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逐次截尾步加试验的最优设计 *

李 雪 程依明

(华东师范大学金融与统计学院, 上海, 200241)

摘 要

本文研究的是一类特殊的步进应力加速寿命试验, 即在试验过程中会有部分未失效产品移去(退出试验), 我们称这种试验为*逐次截尾加速寿命试验*. 本文在两种最优准则下分别研究了k个加速应力下定时与定数逐次截尾步加试验的最优设计问题.

关键词: 指数分布, 逐次截尾, Fisher信息矩阵, 最优设计.

学科分类号: O213.2.

§1. 引 言

在产品的可靠性试验中, 步进应力加速寿命试验是受到人们广泛关注和应用的一种方法. 而为了更进一步的降低试验成本, 可在每一个加速应力阶段末尾移去一定数量的未失效产品, 我们称这种试验为*逐次截尾加速寿命试验*.

步进应力加速寿命试验的最优设计问题早在60年代就已经提出, Miller and Nelson[1]和Bai, Kim and Lee[2]分别针对完全样本和截尾样本, 用极大似然估计理论对指数分布讨论了两个应力下的最优设计问题; 程依明[3]以极大似然估计渐近方差最小为准则, 研究了k个应力k个未知参数的加速方程下一般步进应力加速寿命试验的最优设计问题.

本文在逐次截尾寿命试验情况下, 对k个未知参数、k个应力水平, 研究了定时和定数截尾时, 步进应力加速寿命试验的最优设计问题.

1.1 基本假定

若记s_0为正常应力水平, $s_1 < s_2 < \cdots < s_k$为k个加速应力水平, 则我们在以下几个假定下进行步加试验

假定 1 在应力水平s_i下, 产品寿命服从指数分布$\exp(1/\theta_i)$, $i = 0, 1, \cdots, k$.

假定 2 产品的平均寿命θ_i与应力水平s_i间满足k个未知参数的加速方程

$$\ln\theta_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_{k-1} x_i^{k-1}, \qquad i = 0, 1, \cdots, k,$$

其中x_i为应力水平s_i的已知函数, 一般实际问题中为单调函数.

假定 3 产品的残余寿命仅依赖于当时已累积的失效部分和当时的应力水平, 而与累积方式无关.

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1.2 基本符号

取n个样品放在$s_1 < s_2 < \cdots < s_k$下进行步加试验, 记s_i下的试验持续时间为τ_i, $i = 0, 1, \cdots, k$, 其中$\tau_0 = 0$.

记n_i为在应力水平s_i下产品的失效个数, c_i为应力水平从s_i提高到s_{i+1}时未失效产品被移去的个数, $i = 1, 2, \cdots, k$, 其中c_k为试验结束时总的未失效产品个数, 即$c_k = n - \sum\limits_{i=1}^{k-1}(n_i + c_i) - n_k$. 由此知$\sum\limits_{i=1}^{k} n_i + \sum\limits_{i=1}^{k} c_i = n$. 又记$\pi_i = c_i/n$为第$i$次移去的未失效产品数的比例.

若记N_i为在应力水平s_i下的受试产品个数, 则有$N_1 = n$, 而当$i \geq 2$时, $N_i = n - \sum\limits_{j=1}^{i-1}(n_j + c_j)$.

由假定3知: 前$i-1$个应力水平$s_1, \cdots, s_{i-1}$下的试验持续时间$\tau_1, \cdots, \tau_{i-1}$折算到$s_i$下的总折算时间为

$$a_{i-1} = \theta_i \sum_{j=1}^{i-1} \frac{\tau_j}{\theta_j}, \qquad i = 1, 2, \cdots, k,$$

其中$a_0 = 0$. 又记$\overline{\tau}_i = \sum\limits_{j=1}^{i} \tau_j$, $i = 1, 2, \cdots, k$, $\overline{\tau}_0 = 0$. 由基本假定1、3可知样本在k个应力下的寿命分布密度函数为

$$f(t) = \frac{1}{\theta_i} \exp\Big\{-\frac{t - \overline{\tau}_{i-1} + a_{i-1}}{\theta_i}\Big\}, \qquad \overline{\tau}_{i-1} \leq t < \overline{\tau}_i, \qquad i = 1, 2, \cdots, k. \tag{1.1}$$

