in \mathbb{R}$.

The paper is organized as follows. In Section 2, some basic concepts of random dynamic systems with jumps are given. In Section 3, we prove the existence of random attractors of stochastic Duffing-van der Pol equation with jumps. Finally, the stochastic bifurcation behavior are researched in Section 4.

§2. Random Dynamical Systems with Jumps

Arnold had studied continuous random dynamical systems in [1]. In order to use Arnold's theory to research random dynamical systems with jumps, we first give some preliminaries in the following.

Definition 2.1 Let $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system. That is, $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space with a flow $(\theta_t)_{t \in \mathbb{R}}$ on Ω which is measurable and measure preserving, i.e.

$$\theta_0 = \text{id}_\Omega, \quad \theta_{s+t} = \theta_t \circ \theta_s, \quad \forall s, t \in \mathbb{R}$$

and $\theta_t P = P$ for all $t \in \mathbb{R}$.

Random dynamical system (RDS) φ with jumps over $((\Omega, \mathcal{F}, P), (\theta_t)_{t \in T})$ on \mathbb{R}^d is defined as a measurable mapping

$$\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x)$$

such that

$$(i) \quad \varphi(0, \omega) = \text{id}_{\mathbb{R}^d} \text{ and}$$

$$\varphi(s+t, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \quad \forall s, t \in \mathbb{R},$$

where $\varphi(t, \omega)$ is defined by $\varphi(t, \omega)x = \varphi(t, \omega, x)$;

(ii) For P-a.s. ω and for all $t \in \mathbb{R}$, the map $(x, t) \rightarrow \varphi(t, \omega)x$ is càdàg with respect to $t \in \mathbb{R}$ and diffeomorphism on \mathbb{R}^d .

Definition 2.2 φ is called a local random dynamical system with jumps over $((\Omega, \mathcal{F}, P), (\theta_t)_{t \in \mathbb{R}})$ on \mathbb{R}^d , if it satisfies the following conditions:

(i) for P-a.s. ω and all $t \in \mathbb{R}$, $\varphi(t, \omega) : D_t(\omega) \rightarrow R_t(\omega)$ is a (local) diffeomorphism of the random (open) domain $D_t(\omega) \subset \mathbb{R}^d$ onto the random range $R_t(\omega) = \varphi(t, \omega)D_t(\omega) \subset \mathbb{R}^d$ where $D_t(\omega) \downarrow$ as $t \uparrow \infty$, and $\varphi(0, \omega) = \text{Id}_{\mathbb{R}^d}$;

(ii) for P-a.s. ω the family $(\varphi(t, \omega))_{t \in \mathbb{R}}$ of local diffeomorphisms is a local cocycle in the following sense:

Let $s, t \in \mathbb{R}$ and $x \in D_s(\omega)$. then $x \in D_{s+t}(\omega)$ if and only if $\varphi(s, \omega)x \in D_t(\theta_s \omega)$, and in the case,

$$\varphi(s+t, \omega)x = \varphi(t, \theta_s \omega)\varphi(s, \omega)x. \quad (2.1)$$

Putting $t = -s$ in (ii) and using $D_0(\omega) = \mathbb{R}^d$. We find that if $x \in D_s(\omega)$ then $\varphi(s, \omega)x \in D_s(\theta_s \omega)$. That is $R_t(\omega) = D_{-t}(\theta_t \omega)$ for all $t \in \mathbb{R}$, and (2.1) reads

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega) : D_{-t}(\theta_t \omega) \rightarrow D_t(\omega).$$

Set $E(\omega) = \bigcap_{t \in \mathbb{R}} D_t(\omega) = \{x : \varphi(t, \omega, x) \text{ exists for all times}\}$. $D(\omega)$ is the set of all initial values x whose orbits never explode. It may happen that $D(\omega) = \emptyset$.

In the following we will define random attractor, universe of sets, and domain of attraction. Some remarks on measurable set-valued maps are given.

A family $(A(\omega))_{\omega \in \Omega}$ of closed subsets of \mathbb{R}^d is called a random set with respect to a measurable space (Ω, \mathcal{F}) , if for any fixed $x \in \mathbb{R}^d$, the random variable

$$\omega \rightarrow d(x, A(\omega)) = \inf\{d(x, y) : y \in A(\omega)\}$$

is $\mathcal{F}/\mathcal{B}(\mathbb{R}^+)$ -measurable, where $d(x, y)$ is distance between x and y . We make the convention $d(x, \emptyset) = \infty$.

If, in addition, $A(\omega) \neq \emptyset$ a.s., then A is called a non-empty random set.

Definition 2.3 Let φ be a two-sided local RDS with jumps on \mathbb{R}^n .

