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Some Results on Tail Probabilities with Applications of Generalized Comparison Method

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Abstract

We extend the comparison method and present a new method to derive sharp closed-form semiparametric bounds on small value probability $\mathsf{P}(X \leq t)$, where $X \in [0, M]$ is a random variable with $\mathsf{E}X = m_1$ and $\mathsf{E}X^2 = m_2$ fixed. The proofs of our results are elementary.

Keywords: Comparison method, moment problem, semiparametric bounds, probability estimation.

AMS Subject Classification: 60E99.

§1. Introduction

Bounds of small value probability arise naturally in many areas of probability, statistics, economics, and operations research. There is a long history of studying them, back to works of Chebyshev. Although Markoff[1] (see also [2], [3]) obtained $\sup \int_{-1}^{x} f(t)\mathrm{d}\psi(t)$ and $\inf \int_{-1}^{x} f(t)\mathrm{d}\psi(t)$ given moments $\int_{-1}^{1} t^n \mathrm{d}\psi(t)$ and stated that the bounds are sharp at discrete distributions, his results have no expressions and are hard to use. Since then, various applicable approaches have been addressed. Among these attempt, Karlin and Studden[4] and Isii[5] independently propose duality theory to deal with moment problems. The later then generalizes the duality results to the multivariate case. Bertsimas and Popescu[6] present a semidefinite optimization program to transform a moment problems into semidefinite metrics and obtained $\sup \mathsf{P}(0 \leq X \leq t)$ for $X \geq 0$. In 2007, Vandenberge[7] obtained $\inf \mathsf{P}(X \in C)$ for $C \subseteq R^n$. Other results related moment problems, see [8], [9], [10] and the references therein.

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In this paper, we are interested in bounding the small value probability $\mathsf{P}(X \le t)$, where $X \in [0, M]$ is a random variable with $\mathsf{E}X = m_1$ and $\mathsf{E}X^2 = m_2$ fixed . In this attempt, we first extend comparison method, define a new random variable X_0 related to X , find the relationship between $\mathsf{E}X^k$ and $\mathsf{E}X_0^k$, and then give elementary proofs of the closed-form semiparametric bounds on $\mathsf{P}(X \le t)$. Our results strengthen the following two known results (see e.g. [6], [11], [12]): $\mathsf{P}(X = 0) \le 1 - m_1^2/m_2$ and $\mathsf{P}(X \le t) \le (m_2 - m_1^2)/(m_2 - 2m_1 t + t^2)$.

The remaining of this paper is organized as follows. As a warm up, we present the known comparison method and illustrate an example in Section 2. The statements of our main results, Theorem 3.1, Theorem 3.2, and Theorem 3.3 are given in Section 3 together with their proofs.

§2. Preliminaries

We begin with the known comparison method and then illustrate an example as application.

Lemma 2.1 (see [10], [11]) Let $\mathsf{P}(X = 0) = q_0 < 1$ for any random variable $X \in [0, M]$ with $\mathsf{E}X^k = m_k$ fixed and define a nonnegative random variable X_0 with $\mathsf{P}(X_0 \le x) = (1 - q_0)^{-1}\mathsf{P}(0 < X \le x)$ for $x > 0$. Then the following holds.

(i) $X \overset{D}{=} X_0 \cdot 1_{U > q_0}$ in distribution, where X_0 and U are independent, U is uniformly distributed over $(0, 1)$.

(ii) $\mathsf{E}X^k = \mathsf{E}X_0^k(1 - q_0)$.

Proof (i) if $x > 0$, then

$$\mathsf{P}(X_0 \cdot 1_{U>q_0} > x) = \mathsf{P}(X_0 > x, 1_{U>q_0} = 1) = \mathsf{P}(X_0 > x)\mathsf{P}(1_{U>q_0} = 1) = \mathsf{P}(X > x).$$

(ii) $\mathsf{E}X_0^k = \displaystyle\int_0^\infty kx^{k-1}\mathsf{P}(X_0 > x)dx = \mathsf{E}X^k/(1 - q_0).$ □

Example 1 If X is a non-negative integer valued random variable with $\mathsf{E}X^k = m_k$ $(k = 1, 2)$ given and $(n - 1)m_1 \le m_2 \le n \cdot m_1 \le n(n - 1)$ for positive integer n, then

$$\mathsf{P}(X = 0) \le 1 - \frac{(2n - 1)m_1 - m_2}{n(n - 1)} \le 1 - \frac{m_1^2}{m_2}.$$