1.3 似然函数

设y_{ij}是试验样本在应力水平s_i下的第j个失效时间, 由(1.1)式可得似然函数为

$$\begin{aligned} & L(\theta_1, \theta_2, \cdots, \theta_k) \\ = & \prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!} \prod_{j=1}^{n_i} \frac{1}{\theta_i} \exp\Big\{-\frac{y_{ij} - \overline{\tau}_{i-1} + a_{i-1}}{\theta_i}\Big\} \Big[\exp\Big\{-\frac{\tau_i + a_{i-1}}{\theta_i}\Big\}\Big]^{c_i} \\ = & C \prod_{i=1}^{k} \theta_i{}^{-n_i} \exp\Big\{-\frac{u_i}{\theta_i}\Big\} = C \exp\Big\{-\sum_{i=1}^{k} \frac{u_i}{\theta_i}\Big\} \prod_{i=1}^{k} \theta_i{}^{-n_i}, \end{aligned}$$

其中

$$C = \prod_{j=1}^{k} \frac{N_j!}{(N_j - n_j)!}, \qquad u_i = \sum_{j=1}^{n_i}(y_{ij} - \overline{\tau}_{i-1}) + (N_i - n_i)\tau_i.$$

在定数截尾场合, 上述似然函数中$\tau_i = y_{in_i} - y_{i-1n_{i-1}}$.

§2. 定时逐次截尾步加试验的最优设计

设y_{ij}是试验样本在应力水平s_i下的第j个失效时间, 记$F_i(t)$是均值为θ_i的指数分布的分布函数. 在定时逐次截尾步加试验场合, 有

引理 2.1 $\mathsf{E}(u_i)=\theta_i F_i(\tau_i)\mathsf{E}(N_i)=n\theta_i F_i(\tau_i)\Big[1-\sum\limits_{j=1}^{i-1}(\pi_i/G_i)\Big]G_{i-1}$, 其中

$$G_j=\prod_{i=1}^{j}[1-F_i(\tau_i)]=\prod_{i=1}^{j}\exp\Big\{-\frac{\tau_i}{\theta_i}\Big\},\qquad j=1,2,\cdots,k.$$

证明: 因为对每个$i\,(=1,2,\cdots,k)$, $y_{ij}-\sum\limits_{l=1}^{i-1}\tau_l$, $(j=1,2,\cdots,n_i)$是来自$(0,\tau_i)$上的截头指数分布的样本, 所以

$$\mathsf{E}\Big(y_{ij}-\sum_{l=1}^{i-1}\tau_l\Big)=\theta_i-\frac{\tau_i[1-F_i(\tau_i)]}{F_i(\tau_i)}.$$

又因为给定$n_1,n_2,\cdots,n_{i-1}$, 随机变量n_i服从二项分布$b(N_i,F_i(\tau_i))$, 所以$\mathsf{E}n_i=F_i(\tau_i)\mathsf{E}(N_i)$. 又可用数学归纳法证明

$$\mathsf{E}N_i=n\Big[1-\sum_{j=1}^{i-1}\frac{\pi_j}{G_j}\Big]G_{i-1},$$

所以

$$\begin{aligned}\mathsf{E}u_i&=\mathsf{E}n_i\mathsf{E}\Big(y_{ij}-\sum_{l=1}^{i-1}\tau_l\Big)+(\mathsf{E}N_i-\mathsf{E}n_i)\tau_i=\theta_iF_i(\tau_i)\mathsf{E}N_i\\&=n\theta_iF_i(\tau_i)\Big[1-\sum_{j=1}^{i-1}\frac{\pi_i}{G_i}\Big]G_{i-1}.\end{aligned}$$

□

2.1 Fisher信息矩阵

下面将$\theta_i=\exp\{\beta_0+\beta_1x_i+\cdots+\beta_{k-1}x_i^{k-1}\}$代入似然函数中得

$$\begin{aligned}L(\beta_0,\beta_1\cdots\beta_{k-1})&=\prod_{i=1}^{k}\exp\Big\{-n_i\Big(\beta_0+\sum_{j=1}^{k-1}\beta_jx_i^j\Big)\Big\}\\&\quad\cdot\exp\Big\{-\sum_{i=1}^{k}u_i\exp\Big\{-\Big(\beta_0+\sum_{j=1}^{k-1}\beta_jx_i^j\Big)\Big\}\Big\},\\ \ln L&=-\beta_0\sum_{i=1}^{k}n_i-\beta_1\sum_{i=1}^{k}n_ix_i-\cdots-\beta_{k-1}\sum_{i=1}^{k}n_ix_i^{k-1}\\&\quad-\sum_{i=1}^{k}u_i\exp\Big\{-\Big(\beta_0+\sum_{j=1}^{k-1}\beta_jx_i^j\Big)\Big\},\end{aligned}$$