(i) A universe of sets \mathcal{D} is a collection of families $(D(\omega))_{\omega \in \Omega}$ of non-empty subsets of \mathbb{R}^n , such that \mathcal{D} is closed with respect to set inclusion, i.e. if $D_1 \in \mathcal{D}$ and $D_2(\omega) \subset D_1(\omega)$ for all $\omega \in \Omega$, then $D_2 \in \mathcal{D}$;

(ii) A set $B \in \mathcal{D}$ is called absorbing in \mathcal{D} for φ , if B absorbs all sets in \mathcal{D} , i.e. for any $D \in \mathcal{D}$ there is a time $t_0(\omega, D)$ such that, for all $t \geq t_0(\omega, D)$, $\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega)$;

(iii) A random attractor of φ in \mathcal{D} is a random compact set $A \in \mathcal{D}$ with the following properties:

- (a) A is strictly invariant, $\varphi(t, \omega)A(\omega) = A(\theta_t\omega)$ for all $t \in \mathbb{R}$,
- (b) A attracts all sets in \mathcal{D} , i.e.

$$\lim_{t \rightarrow \infty} d(\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega)|A(\omega)) = 0.$$

The universe \mathcal{D} is called a domain of attraction of A , where $d(A|B) = \sup\{\inf\{d(x, y) : y \in B\} : x \in A\}$ is the Hausdorff semi-metric, $d(x, A) = d(\{x\}|A)$. Denote $d(A|B)$ is not a metric.

§3. The Random Attractors

In this section we will give the Duffing-van der Pol equation perturbed by the Poisson noise and prove the existence of random attractors for this particular system.

Let Ω be the set of càdlàg function ω_t on \mathbb{R} with $\omega_0 = 0$. Put $\theta_t\omega_s = \omega_{s+t} - \omega_t$ for $\omega \in \Omega$. It's easy to see that θ is a dynamical system on (Ω, \mathcal{F}) , where \mathcal{F} is the Borel σ -field on Ω with Skorokhod topology. Suppose that there exists a probability measure P on (Ω, \mathcal{F}) such that $\theta_t P = P$ for all $t \in \mathbb{R}$. We then have that $((\Omega, \mathcal{F}, P), (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system.

For the deterministic nonlinear Duffing-van der Pol equation

$$\ddot{y} = \alpha y + \beta \dot{y} - y^3 - y^2 \dot{y}, \quad \alpha, \beta \in \mathbb{R},$$

we have that its first order system for $x = (x_1, x_2) = (y, \dot{y})$ takes the form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \alpha x_1 + \beta x_2 - x_1^3 - x_1^2 x_2. \end{cases} \quad (3.1)$$

We now perturb the parameters α , β and the right-hand side of (3.1) by Poisson noise and consider the following SDE with jumps:

$$\begin{cases} dX_t = Y_t dt, \\ dY_t = [\alpha X_t + \beta Y_t - X_t^3 - X_t^2 Y_t] dt + \gamma_1 X_{t-} dN_t^1 + \gamma_2 Y_{t-} dN_t^2 + \gamma_3 dN_t^3, \end{cases} \quad (3.2)$$

where $\gamma_i \in \mathbb{R}$ ($i = 1, 2, 3$) are strength parameters, N^i is a Poisson process with intensity λ_i over $(\Omega, \mathcal{F}, \mathbb{P})$ on \mathbb{R} and N^1 , N^2 and N^3 are independent. A Poisson process N over $(\Omega, \mathcal{F}, \mathbb{P})$ on \mathbb{R} means that $\{N_t\}_{t \geq 0}$ and $\{-N_{-t}\}_{t \geq 0}$ are independent Poisson processes with $N_0 = 0$. The assumptions imply that N^1 , N^2 and N^3 do not have the jump points at the same time. Hence, $N^1 + N^2 + N^3$ is also a Poisson process with intensity $\lambda_1 + \lambda_2 + \lambda_3$.

We can prove that the results of Crauel and Flandoli^[6] are true for random dynamical system with jumps. As the proof in Theorem 4.4 of Schenk-Hoppé^[10] we have the follows:

Theorem 3.1 Let φ be a random dynamical system with jumps, and let \mathcal{U} denote a universe of sets. Suppose that there is a compact set $B \in \mathcal{U}$ such that

- (i) B is forward invariant, i.e. $\varphi(t, \omega)B(\omega) \subset B(\theta_t \omega)$ for all $t \geq 0$;
- (ii) B absorbs any set $D \in \mathcal{U}$: there is a time $t(\omega, D) > 0$ such that $\varphi(t, \theta_{-t} \omega)D(\theta_{-t} \omega) \subset B(\omega)$ for $t \geq t(\omega, D)$;
- (iii) there is a neighborhood of B in \mathcal{U} .

We then have that $A(\omega) = \bigcap_{n \in \mathbb{N}} \varphi(n, \theta_{-n} \omega)B(\theta_{-n} \omega)$ is the unique attractor for φ with domain of attraction $\mathcal{D}(A)$ containing \mathcal{U} .

In addition, we have also the following:

- (a) if $B(\omega)$ is measurable, then so is $A(\omega)$ and thus A is a random attractor;
- (b) if $\varphi(t, \theta_{-t} \omega)x$ is \mathcal{F}_{-t}^0 -measurable and B is $\mathcal{F}_{-\infty}^0$ -measurable, then A is $\mathcal{F}_{-\infty}^0$ -measurable too, where $\mathcal{F}_s^t = \sigma\{N_u^1, N_u^2, N_u^3, s \leq u \leq t\}$;
- (c) if $B(\omega)$ is connected, then so is $A(\omega)$.