Proof Let $\mathsf{P}(X = 0) = q_0 < 1$ and denote X_0 as in Lemma 2.1, then $\mathsf{P}(X_0 = 0) = 0$. Define a three points distribution V as follows

$$V = \begin{cases} 0 & \text{W.P. } 1 - \mathsf{P}_1 - \mathsf{P}_2, \\ n - 1 & \text{W.P. } \mathsf{P}_1, \\ n & \text{W.P. } \mathsf{P}_2, \end{cases}$$

where $\mathsf{P}_1 = (nm_1 - m_2)/(n-1)$, $\mathsf{P}_2 = (m_1 + m_2 - nm_1)/n$, $\mathsf{P}_1 + \mathsf{P}_2 = ((2n-1)m_1 - m_2)/n(n-1)$.

Then it is easy to check that $\mathsf{E}X^k = \mathsf{E}V^k$, $k = 1, 2$. We next define

$$V_0 = \begin{cases} 0 & \text{W.P. } 1 - (1-q_0)^{-1}(\mathsf{P}_1 + \mathsf{P}_2), \\ n-1 & \text{W.P. } (1-q_0)^{-1}\mathsf{P}_1, \\ n & \text{W.P. } (1-q_0)^{-1}\mathsf{P}_2, \end{cases}$$

or equivalently in distribution,

$$V \stackrel{d}{=} V_0 \cdot 1_{(U>q_0)}, \quad \text{where } V_0 \text{ and } U \text{ are independent, } U \sim U(0,1).$$

Thus by Lemma 2.1 (i),

$$\begin{aligned} \mathsf{P}(X=0) = q_0 &= \mathsf{P}(1_{(U>q_0)} = 0) + \mathsf{P}(1_{(U>q_0)} = 1)\mathsf{P}(X_0 = 0) \\ &\le \mathsf{P}(1_{(U>q_0)} = 0) + \mathsf{P}(1_{(U>q_0)} = 1)\mathsf{P}(V_0 = 0) \\ &= \mathsf{P}(V=0) = 1 - (\mathsf{P}_1 + \mathsf{P}_2) = 1 - \frac{(2n-1)m_1 - m_2}{n(n-1)}. \end{aligned}$$

This finishes the proof. $\square$

§3. Main Results

In this section, we extend comparison method, obtain the sharp closed-form semi-parametric bounds on $\mathsf{P}(X \le t)$ and give elementary proofs of the results.

Theorem 3.1 Let $\mathsf{P}(X \le t) = q < 1$ for any random variable $X \in [0, M]$ with $\mathsf{E}X^k = m_k$ fixed and define a random variable $t < X_0 \le M$ with $\mathsf{P}(X_0 \le x) = (1-q)^{-1}\mathsf{P}(t < X \le x)$ for $x > t$ and $\mathsf{P}(X_0 \le x) = 0$ otherwise. Then the following holds.

(i) $(X-t)_+ \stackrel{D}{=} (X_0 - t) \cdot 1_{U>q}$ in distribution, where X_0 and U are independent, U is uniformly distributed over $(0,1)$. Notation x_+ means $\max(0,x)$.

(ii) Denote $h_k = (1-q)^{-1}\displaystyle\int_0^t kx^{k-1}\mathsf{P}(X \le x)\mathrm{d}x$, then

$$\mathsf{E}X^k = (\mathsf{E}X_0^k - h_k)(1-q) + qt^k. \tag{3.1}$$

Proof (i) It is easy to check that

$$\mathsf{P}((X-t)_+ \le x) = \begin{cases} 0, & \text{if } x < 0, \\ q, & \text{if } x = 0, \\ q + \mathsf{P}(t < X \le x + t), & \text{if } x > 0. \end{cases}$$

By independence of X_0 and U, $U \sim U(0,1)$, we obtain

$$\mathsf{P}((X_0 - t)\cdot 1_{(U>q)} \le x) = \begin{cases} 0, & \text{if } x < 0, \\ q, & \text{if } x = 0, \\ q + \mathsf{P}(t < X \le x + t), & \text{if } x > 0. \end{cases}$$

Hence, $(X-t)_+ \overset{D}{=} (X_0 - t)\cdot 1_{(U>q)}$.