所以

$$\begin{aligned}\frac{\partial\ln L}{\partial\beta_l}&=-\sum_{i=1}^{k}n_ix_i^l+\sum_{i=1}^{k}x_i^lu_i\exp\{-(\beta_0+\beta_1x_0+\cdots+\beta_{k-1}x_i^{k-1})\},\\ \frac{\partial^2\ln L}{\partial\beta_l\partial\beta_j}&=-\sum_{i=1}^{k}x_i^{l+j}u_i\exp\{-(\beta_0+\beta_1x_0+\cdots+\beta_{k-1}x_i^{k-1})\}\\&=-\sum_{i=1}^{k}\frac{1}{\theta_i}x_i^{l+j}u_i,\qquad l=0,1,2,\cdots,k-1;\ j=0,1,\cdots,k-1.\end{aligned}$$

由引理2.1得

$$\mathsf{E}\Big[-\frac{\partial^2 \ln L}{\partial\beta_l\partial\beta_j}\Big]=n\sum_{i=1}^{k}x_i^{l+j}A_i,$$

其中

$$A_i=\Big[1-\sum_{j=1}^{i-1}\frac{\pi_j}{G_j}\Big]G_{i-1}F_i(\tau_i),\qquad G_j=\prod_{i=1}^{j}[1-F_i(\tau_i)]=\prod_{i=1}^{j}\exp\Big\{-\frac{\tau_i}{\theta_i}\Big\}.$$

令$\beta=(\beta_0,\beta_1,\cdots,\beta_{k-1})'$, $X_0=(1,x_0,\cdots,x_0^{k-1})'$, 则$\beta$的Fisher信息矩阵为

$$I=n\begin{pmatrix}\sum\limits_{i=1}^{k}A_i & \sum\limits_{i=1}^{k}A_ix_i & \cdots & \sum\limits_{i=1}^{k}A_ix_i^{k-1}\\ \sum\limits_{i=1}^{k}A_ix_i & \sum\limits_{i=1}^{k}A_ix_i^2 & \cdots & \sum\limits_{i=1}^{k}A_ix_i^{k}\\ \vdots & \vdots & \ddots & \vdots\\ \sum\limits_{i=1}^{k}A_ix_i^{k-1} & \sum\limits_{i=1}^{k}A_ix_i^{k} & \cdots & \sum\limits_{i=1}^{k}A_ix_i^{2k-2}\end{pmatrix}. \tag{2.1}$$

2.2　最优准则

在逐次截尾试验中, 由于移去部分所占比例$\pi_j=c_j/n$的存在, 所以从以上A_i的表达式中可以看出: 对于某些τ_i来说, A_i取值有可能是负的, 由此会造成I的行列式为负或者产生一个负的方差函数, 这说明了逐次截尾试验带来的复杂性. 因此我们必须限制在区域$c_\tau=\{\tau:A_i>0,i=2,3,\cdots,k\}$中寻找最优的$\tau_i^*$.

由引理2.1可知$A_i>0\Leftrightarrow \mathsf{E}N_i>0$, 又因为$N_i=n-\sum\limits_{j=1}^{i-1}n_j-\sum\limits_{j=1}^{i-1}c_j$, 所以$\mathsf{E}N_i>0$意味着: 每一个试验阶段末的未失效个数必须大于移去部分个数, 这是实际操作的必然要求, 所以c_τ实际是符合要求的.

本文是在以下两个最优准则下寻找最优τ_i^*的:

(1) V-最优: 即寻找最优的τ_i^*, 使得$\mathrm{AsVar}(\ln\widehat{\theta}_0)$达到最小.

(2) D-最优: 即寻找最优的τ_i^*, 使得$|I|$达到最大.

注意到$\beta=(\beta_0,\beta_1,\cdots,\beta_{k-1})$的渐近联合置信域大小与$|I^{-1}(\beta)|^{1/2}$成正比, 所以$|I^{-1}|$越小(即$|I|$越大), 精度越高.

两准则的侧重点不同, 如果想使初始状态下θ_0的估计$\widehat{\theta}_0$精度最高, 则选用V-最优准则, 如果要使加速方程系数估计的总体精度最高则选用D-最优准则.

2.3　V-最优准则下的逐次截尾步加试验最优设计

定理 2.1

$$g(\tau_1,\cdots,\tau_{k-1})=n^2\mathrm{AsVar}(\ln\widehat{\theta}_0)=\frac{\xi_1^2}{A_1}+\frac{\xi_2^2}{A_2}+\cdots+\frac{\xi_k^2}{A_k}.$$

证明: 引用2.1节中的结论和记号, 我们有

$$I=n\begin{pmatrix}A_1 & A_2 & \cdots & A_k\\ A_1x_1 & A_2x_2 & \cdots & A_kx_k\\ \vdots & \vdots & \ddots & \vdots\\ A_1x_1^{k-1} & A_2x_2^{k-1} & \cdots & A_kx_k^{k-1}\end{pmatrix}\times\begin{pmatrix}1 & x_1 & \cdots & x_1^{k-1}\\ 1 & x_2 & \cdots & x_2^{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_k & \cdots & x_k^{k-1}\end{pmatrix}\,\widehat{=}\,nAB.$$