A random variable $\eta : \Omega \rightarrow \mathbb{R}^+$ is called tempered if

$$\lim_{t \rightarrow \infty} \frac{\log^+ \eta(\theta_t \omega)}{t} = 0, \quad (3.3)$$

where $\log^+(\cdot) = \max\{\log(\cdot), 0\}$.

A non-empty random set $A(\omega)$ is called tempered if $\eta(\omega) = \sup\{\|x\| : x \in A(\omega)\}$ is tempered.

Obviously, for any random variable η ,

$$\lim_{t \rightarrow \infty} \frac{\log^+ |\eta(\theta_t \omega)|}{t} = 0 \Leftrightarrow \lim_{t \rightarrow \infty} e^{-\varepsilon t} |\eta(\theta_t \omega)| = 0, \quad \forall \varepsilon > 0. \quad (3.4)$$

This equivalence means that η does not grow with an exponential speed. Corollary 4 of O'Brien^[9] implies

$$\lim_{t \rightarrow \infty} \frac{\log^+ |\eta(\theta_t \omega)|}{t} = 0 \Leftrightarrow \lim_{t \rightarrow \infty} \frac{\log^+ |\eta(\theta_{-t} \omega)|}{t} = 0. \quad (3.5)$$

The latter will be used in our proofs.

Suppose that $A(\omega) \neq \emptyset$ is tempered random set. Proposition 3.2 (iii) of Schenk-Hoppé in [10] yields that there exists a dense sequence of measurable selections $s_n(\omega)$ of $A(\omega)$ such that $\overline{\{s_n(\omega) : n \geq 1\}} \supset A(\omega)$. This means that a tempered random set is well-defined, i.e. η is measurable.

Theorem 3.2 The equation (3.2) generates a smooth random dynamical system φ with jumps which is global to the forward, that is, $D_t(\omega) = \mathbb{R}^d$ for all $t > 0$. Moreover, $\varphi(t, \theta_{-t}\omega, x)$ is \mathcal{F}_t^0 -measurable.

Proof As the proof of Kunita in [8] we can prove that the random dynamical system $\varphi(t, \omega, x)$ generated by (3.2) is smooth with respect to $x \in \mathbb{R}^2$ and càdlàg with $t \in \mathbb{R}$.

Choose a function $V(x, y) = x^4 + 2y^2$. Itô's formula yields

$$\begin{aligned} V(X_t, Y_t) &= X_0^4 + 2Y_0^2 + 4 \int_0^t (\alpha X_s Y_s + \beta Y_s^2 - X_s^2 Y_s^2) ds \\ &\quad + \int_0^t (4\gamma_1 X_s Y_{s-} + 2\gamma_1^2 X_s^2) dN_s^1 \\ &\quad + (4\gamma_2 + 2\gamma_2^2) \int_0^t Y_{s-}^2 dN_s^2 + \int_0^t (4\gamma_3 Y_{s-} + 2\gamma_3^2) dN_s^3. \end{aligned} \quad (3.6)$$

Using

$$\alpha xy + \beta y^2 \leq \frac{|\alpha|}{2} x^2 + \left(\frac{|\alpha|}{2} + |\beta| \right) y^2 \leq \frac{1}{2} \left(|\alpha| \vee \left(\frac{|\alpha|}{2} + |\beta| \right) \right) V(x, y) + \frac{1}{2} |\alpha|$$

and

$$4\gamma_1 xy + 2\gamma_1^2 x^2 \leq 2(|\gamma_1| + \gamma_1^2) V(x, y) + 2(|\gamma_1| + \gamma_1^2),$$

(3.6) implies

$$\begin{aligned} V(X_t, Y_t) &\leq V(X_0, Y_0) + \int_0^t [(2|\alpha|) \vee (|\alpha| + 2|\beta|)] V(X_s, Y_s) + 2|\alpha| ds \\ &\quad + \int_0^t [2(|\gamma_1| + \gamma_1^2) V(X_{s-}, Y_{s-}) + 2(|\gamma_1| + \gamma_1^2)] dN_s^1 \\ &\quad + (2|\gamma_2| + \gamma_2^2) \int_0^t V(X_{s-}, Y_{s-}) dN_s^2 \\ &\quad + \int_0^t (2|\gamma_3| V(X_{s-}, Y_{s-}) + 4|\gamma_3| + 2\gamma_3^2) dN_s^3. \end{aligned} \quad (3.7)$$

Suppose that Z satisfies the following equation with jumps

$$\begin{aligned} Z_t = & V(X_0, Y_0) + \int_0^t [(2|\alpha|) \vee (2|\alpha| + 4|\beta|)Z_s + 2|\alpha|]ds \\ & + \int_0^t [2(|\gamma_1| + \gamma_1^2)Z_{s-} + 2(|\gamma_1| + \gamma_1^2)]dN_s^1 \\ & + (2|\gamma_2| + \gamma_2^2) \int_0^t Z_{s-}dN_s^2 + \int_0^t (2|\gamma_3|Z_{s-} + 4|\gamma_3| + 2\gamma_3^2)dN_s^3, \end{aligned} \quad (3.8)$$

then Z is finite on any finite interval $[0, T]$ for any $T > 0$. Hence, (3.7) and (3.8) yield that the process $V(X_t, Y_t)$ is finite on $[0, T]$ for any $0 < T < \infty$, that is, the random dynamical system $\varphi(t, \omega, x)$ is global to the forward. \square

Corollary 3.1 The random dynamical system $\varphi(t, \omega, x)$ generated by (3.2) has the following additional properties:

- (i) $D_t(\theta_{-t}\omega) = \mathbb{R}^2$ for all $t \geq 0$;
- (ii) $R_t(\theta_{-t}\omega) \downarrow E(\omega)$ as $t \uparrow \infty$.