(ii) Notice that the definition of X_0 implies

$$\mathsf{P}(X_0 > x) = 1 - \mathsf{P}(X_0 \le x) = \begin{cases} (1-q)^{-1}\mathsf{P}(X > x), & \text{if } x > t, \\ 1, & \text{if } 0 \le x \le t, \end{cases}$$

we then have

$$\begin{aligned} \mathsf{E}X_0^k &= \int_0^\infty kx^{k-1}\mathsf{P}(X_0 > x)\mathrm{d}x \\ &= \int_0^t kx^{k-1}\mathrm{d}x + \int_t^\infty kx^{k-1}(1-q)^{-1}\mathsf{P}(X > x)\mathrm{d}x \\ &= t^k + (1-q)^{-1}\cdot\Big(\int_0^\infty kx^{k-1}\mathsf{P}(X > x)\mathrm{d}x - \int_0^t kx^{k-1}\mathsf{P}(X > x)\mathrm{d}x\Big) \\ &= t^k + (1-q)^{-1}\Big(\mathsf{E}X^k - \int_0^t kx^{k-1}\mathsf{P}(X > x)\mathrm{d}x\Big) \\ &= (1-q)^{-1}(\mathsf{E}X^k - qt^k) + (1-q)^{-1}\int_0^t kx^{k-1}\cdot\mathsf{P}(X \le x)\mathrm{d}x \\ &= (1-q)^{-1}(\mathsf{E}X^k - qt^k) + h_k. \end{aligned}$$

This finishes the proof of the theorem. □

Theorem 3.2 Given any random variable $X \in [0, M]$ with $\mathsf{E}X = m_1$ and fixed $t > 0$, then

(i) If $0 \le t \le m_1$, then $\mathsf{P}(X \le t) \le (M - m_1)/(M - t)$, and the equality holds if and only if X takes only two values t and M, with $\mathsf{P}(X = t) = (M - m_1)/(M - t)$.

(ii) If $m_1 \le t \le M$, then trivially $\mathsf{P}(X \le t) \le 1$, and the equality holds if and only if X takes only two values 0 and t, with $\mathsf{P}(X = t) = m_1/t$.

Proof Since part (ii) is trivial, we only show part (i). Denote $\mathsf{P}(X \le t) = q < 1$, then by (3.1)

$$\begin{aligned} (1-q)^{-1}\frac{m_1 - t}{M - t} &= \frac{(1-q)^{-1}}{M-t}\cdot((\mathsf{E}X_0 - h_1)(1-q) + qt - t) \\ &= \frac{\mathsf{E}X_0 - t - h_1}{M - t} = \frac{\mathsf{E}X_0 - t}{M - t} - \frac{h_1}{M - t} \le 1 \end{aligned}$$

since $\mathsf{E}X_0 \le M$ and $h_1 \ge 0$.

Hence, $q = \mathsf{P}(X \le t) \le 1 - (m_1 - t)/(M - t) = (M - m_1)/(M - t)$. This finishes the proof of (i). $\square$

Theorem 3.3 Given any random variable $X \in [0, M]$ with $\mathsf{E}X = m_1$, $\mathsf{E}X^2 = m_2$ and $t > 0$ fixed, then

(i) If $0 \le t \le (Mm_1 - m_2)/(M - m_1)$, then $\mathsf{P}(X \le t) \le (m_2 - m_1^2)/(m_2 - 2m_1t + t^2)$ and the equality holds if and only if X takes only two values, t and $(m_2 - m_1t)/(m_1 - t)$, with $\mathsf{P}(X = t) = (m_2 - m_1^2)/(m_2 - 2m_1t + t^2)$.

(ii) If $(Mm_1 - m_2)/(M - m_1) \le t \le m_2/m_1$, then

$$\mathsf{P}(X \le t) \le 1 - \frac{m_2 - m_1t}{M(M - t)}$$

and the equality holds if and only if X takes only three values, 0, t and M, with $\mathsf{P}(X = 0) = ((M - m_1)t - (Mm_1 - m_2))/Mt$, $\mathsf{P}(X = t) = (Mm_1 - m_2)/(M - t)t$.

(iii) If $m_2/m_1 \le t \le M$, then $\mathsf{P}(X \le t) = 1$ for two point distribution at 0 and m_2/m_1, with $\mathsf{P}(X = m_2/m_1) = m_1^2/m_2$.