所以

$$\begin{aligned}g(\tau_1,\cdots,\tau_{k-1}) &= n^2\mathrm{AsVar}(\ln\widehat{\theta}_0)=n^2\mathrm{AsVar}(\widehat{\beta}_0+\widehat{\beta}_1x_0+\cdots+\widehat{\beta}_{k-1}x_0^{k-1})\\ &= n^2\mathrm{AsVar}(\widehat{\beta}'X_0)=nX_0I^{-1}X_0=X_0'B^{-1}A^{-1}X_0.\end{aligned}$$

记$A^*=(a_{ij}^*)_{kk}$, $B^*=(b_{ij}^*)_{kk}$分别为A, B的伴随矩阵, 则$A^{-1}=A^*/|A|$, $B^{-1}=B^*/|B|$. 所以$g(\tau_1,\cdots,\tau_{k-1})=X_0'B^*A^*X_0/(|A||B|)$. 又记$X_0'B^*=(b_1,b_2,\cdots,b_k)$, $A^*X_0=(a_1,a_2,\cdots,a_k)'$, 则

$$b_i = \sum_{j=1}^{k}x_0^{j-1}b_{ij}^*=\begin{vmatrix}1 & x_1 & \cdots & x_1^{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_{i-1} & \cdots & x_{i-1}^{k-1}\\ 1 & x_0 & \cdots & x_0^{k-1}\\ 1 & x_{i+1} & \cdots & x_{i+1}^{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_k & \cdots & x_k^{k-1}\end{vmatrix}=\prod_{j=0,j\neq i,l>j}^{k-1}(x_l-x_j),$$

$$\begin{aligned}a_i &= \sum_{j=1}^{k}x_0^{j-1}a_{ji}^*=\begin{vmatrix}A_1 & \cdots & A_{i-1} & 1 & A_{i+1} & \cdots & A_k\\ A_1x_1 & \cdots & A_{i-1}x_{i-1} & x_0 & A_{i+1}x_{i+1} & \cdots & A_kx_k\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ A_1x_1^{k-1} & \cdots & A_{i-1}x_{i-1}^{k-1} & x_0^{k-1} & A_{i+1}x_{i+1}^{k-1} & \cdots & A_kx_k^{k-1}\end{vmatrix}\\ &= \Big(\prod_{j\neq i}A_j\Big)\prod_{j=0,j\neq i,l>j}^{k-1}(x_l-x_j),\end{aligned}$$

且$|A|=\Big(\prod\limits_{j=1}^{k}A_j\Big)\prod\limits_{j=1,l>j}^{k-1}(x_l-x_j)$, $|B|=\prod\limits_{j=1,l>j}^{k-1}(x_l-x_j)$. 所以

$$\begin{aligned}g(\tau_1,\tau_2,\cdots,\tau_{k-1}) &= \frac{1}{|A||B|}\sum_{i=1}^{k}a_ib_i=\frac{\sum\limits_{i=1}^{k}\Big(\prod\limits_{j\neq i}A_j\Big)\Big(\prod\limits_{j=0,j\neq i,l>j}^{k-1}(x_l-x_j)^2\Big)}{\Big(\prod\limits_{j=1}^{k}A_j\Big)\Big(\prod\limits_{j=1,l>j}^{k-1}(x_l-x_j)^2\Big)}\\ &= \sum_{i=1}^{k}\frac{\prod\limits_{j\neq i}(x_j-x_0)^2}{A_i\prod\limits_{j\neq i}(x_j-x_i)^2}=\frac{\xi_1^2}{A_1}+\frac{\xi_2^2}{A_2}+\cdots+\frac{\xi_k^2}{A_k}.\end{aligned}$$

□

定理 2.2　定时逐次截尾试验时$g(\tau_1,\tau_2,\cdots,\tau_{k-1})$为严格凸函数.