Proof For $t \geq 0$, Theorem 3.2 yields that $\varphi(t, \omega, x)$ is non-explosive, we have the property (i) by replacing ω by $\theta_{-t}\omega$. Theorem 3.2 and Definition 2.2 (i) imply that $R_t(\theta_{-t}\omega) = \mathbb{R}^2$ for $t \leq 0$ and $R_t(\theta_t\omega) \subset R_s(\theta_s\omega)$ for $0 \leq s \leq t$, respectively. Since $R_t(\theta_{-t}\omega) = D_{-t}(\omega)$, the definition of $E(\omega)$ yields that (ii) is true. \square

In the following we will research the existence of random attractor of the random dynamical system $\varphi(t, \omega, x)$ generated by (3.2). Firstly, we consider the invariant measures for solution of stochastic differential equation with jumps.

Lemma 3.1 Suppose $h > -1$. If $a + \lambda \log(1+h) < 0$, then the stochastic differential equation with jumps

$$dX_t = (aX_t + c)dt + (hX_{t-} + k)dN_t, \quad X_0 = x, \quad (3.9)$$

has the unique invariant measure which is a Dirac measure supported by

$$\xi(\omega) = c \int_{-\infty}^0 e^{-as-bN_s}ds + k \int_{-\infty}^0 e^{-as-bN_s}dN_s \quad (3.10)$$

and $\xi(\omega)$ attracts all points with exponential speed. Where $b = \log(1+h)$ and N is a poisson process with intensity λ on \mathbb{R} .

Proof By Itô's formula we have that

$$\varphi(t, \omega)x = e^{at+bN_t} \left[x + c \int_0^t e^{-as-bN_s}ds + k \int_0^t e^{-as-bN_s}dN_s \right] \quad (3.11)$$

is the random dynamical system generated by (3.9). Since $\lim_{t \rightarrow -\infty} N_t/(\lambda t) = 1$ and $a + \lambda b < 0$, we have the integrals

$$\int_{-\infty}^0 e^{-as-bN_s} ds < \infty \quad \text{and} \quad \int_{-\infty}^0 e^{-as-bN_s} dN_s < \infty.$$

(3.9) is affine with stable linear part. Hence, (3.11) and Theorem 5.6.1 of Arnold in [1] imply that the unique invariant measure of (3.9) is the Dirac measure supported by

$$\xi(\omega) = \lim_{t \rightarrow \infty} \varphi(-t, \omega)^{-1} x = c \int_{-\infty}^0 e^{-as-bN_s} ds + k \int_{-\infty}^0 e^{-as-bN_s} dN_s$$

for all $x \in \mathbb{R}^2$ and $\xi(\omega)$ attracts all points with exponential speed. \square

Theorem 3.3 Suppose that the coefficients in (3.2) satisfy $\beta < -1$, $\gamma_1 > 0$, $\gamma_2 \geq 0$, $\gamma_3 = 0$ and $r_0 = \frac{\gamma_2(1+\gamma_2)}{\gamma_1+\gamma_2} < 1$. For $1 < \delta < 1.75$, if $2\beta + \delta < -1$ and

$$(\lambda_1 + \lambda_2) \log(1 + c(\gamma_1 + \gamma_2)) < \delta \quad (3.12)$$

for some $c \in (r_0, 1)$, then the random dynamical system generated by (3.2) possesses the unique parameter dependent tempered random attractor A with domain of attraction $\mathcal{D}(A)$ containing the universe of sets $Cl(\mathcal{U})$, generated by

$$\mathcal{U} = \{(D(\omega))_{\omega \in \Omega} : D(\omega) \subset \mathbb{R}^2 \text{ is a tempered random set}\}.$$

Moreover, the random attractor A is measurable with respect to $\mathcal{F}_{-\infty}^0 = \sigma\{N_t^1, N_t^2, t \leq 0\}$.