Proof Since part (iii) is trivial, we only show part (i) and (ii). There are at least two ways to prove part (i). One is based on the extended comparison method given below in the proof of part (ii), so we omit it here. The other is based on the so called shift Chebyshev's inequality, a somewhat special technique. Namely, for any $\lambda > t$, we have by Chebyshev's inequality

$$\mathsf{P}(X \le t) = \mathsf{P}(\lambda - X \ge \lambda - t) \le \frac{\mathsf{E}(\lambda - X)^2}{(\lambda - t)^2} = \frac{\lambda^2 - 2\lambda m_1 + m_2}{(\lambda - t)^2}.$$

Hence

$$\mathsf{P}(X \le t) \le \inf_{\lambda > t} \frac{\lambda^2 - 2\lambda m_1 + m_2}{(\lambda - t)^2} = \frac{m_2 - m_1^2}{m_2 - 2m_1t + t^2},$$

where the infimum is achieved at

$$\lambda = \lambda_0 = \frac{m_2 - m_1t}{m_1 - t} > t \qquad \text{for } t < \frac{Mm_1 - m_2}{M - m_1}$$

by simple calculus.

To prove (ii), denote $\mathsf{P}(X \le t) = q < 1$, then by (3.1)

$$\begin{aligned}
(1 - q)^{-1}\frac{m_2 - m_1t}{M(M - t)} &= \frac{(1-q)^{-1}}{M(M-t)}((1 - q)(\mathsf{E}X_0^2 - h_2) + qt^2 - (1 - q)(\mathsf{E}X_0 - h_1)t - qt^2) \\
&= \frac{\mathsf{E}X_0^2 - \mathsf{E}X_0t - h_2 + h_1t}{M(M - t)} \\
&= \frac{\mathsf{E}X_0(X_0 - t)}{M(M - t)} - \frac{h_2 - h_1t}{M(M - t)} \le 1 - \frac{h_2 - h_1t}{M(M - t)}
\end{aligned}$$

since $t < X_0 \leq M$.

Also

$$\begin{aligned} h_2 - h_1 t &= (1-q)^{-1}\Big(\int_0^t 2x\mathsf{P}(X \leq x)\mathrm{d}x - \int_0^t t\mathsf{P}(X \leq x)\mathrm{d}x\Big) \\ &= (1-q)^{-1}\Big(-\int_{t/2}^t (2y-t)\mathsf{P}(X \leq t-y)\mathrm{d}y + \int_{t/2}^t (2x-t)\mathsf{P}(X \leq x)\mathrm{d}x\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by putting } \ y = t - x) \\ &= (1-q)^{-1}\int_{t/2}^t (2x-t)(\mathsf{P}(X \leq x) - \mathsf{P}(X \leq t-x))\mathrm{d}x. \end{aligned}$$

Notice that as $t/2 < x < t$, $x > t - x$, one can obtain $\mathsf{P}(X \leq x) \geq \mathsf{P}(X \leq t - x)$, thus $h_2 - h_1 t \geq 0$.

Hence

$$(1-q)^{-1} \cdot \frac{m_2 - m_1 t}{M(M-t)} \leq 1,$$

that is

$$q = \mathsf{P}(X \leq t) \leq 1 - \frac{m_2 - m_1 t}{M(M-t)}.$$

This finishes the proof of the theorem. □

As applications, we give the following three remarks to illustrate applications of our results in European call option and inventory management problems (the interested reader, see, e.g. [6], [13]).

Remark 1 The lower bound of small value probability $\mathsf{P}(X \leq t)$ can be easily obtained by applying

$$\inf \mathsf{P}(X \leq t) = \inf \mathsf{P}(M - X \geq M - t) = 1 - \sup \mathsf{P}(M - X \leq M - t),$$

so we omit the discussion about lower bound in the present paper.

Remark 2 The sharp upper and lower bounds on European call option $\max(0, X - K)$ with K fixed, is immediately obtained by Theorem 3.3 and through the following

$$\mathsf{P}(\max(0, X-K) \leq t) = \mathsf{P}(X \leq K + t), \qquad \text{for } \ t \geq 0.$$

Remark 3 Let x be the inventory of a single product, c be the product's unit cost, and r the product's unit price. The upper and lower bounds on profit over all demands (represented by the random variable X) with given $\mathsf{E}X = m_1$ and $\mathsf{E}X^2 = m_2$, can be obtained by Theorem 3.3 and through the following

$$\mathsf{P}(r\min\{x, X\} - cx) \leq t) = \mathsf{P}(X \leq (t + cx)/r), \qquad \text{for } \ t \leq (r-c)x.$$

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应用推广的比较法得到的概率估计的几个结果

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我们推广了比较法, 提出一个新方法得到概率$\mathsf{P}(X \le t)$的可达的半参数界, 这里$X \in [0, M]$有给定的$\mathsf{E}X = m_1$和$\mathsf{E}X^2 = m_2$. 我们的证明是初等的.

关键词: 比较法, 矩问题, 半参数界, 概率估计.

学科分类号: O212.5.