证明:　记$H(A_i)$为A_i的Hession矩阵. 因为凸函数与严格凸函数的正线性组合为严格凸函数, 所以由定理2.1, 为了证明$g(\tau_1,\cdots,\tau_{k-1})$是严格凸函数, 只需证当$i\le k-1$时, $H(1/A_i)$为半正定, 且$H(1/A_k)$正定即可. 又因为

$$\frac{\partial(1/A_i)}{\partial\tau_l}=-\frac{1}{A_i^2}\frac{\partial A_i}{\partial\tau_l};\qquad \frac{\partial^2(1/A_i)}{\partial\tau_l\partial\tau_j}=\frac{2}{A_i^3}\frac{\partial A_i}{\partial\tau_l}\frac{\partial A_i}{\partial\tau_j}-\frac{1}{A_i^2}\frac{\partial^2 A_i}{\partial\tau_l\partial\tau_j}.$$

由此可知

$$A_i^3H(1/A_i)=-A_iH(A_i)+2\nabla A_i\nabla A_i',\qquad i=1,2,\cdots,k,\tag{2.2}$$

其中$\nabla A_i=(\partial A_i/\partial\tau_1,\cdots,\partial A_i/\partial\tau_{k-1})'$为$A_i$的梯度向量.

因为在(2.2)式中, $A_i>0$, 且$\nabla A_i\nabla A_i'$为半正定, 所以我们只须证: $H(A_k)$是负定的; 且当$i\le k-1$时, $H(A_i)$是半负定的. 为此记$B_i=G_{i-1}F_i(\tau_i)$, $D_i=1-\sum\limits_{j=1}^{i}(\pi_j/G_j)$, 显然有$D_1>D_2>\cdots>D_k$. 又因为

$$A_k=\Big[1-\sum_{j=1}^{k-1}\frac{\pi_j}{G_j}\Big]G_{k-1}F_k(\tau_k)=G_{k-1}F_k(\tau_k)-\sum_{j=1}^{k-1}\frac{G_{k-1}}{G_j}\pi_jF_k(\tau_k),$$

所以求出$\partial^2A_k/(\partial\tau_i\partial\tau_j)$后可得

$$H(A_k)=\Big(\frac{\partial^2A_k}{\partial\tau_i\partial\tau_j}\Big)=\begin{pmatrix}\frac{B_k}{\theta_1^2}&\frac{B_k}{\theta_1\theta_2}&\frac{B_k}{\theta_1\theta_3}&\cdots&\frac{B_k}{\theta_1\theta_{k-1}}\\ \frac{B_k}{\theta_2\theta_1}&\frac{B_kD_1}{\theta_2^2}&\frac{B_kD_1}{\theta_2\theta_3}&\cdots&\frac{B_kD_1}{\theta_2\theta_{k-1}}\\ \frac{B_k}{\theta_3\theta_1}&\frac{B_kD_1}{\theta_3\theta_2}&\frac{B_kD_2}{\theta_3^2}&\cdots&\frac{B_kD_2}{\theta_3\theta_{k-1}}\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ \frac{B_k}{\theta_{k-1}\theta_1}&\frac{B_kD_1}{\theta_{k-1}\theta_2}&\frac{B_kD_2}{\theta_{k-1}\theta_3}&\cdots&\frac{B_kD_{k-2}}{\theta_{k-1}^2}\end{pmatrix}.$$

记

$$M=\begin{pmatrix}\frac{1}{\theta_1}&0&\cdots&0\\ 0&\frac{1}{\theta_2}&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&\frac{1}{\theta_{k-1}}\end{pmatrix},\qquad T=\begin{pmatrix}1&1&1&1&\cdots&1&1\\ 1&D_1&D_1&D_1&\cdots&D_1&D_1\\ 1&D_1&D_2&D_2&\cdots&D_2&D_2\\ 1&D_1&D_2&D_3&\cdots&D_3&D_3\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 1&D_1&D_2&D_3&\cdots&D_{k-3}&D_{k-2}\end{pmatrix},$$

则$H(A_k)=B_kM'TM$. 由于$B_k>0$, M正定, 所以下证T为负定阵. 记T_i为T的i阶顺序主子式, 则$T_1=1$; $T_2=D_1-1$; $T_i=(D_1-1)(D_2-D_1)\cdots(D_{i-1}-D_{i-2})$, $i=3,\cdots,k$. 由$1>D_1>D_2>\cdots>D_k$得: T的奇数阶顺序主子式为正、偶数阶顺序主子式为负, 所以T是负定的, 由此得$H(A_k)$是负定的.

注意, 当$k\geq 3$时, 与上面类似地可得$H(A_{k-1})$是负定的, 而且可计算得

$$H(A_{k-2})=M'\begin{pmatrix} B_{k-2} & B_{k-2} & \cdots & B_{k-2} & -G_{k-2} & 0\\ B_{k-2} & B_{k-2}D_1 & \cdots & B_{k-2}D_1 & -G_{k-2}D_1 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ B_{k-2} & B_{k-2}D_1 & \cdots & B_{k-2}D_{k-4} & -G_{k-2}D_{k-3} & 0\end{pmatrix}M\triangleq M'H_{k-2}M.$$

与上面的分析方法类似, 且注意到H_{k-2}中出现了为零的列, 可推知$H(A_{k-2})$为半负定. 进一步, 当$i\leq k-2$时, 由$H(A_i)=M'H_iM$, 且H_i中有为零的列, 所以$H(A_i)$为半负定. 至此定理得证. □

有了以上定理2.2, 使得$\mathrm{AsVar}(\ln\widehat{\theta}_0)$达到最小的最优持续时间$\tau_i^*$可从$\partial g/\partial\tau_i=0$中唯一解得. 下面我们给出$\tau_i^*$所满足的关系式.