Proof Define the Lyapunov function on \mathbb{R}^2

$$V(x, y) = x^4 + x^2 + 2xy + 2y^2.$$

Itô's formula yields

$$\begin{aligned} V(X_t, Y_t) &= V(X_0, Y_0) \\ &+ \int_0^t [-2X_s^4 - 2X_s^3 Y_s - 4X_s^2 Y_s^2 + 2\alpha X_s^2 + (4\beta + 2)Y_s^2 + (4\alpha + 2\beta + 2)X_s Y_s] ds \\ &+ \int_0^t (4\gamma_1 X_{s-} Y_{s-} + 2\gamma_1(1 + \gamma_1)X_{s-}^2) dN_s^1 \\ &+ \int_0^t (2\gamma_2(1 + \gamma_2)Y_{s-}^2 + \gamma_2 X_{s-} Y_{s-}) dN_s^2. \end{aligned}$$

This implies

$$\begin{aligned} V(X_t, Y_t) &= V(X_0, Y_0) + \int_0^t [-\delta V(X_s, Y_s) + A(X_s, Y_s)] ds \\ &+ c(\gamma_1 + \gamma_2) \int_0^t V(X_{s-}, Y_{s-}) d(N_s^1 + N_s^2) \\ &+ \int_0^t B(X_{s-}, Y_{s-}) dN_s^1 + \int_0^t C(X_{s-}, Y_{s-}) dN_s^2, \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} A(x, y) &= -(2 - \delta)x^2[(x + (2 - \delta)^{-1}y)^2 + (4(2 - \delta)^{-1} - (2 - \delta)^{-2})y^2] \\ &\quad + (\delta + 2\alpha)x^2 + (4\alpha + 2\beta + 2 + 2\delta)xy + (4\beta + 2 + 2\delta)y^2, \\ B(x, y) &= -c(\gamma_1 + \gamma_2)x^4 + [\gamma_1(2 - c) + 2\gamma_1^2 - c\gamma_2]x^2 \\ &\quad + 2[\gamma_1 + c(\gamma_1 + \gamma_2)]xy - 2c(\gamma_1 + \gamma_2)y^2 \end{aligned}$$

and

$$\begin{aligned} C(x, y) &= -c(\gamma_1 + \gamma_2)x^4 - c(\gamma_1 + \gamma_2)x^2 + 2[\gamma_2 - c(\gamma_1 + \gamma_2)]xy \\ &\quad + 2[\gamma_2(1 + \gamma_2) - c(\gamma_1 + \gamma_2)]y^2. \end{aligned}$$

The hypotheses yield $2\beta + \delta + 1 < 0$ and $\gamma_2(1 + \gamma_2) - c(\gamma_1 + \gamma_2) < 0$. We then have $A(x, y) \leq K$, $B(x, y) \leq K$ and $C(x, y) \leq K$, where $K = K(\alpha, \beta, c, \delta, \gamma_1, \gamma_2)$ is a positive constant independent of $(x, y) \in \mathbb{R}^2$. (3.13) implies

$$\begin{aligned} V(X_t, Y_t) &\leq V(X_0, Y_0) + \int_0^t [-\delta V(X_s, Y_s) + K]ds \\ &\quad + \int_0^t [c(\gamma_1 + \gamma_2)V(X_{s-}, Y_{s-}) + K]d(N_s^1 + N_s^2). \end{aligned} \quad (3.14)$$

Denote $\varphi(t, \omega, (x, y))$ and $\psi(t, \omega, V(x, y))$ the random dynamical systems which are generated by (3.2) and

$$Z_t = V(X_0, Y_0) + \int_0^t (-\delta Z_s + K)ds + \int_0^t (c(\gamma_1 + \gamma_2)Z_{s-} + K)d(N_s^1 + N_s^2), \quad (3.15)$$

respectively. The random dynamical system generated by (3.15) is

$$\begin{aligned} &\psi(t, \omega)x \\ &= e^{-\delta t + (N_t^1 + N_t^2) \log(1 + c(\gamma_1 + \gamma_2))} \left[x + K \int_0^t e^{\delta s - (N_s^1 + N_s^2) \log(1 + c(\gamma_1 + \gamma_2))} d(s + N_s^1 + N_s^2) \right]. \end{aligned} \quad (3.16)$$

(3.14) yields

$$V(\varphi(t, \omega, (x, y))) \leq \psi(t, \omega)V(x, y). \quad (3.17)$$

Lemma 3.1 and (3.12) imply that $\psi(t, \omega)$ has the unique invariant measure which is a Dirac measure supported by

$$\xi(\omega) = K \int_{-\infty}^0 e^{\delta s - (N_s^1 + N_s^2) \log(1 + c(\gamma_1 + \gamma_2))} d(s + N_s^1 + N_s^2) > 0.$$

It is easy to prove that $\psi(t, \theta_{-t}\omega)x(\theta_{-t}\omega) \rightarrow \xi(\omega)$ as $t \rightarrow \infty$ for any initial value $x(\omega) \in \mathbb{R}_+$ such that $e^{-\delta t}x(\theta_{-t}\omega) \rightarrow 0$ for some $\delta > 0$. Hence, we may define the universe of sets

$$\bar{\mathcal{U}} = \{I(\omega) \subset \mathbb{R}^+ \text{ is a tempered random set}\},$$

that is, the random variable $\eta(\omega) = \sup_{x \in I(\omega)} x$ satisfies $\lim_{t \rightarrow \infty} \log^+[\eta(\theta_t\omega)/t] = 0$ for any $I \in \bar{\mathcal{U}}$.

(3.4) and (3.5) yield $\lim_{t \rightarrow \infty} e^{-\varepsilon t}\eta(\theta_{-t}\omega) = 0$ for any $\varepsilon > 0$.

It is obvious that $\bar{\mathcal{U}}$ is closed under inclusion and $\Omega \times \{x\} \in \bar{\mathcal{U}}$ for all $x \in \mathbb{R}^+$.

Next, we will prove that the random set $[0, (1+\varepsilon)\xi]$ is forward invariant and absorbing for $\psi(t, \omega)$ with respect to the universe $\bar{\mathcal{U}}$ for any $\varepsilon > 0$.

In fact, the absorbing property follows from the definition of $\bar{\mathcal{U}}$ and (3.5). We only prove that $\psi(t, \omega)[0, (1+\varepsilon)\xi(\omega)] \subset [0, (1+\varepsilon)\xi(\theta_t\omega)]$ for all $t \geq 0$. Since $\psi(t, \omega)$ is non-negative function on \mathbb{R} and $\psi(t, \omega)x < \psi(t, \omega)y$ for $x < y$, it suffices to show $\psi(t, \omega)(1+\varepsilon)\xi(\omega) \leq (1+\varepsilon)\xi(\theta_t\omega)$, this is equivalent to prove

$$\psi(t, \theta_{-t}\omega)(1+\varepsilon)\xi(\theta_{-t}\omega) \leq (1+\varepsilon)\xi(\omega). \quad (3.18)$$

By (3.16), we have

$$\begin{aligned} & \psi(t, \theta_{-t}\omega)(1+\varepsilon)\xi(\theta_{-t}\omega) \\ &= \xi(\omega) + \varepsilon K \int_{-\infty}^{-t} e^{\delta s - (N_s^1 + N_s^2) \log(1+c(\gamma_1+\gamma_2))} d(s + N_s^1 + N_s^2) \leq (1+\varepsilon)\xi(\omega). \end{aligned}$$

This shows that (3.18) is true.

Since

$$\begin{aligned} \xi(\theta_{-t}\omega) &= K e^{\delta t + (N_{-t}^1 + N_{-t}^2) \log(1+c(\gamma_1+\gamma_2))} \\ &\quad \cdot \int_{-\infty}^{-t} e^{\delta s - (N_s^1 + N_s^2) \log(1+c(\gamma_1+\gamma_2))} d(s + N_s^1 + N_s^2) \leq \xi(\omega), \end{aligned}$$

we have

$$e^{-\varepsilon_1 t} \xi(\theta_{-t}\omega) \rightarrow 0, \quad t \rightarrow \infty$$

for any $\varepsilon_1 > 0$. This means that $[0, (1+\varepsilon)\xi(\omega)] \in \bar{\mathcal{U}}$ for any $\varepsilon > 0$.

For any $\varepsilon > 0$, define the subset $B(\omega)$ of \mathbb{R}^2 by

$$B(\omega) = V^{-1}([0, (1+\varepsilon)\xi(\omega)]).$$

This is a non-empty compact set by the surjectivity and continuity of $V(x, y)$, and the fact that pre-images of bounded sets are bounded under $V(x, y)$. We will prove that the random set $B(\omega)$ satisfies the assumptions of Theorem 3.1.

The measurability of $B(\omega)$ may be implied by Lemma 5.7 of Schenk-Hoppé^[10].

Using the surjectivity and non-negativity of V and (3.17) we have

$$\begin{aligned} & \psi(t, \theta_{-t}\omega)[0, (1 + \varepsilon)\xi(\theta_{-t}\omega)] \subset [0, (1 + \varepsilon)\xi(\omega)] \\ \Leftrightarrow & \psi(t, \theta_{-t}\omega)V(B(\theta_{-t}\omega)) \subset V(B(\omega)) \\ \Rightarrow & V(\phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)) \subset V(B(\omega)) \end{aligned} \quad (3.19)$$

for all $t \geq 0$. This implies

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset V^{-1}(V(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega))) \subset V^{-1}(V(B(\omega))) = B(\omega), \quad (3.20)$$

that is, B is forward invariant.

For any $D \in \mathcal{U}$, we have that the random variable $\eta(\omega) = \sup\{\|(x, y)\|^4 : (x, y) \in D(\omega)\}$ grows sub-exponentially fast by the definition of \mathcal{U} . $V(x, y) \leq 3(x^4 + y^4) + 5 \leq 3\|(x, y)\|^4 + 5$ yields the random variable $\sup\{x \in V(D(\omega))\} \leq \eta(\omega) + 5$ grows sub-exponentially fast, that is, $V(D) \in \overline{\mathcal{U}}$.

As the proof of (3.19) and (3.20) we have that there is a $t(\omega, D) > 0$ such that, for all $t > t(\omega, D)$,

$$V(\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega)) \subset \psi(t, \theta_{-t}\omega)V(D(\theta_{-t}\omega)) \subset [0, (1 + \varepsilon)\xi(\omega)] = V(B(\omega))$$

by (3.18) and the set $[0, (1 + \varepsilon)\xi(\omega)]$ absorbing any set in $\overline{\mathcal{U}}$. This shows the absorption of any set in \mathcal{U} .