定理 2.3 在定时逐次截尾试验时, 最优持续时间τ_i^*满足如下关系式

$$\frac{\xi_i^2}{A_i^2}G_i=\sum_{j=i+1}^{k}\frac{\xi_j^2}{A_j^2}B_j,\qquad i=1,2,\cdots,k-1.$$

证明: 因为$g(\tau_1,\tau_2,\cdots,\tau_{k-1})=(\xi_1^2,\xi_2^2,\cdots,\xi_k^2)(1/A_1,1/A_2,\cdots,1/A_k)'$, 所以有

$$\frac{\partial g}{\partial\tau_i}=(\xi_1^2,\xi_2^2,\cdots,\xi_k^2)\begin{pmatrix} -\dfrac{\partial A_1}{\partial\tau_i} & 0 & \cdots & 0\\ 0 & -\dfrac{\partial A_2}{\partial\tau_i} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -\dfrac{\partial A_k}{\partial\tau_i}\end{pmatrix}\begin{pmatrix}\dfrac{1}{A_1^2}\\ \dfrac{1}{A_2^2}\\ \vdots\\ \dfrac{1}{A_k^2}\end{pmatrix},\qquad i\leq k-1.$$

经计算可得

$$\frac{\partial A_j}{\partial\tau_i}=\begin{cases}0, & j<i;\\ \dfrac{1}{\theta_i}G_jD_{j-1}, & j=i;\\ -\dfrac{1}{\theta_i}B_jD_{i-1}, & j>i,\end{cases}\qquad i=1,2,\cdots,k-1;\ j=1,2,\cdots,k. \tag{2.3}$$

因此由$\partial g/\partial \tau_{k-1}=0$得

$$\begin{pmatrix}\xi_1^2 & \xi_2^2 & \cdots & \xi_k^2\end{pmatrix}\begin{pmatrix}0 & \cdots & 0 & 0 & 0\\ 0 & \cdots & 0 & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & 0 & 0\\ 0 & \cdots & 0 & -\dfrac{1}{\theta_{k-1}}G_{k-1}D_{k-2} & 0\\ 0 & \cdots & 0 & 0 & \dfrac{1}{\theta_k}B_kD_{k-2}\end{pmatrix}\begin{pmatrix}\dfrac{1}{A_1^2}\\ \dfrac{1}{A_2^2}\\ \vdots\\ \dfrac{1}{A_k^2}\end{pmatrix}=0.$$

整理可得

$$\frac{\xi_k^2}{A_k^2}B_k=\frac{\xi_{k-1}^2}{A_{k-1}^2}G_{k-1}.$$

同理由$\partial g/\partial \tau_{k-2}=0$推知

$$\frac{\xi_k^2}{A_k^2}B_k=\frac{\xi_{k-2}^2}{A_{k-2}^2}G_{k-2}-\frac{\xi_{k-1}^2}{A_{k-1}^2}B_{k-1};$$

由$\partial g/\partial \tau_i=0$推知

$$\frac{\xi_k^2}{A_k^2}B_k=\frac{\xi_i^2}{A_i^2}G_i-\sum_{j=i+1}^{k-1}\frac{\xi_j^2}{A_j^2}B_j,\qquad j=1,2,\cdots,k-3.$$

再进行整理, 即可得结论. $\square$

2.4 D-最优准则下的定时逐次截尾步加试验最优设计

根据D-最优定义知, 我们的目的就是找到最优的τ_i^*使$|I|$最大.