Finally, we prove the existence of a neighborhood of B in \mathcal{U} . In fact, for any $B \in \mathcal{U}$, Proposition 3.2 (iii) of Schenk-Hoppé^[10] implies that there exists a random variable $\rho(\omega) > 0$ such that $B(\omega) \subset S(\rho(\omega)) = \{x \in \mathbb{R}^2, \|x\| \leq \rho(\omega)\}$. It is obvious that $S(a\rho(\omega)) \in \mathcal{U}$ and that $S(a\rho(\omega))$ is a neighborhood of B for all $a > 1$.

By Theorem 3.1 we finish the proof. Because of $\xi(\omega) \in \mathcal{F}_{-\infty}^0$, we have that the attractor A is $\mathcal{F}_{-\infty}^0$ -measurable. \square

§4. The Stochastic Bifurcation

If φ is a random dynamical system with jumps and satisfies that $\varphi(t, \omega)x$ is a diffeomorphism on \mathbb{R}^d , we have

$$\Theta_r : \Omega \times \mathbb{R}^d \rightarrow \Omega \times \mathbb{R}^d, \quad (\omega, x) \rightarrow (\theta_t\omega, \varphi(t, \omega)x)$$

is a flow, called skew product flow induced by φ .

A probability measure μ on $\Omega \times \mathbb{R}^d$ is said to be invariant for the random dynamical system φ with jumps if

- (i) the marginal of μ on Ω is P ;
- (ii) μ satisfies $\Theta_t \mu = \mu$ for all $t \in \mathbb{R}$.

Using (i) and the disintegration $\mu(d\omega, dx) = \mu_\omega(dx)P(d\omega)$, (ii) is equivalent to $\varphi(t, \omega)\mu_\omega = \mu_{\theta_t \omega}$ for all $t \in \mathbb{R}$, P -a.s..

An invariant measure μ is supported by a random set A , if $\mu_\omega(A(\omega)) = 1$. For a closed set A , this is equivalent to $\text{Supp } \mu_\omega \subset A(\omega)$. Notice that $E(\omega) \neq \emptyset$ implies $\mu_\omega(E(\omega)) = 1$ for any invariant measure μ .

Let φ be a measurable random dynamical system on \mathbb{R}^d , and μ be an invariant measure for φ . Put $M^+(\omega, x) = \{y \in \mathbb{R}^d : \|\varphi(-t, \omega, y) - \varphi(-t, \omega, x)\| \rightarrow 0, \text{ as } t \rightarrow \infty\}$ and

$$M_\mu^+(\omega) = \bigcup_{x \in E(\omega) \cap \text{Supp } \mu_\omega} M^+(\omega, x).$$

$M_\mu^+(\omega)$ is called the unstable set of μ .

Proposition 4.1 Suppose that the conditions of Theorem 3.2 are satisfied. Denote $A(\omega)$ the random attractor of the random dynamical system φ generated by stochastic Duffing-van der Pol equation (3.2). Then

- (i) the random attractor A supports all invariant measures;
- (ii) the unstable set of any invariant measure μ is contained in random attractor, that is, $M_\mu^+(\omega) \subset A(\omega)$;
- (iii) there exists an invariant Markov measure μ supported by $\partial A(\omega)$, that is, μ_ω is $\mathcal{F}_{-\infty}^0$ -measurable.

Proof It is the same as the proof in Proposition 7.5 of Schenk-Hoppé^[10]. \square

Proposition 4.2 Suppose $\gamma_1 > \gamma_2 \geq 0$. For $-2 < \beta < -1$, we have that the top Lyapunov exponents l of the linear stochastic differential equation

$$\begin{cases} dX_t = Y_t dt, \\ dY_t = (-X_t + \beta Y_t) dt + \gamma_1 X_t dN_t^1 + \gamma_2 Y_t dN_t^2 \end{cases} \quad (4.1)$$

is bounded by

$$l \leq \frac{1}{2}\beta + \frac{1}{2}(\lambda_1 + \lambda_2) \log \left(1 + \frac{1 + \gamma_1^2 + 2\gamma_2 - \gamma_1\beta}{1 + \beta/2} \right). \quad (4.2)$$

Proof Put $V(x, y) = x^2 - \beta xy + y^2$. $\beta \in (-2, -1)$ yields $V(x, y) \geq 0$ and $V(x, y) = 0$

if and only if $x = y = 0$. Itô's formula implies

$$\begin{aligned} dV(X_t, Y_t) &= \beta V(X_t, Y_t)dt + [2\gamma_1 X_{t-} Y_{t-} - (\gamma_1 \beta - \gamma_1^2) X_{t-}^2] dN_t^1 \\ &\quad + [(\gamma_2^2 + 2\gamma_2) Y_{t-}^2 - \gamma_2 \beta X_{t-} Y_{t-}] dN_t^2. \end{aligned} \quad (4.3)$$

Since

$$2xy - (\gamma_1 \beta - \gamma_1^2) x^2 \leq (1 + \gamma_1^2 - \gamma_1 \beta)(x^2 + y^2), \quad (4.4)$$

$$(\gamma_2^2 + 2\gamma_2) y^2 - \gamma_2 \beta xy \leq \left(1 + \gamma_2^2 + 2\gamma_2 - \frac{1}{2} \gamma_2 \beta\right)(x^2 + y^2), \quad (4.5)$$

$$\left(1 + \frac{\beta}{2}\right)(x^2 + y^2) \leq V(x, y) \leq \left(1 - \frac{\beta}{2}\right)(x^2 + y^2), \quad (4.6)$$

and $\gamma_1 > \gamma_2$, (4.4)-(4.6) imply

$$dV(X_t, Y_t) \leq \beta V(X_t, Y_t)dt + \frac{1 + \gamma_1^2 + 2\gamma_2 - \gamma_1 \beta}{1 + \beta/2} V(X_t, Y_t) d(N_t^1 + N_t^2).$$

The Doléans-Dade's exponent formula implies

$$V(X_t, Y_t) \leq V(X_0, Y_0) \exp\{\beta t + (N_t^1 + N_t^2) \log(1 + a)\},$$

where $a = (1 + \gamma_1^2 + 2\gamma_2 - \gamma_1 \beta)/(1 + \beta/2)$. We then have

$$\begin{aligned} l &= \lim_{t \rightarrow \infty} \frac{(1/2) \log(V(X_t, Y_t))}{t} \leq \lim_{t \rightarrow \infty} \frac{\beta t + (N_t^1 + N_t^2) \log(1 + a)}{2t} \\ &= \frac{1}{2}(\beta + (\lambda_1 + \lambda_2) \log(1 + a)). \end{aligned}$$

(4.2) is proved. \square

Theorem 4.1 Suppose $\alpha = -1$ and $\beta \in (-2, -1)$ such that

$$\beta + (\lambda_1 + \lambda_2) \log \left(1 + \frac{1 + \gamma_1^2 + 2\gamma_2 - \gamma_1 \beta}{1 + \beta/2}\right) < 0. \quad (4.7)$$

Then, the random dynamical system generated by the stochastic Duffing-van der Pol equation (3.2) possesses the unique tempered random attractor $A(\omega) = \{0\}$ with domain of attraction $\mathcal{D}(A)$ containing the universe of $Cl(\mathcal{U})$ given by

$$\mathcal{U} = \{(D(\omega)_{\omega \in \Omega} : D(\omega) \subset \mathbb{R}^2 \text{ is a tempered random set}\},$$

where $Cl(\mathcal{U})$ is the completion universe of \mathcal{U} under inclusion.

In particular, for any $x \in \mathbb{R}^2$, the solutions $\varphi(t, \theta_{-t}\omega)x$ and $\varphi(t, \omega)x$ tend to zero exponentially fast as $t \rightarrow \infty$.

Proof Proposition 4.2 and the assumption (4.7) yield that the random dynamical system generated by (3.2) is stability. Let us define the function

$$V(x, y) = \frac{1 - \beta/2}{4}x^4 + \frac{1}{2}x^2 - \frac{\beta}{2}xy + \frac{1}{2}y^2.$$

Then $V(x, y) = 0$ if and only if $x = y = 0$. Using Itô's formula we have

$$\begin{aligned} dV(X_t, Y_t) &= \frac{1}{2}\beta(X_t^4 + X_t^2 - \beta X_t Y_t + Y_t^2)dt + \left[\frac{1}{2}\gamma_1(\gamma_1 - \beta)X_{t-}^2 + \gamma_1 X_{t-} Y_{t-}\right]dN_t^1 \\ &\quad + \left[\gamma_2(1 + \gamma_2)Y_{t-}^2 - \frac{1}{2}\gamma_2\beta X_{t-} Y_{t-}\right]dN_t^2. \end{aligned}$$

Since $\beta < \beta(2 - \beta)/4$ and $V(x, y) \geq (1/2 + \beta/4)(x^2 + y^2)$, by (4.4) and (4.5) we have

$$dV(X_t, Y_t) \leq \beta V(X_t, Y_t)dt + \frac{1 + \gamma_1^2 + 2\gamma_2 - \gamma_1\beta}{1 + \beta/2}V(X_t, Y_t)d(N_t^1 + N_t^2).$$

Corollary 5.6.3 of Arnold^[1] yields that the random dynamical system generated by the stochastic differential equation

$$dZ_t = \beta Z_t dt + \frac{1 + \gamma_1^2 + 2\gamma_2 - \gamma_1\beta}{1 + \beta/2}Z_{t-}d(N_t^1 + N_t^2)$$

has the unique invariant measure δ_0 which is the Dirac measure at 0. As the proof in Theorem 3.3 we have that $A(\omega) = \{0\} = V^{-1}(\{0\})$ is the unique random attractor attracting any set from \mathcal{U} .

Hence, this implies that, for any $x \in \mathbb{R}^2$, $\varphi(t, \theta_{-t}\omega)x$ and $\varphi(t, \omega)x$ tend to zero as $t \rightarrow \infty$ with an exponential fast. \square

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带跳的随机Duffing-Van Der Pol方程的随机吸引子

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本文利用随机动力系统和随机分析方法, 研究了在一定条件下带跳的随机Duffing-van der Pol方程随机吸引子的存在性和随机分岔.

关键词: 随机吸引子, 随机动力系统, 随机分岔, 带跳的随机Duffing-van der Pol方程.

学科分类号: O211.63.