定理 2.4 $h(\tau_1,\cdots,\tau_{k-1})\hat{=}|I|=n^k|A||B|=n^k\prod\limits_{j=1,l>j}^{k-1}(x_l-x_j)^2\prod\limits_{j=1}^{k}A_j.$

证明: 由定理2.1证明过程可知

$$I=nAB,\qquad |A|=\left(\prod_{j=1}^{k}A_j\right)\prod_{j=1,l>j}^{k-1}(x_l-x_j),\qquad |B|=\prod_{j=1,l>j}^{k-1}(x_l-x_j).$$

所以

$$|I|=n^k|A||B|=n^k\prod_{j=1,l>j}^{k-1}(x_l-x_j)^2\prod_{j=1}^{k}A_j.$$

证毕. $\square$

由以上定理可知, 要使$|I|$最大, 等价于使$A_1A_2\cdots A_k$最大. 因为$A_i>0$, 所以等价于使$1/(A_1A_2\cdots A_k)$最小, 即等价于求

$$h(\tau_1,\tau_2,\cdots,\tau_{k-1})\hat{=}\ln\frac{1}{A_1A_2\cdots A_k}=-\ln A_1-\ln A2-\cdots-\ln A_k$$

的极小点. 以下定理说明: 在D-最优准则下, 可从$\partial h/\partial\tau_i=0$中唯一解得最优的$\tau_i^*$.

定理 2.5 定时逐次截尾试验时, $h(\tau_1,\tau_2,\cdots,\tau_{k-1})$为严格凸函数.

证明: 因为

$$\frac{\partial(-\ln A_i)}{\partial\tau_j}=-\frac{1}{A_i}\frac{\partial A_i}{\partial\tau_j},\qquad \frac{\partial^2(-\ln A_i)}{\partial\tau_j\partial\tau_l}=\Big(\frac{1}{A_i}\Big)^2\frac{\partial A_i}{\partial\tau_j}\frac{\partial A_i}{\partial\tau_l}-\frac{1}{A_i}\frac{\partial^2 A_i}{\partial\tau_j\partial\tau_l},$$

所以

$$A_i^2H(-\ln A_i)=-A_iH(A_i)+\nabla A_i\nabla' A_i,\qquad i=1,2,\cdots,k.$$

仿定理2.2证明过程可知: $H(A_k)$和$H(A_{k-1})$负定; $i\le k-2$时, $H(A_i)$半负定, 由此即可证得$h(\tau_1,\tau_2,\cdots,\tau_{k-1})$为严格凸函数. □

定理 2.6 在D-最优准则下, 定时逐次截尾试验的最优持续时间τ_i^*满足如下关系式

$$\frac{G_i}{A_i}=\sum_{j=i+1}^{k}\frac{B_j}{A_j},\qquad i=1,2,\cdots,k-1.$$

证明: 将(2.3)式代入

$$\frac{\partial h}{\partial\tau_i}=-\frac{1}{A_1}\frac{\partial A_1}{\partial\tau_i}-\frac{1}{A_2}\frac{\partial A_2}{\partial\tau_i}-\cdots-\frac{1}{A_k}\frac{\partial A_k}{\partial\tau_i}=0$$

中, 可得

$$-\frac{1}{A_i}\frac{1}{\theta_i}G_iD_{i-1}+\frac{1}{A_{i+1}}\frac{1}{\theta_i}B_{i+1}D_{i-1}+\cdots+\frac{1}{A_k}\frac{1}{\theta_i}B_kD_{i-1}=0.$$

由此即可得结论. □

§3. 定数逐次截尾步加试验的最优设计

记试验得到的失效时间为$y_{11},y_{12},\cdots,y_{1n_1},y_{21},y_{22},\cdots,y_{2n_2},\cdots,y_{k1},y_{k2},\cdots,y_{kn_k}$, 则它们的联合密度函数为:

$$f(y_{11},y_{12},\cdots,y_{kn_k})=C\exp\Big\{-\sum_{i=1}^{k}\frac{u_i}{\theta_i}\Big\}\prod_{i=1}^{k}\theta_i^{-n_i},$$

其中

$$C=\prod_{j=1}^{k}\frac{N_j!}{(N_j-n_j)!},\qquad u_i=\sum_{j=1}^{n_i}(y_{ij}-y_{i-1n_{i-1}})+(N_i-n_i)(y_{in_i}-y_{i-1n_{i-1}}).$$

令

$$\begin{cases}w_{i1}=N_i(y_{i1}-y_{i-1n_{i-1}}),\\ w_{i2}=(N_i-1)[(y_{i2}-y_{i-1n_{i-1}})-(y_{i1}-y_{i-1n_{i-1}})],\\ \quad\vdots\\ w_{in_i}=(N_i-n_i+1)[(y_{in_i}-y_{i-1n_{i-1}})-(y_{in_{i-1}}-y_{i-1n_{i-1}})],\end{cases}$$

则$u_i=\sum\limits_{j=1}^{n_i} w_{ij}$, 且此变换的Jacobi行列式为

$$|J_i|=\frac{1}{N_i(N_i-1)\cdots(N_i-n_i+1)}=\frac{N_i-n_i}{N_i}.$$

所以

$$\begin{aligned}
f(w_{11},w_{12},\cdots,w_{kn_k}) &= \Big(\prod_{j=1}^{k}\frac{N_j}{N_j-n_j}\Big)\exp\Big\{-\sum_{i=1}^{k}\Big(\sum_{j=1}^{n_i}w_{ij}\Big/\theta_i\Big)\Big\}\prod_{i=1}^{k}\theta_i^{-n_i}\prod_{j=1}^{k}|J_j| \\
&= \prod_{i=1}^{k}\theta_i^{-n_i}\exp\Big\{-\sum_{i=1}^{k}\Big(\sum_{j=1}^{n_i}w_{ij}\Big/\theta_i\Big)\Big\} \\
&= \prod_{j=1}^{n_1}\frac{1}{\theta_1}\exp\Big\{-\frac{w_{1j}}{\theta_1}\Big\}\cdots\prod_{j=1}^{n_k}\frac{1}{\theta_k}\exp\Big\{-\frac{w_{kj}}{\theta_k}\Big\}.
\end{aligned}$$

这说明w_{ij} $(i=1,2,\cdots,k,\ j=1,2,\cdots,n_i)$是独立的、服从参数为$\theta_i$的指数分布. 由此得$u_i=\sum\limits_{j=1}^{n_i}w_{ij}\sim\Gamma(n_i,1/\theta_i)$, 所以$\mathsf{E}u_i=n_i\theta_i$. 将$\theta_i=\exp\{\beta_0+\beta_1x_i+\cdots+\beta_{k-1}x_i^{k-1}\}$代入似然函数可求得

$$\frac{\partial^2\ln L}{\partial\beta_l\partial\beta_j}=-\sum_{i=1}^{k}\frac{1}{\theta_i}x_i^{l+j}u_i,\qquad l,j=0,1,\cdots,k-1.$$

所以

$$\mathsf{E}\Big[-\frac{\partial^2\ln L}{\partial\beta_l\partial\beta_j}\Big]=\sum_{i=1}^{k}n_ix_i^{l+j}=n\sum_{i=1}^{k}m_ix_i^{l+j},$$

其中$m_i=n_i/n$, $(i=1,2,\cdots,k)$. 记$m_c=(1/n)\cdot\sum\limits_{j=1}^{k}c_j$, 则$\sum\limits_{i=1}^{k}m_i+m_c=1$. 由此得$\beta$的Fisher信息矩阵为

$$I=n\begin{pmatrix}
\sum\limits_{i=1}^{k}m_i & \sum\limits_{i=1}^{k}m_ix_i & \cdots & \sum\limits_{i=1}^{k}m_ix_i^{k-1} \\
\sum\limits_{i=1}^{k}m_ix_i & \sum\limits_{i=1}^{k}m_ix_i^2 & \cdots & \sum\limits_{i=1}^{k}m_ix_i^{k} \\
\vdots & \vdots & \ddots & \vdots \\
\sum\limits_{i=1}^{k}m_ix_i^{k-1} & \sum\limits_{i=1}^{k}m_ix_i^{k} & \cdots & \sum\limits_{i=1}^{k}m_ix_i^{2k-2}
\end{pmatrix}.$$

记$X_0=(1,x_0,\cdots,x_0^{k-1})'$, 则

$$\begin{aligned}
g(m_1,m_2,\cdots,m_{k-1}) &= n^2\mathrm{AsVar}(\ln\widehat{\theta}_0)=n^2\mathrm{AsVar}(\widehat{\beta}_0+\widehat{\beta}_1x_0+\cdots+\widehat{\beta}_{k-1}x_0^{k-1}) \\
&= n^2\mathrm{AsVar}(\widehat{\beta}'X_0)=nX_0'I^{-1}X_0.
\end{aligned}$$

可见$\mathrm{AsVar}(\ln\widehat{\theta}_0)$与$c_i$无关, 所以参考文献[3]中结论成立, 即

定理 3.1 在定数逐次截尾步加试验时，使$\mathrm{AsVar}(\ln\widehat{\theta}_0)$达到最小的$m_i^*$为

$$m_i^* = (1-m_c)\frac{|\xi_i|}{\sum\limits_{j=1}^{k}|\xi_j|}.$$

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Optimal Design of Step-Stress Accelerated Life Testing with Progressive Censoring

Li Xue Cheng Yiming

(*School of Finance and Statistics, East China Normal University, Shanghai, 200241*)

This paper investigates optimal designs of accelerated life tests (ALT) for exponentially distributed lifetimes under progressive censoring. We search for the optimal ALT plans which minimize the asymptotic variance of the estimated mean or maximize the determinant of Fisher matrix.

Keywords: Exponential distribution, progressive censoring, Fisher information matrix, optimal design.

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