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Some New Results for Weakly Dependent Random Variable Sequences *

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Abstract

Let $\{X_n, n \geq 1\}$ be a $\widetilde{\rho}$-mixing random variable sequence. By using the truncation method of random variables and three series theorem of $\widetilde{\rho}$-mixing sequence, the convergence properties of $\widetilde{\rho}$-mixing sequence are discussed, and a class of strong limit theorems for $\widetilde{\rho}$-mixing sequence are obtained, which generalize the corresponding results of independent sequence. At last, the strong stability for weighted sums of $\widetilde{\rho}$-mixing sequence is studied.

Keywords: Strong limit theorems, $\widetilde{\rho}$-mixing sequence, convergence property, strong stability, weighted sums.

AMS Subject Classification: 60F15.

§1. Introduction

Let $\{X_n, n \geq 1\}$ be a random variable sequence defined on a fixed probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Write $\mathcal{F}_S = \sigma(X_i, i \in S \subset N)$. Given σ-algebras $\mathcal{B}$, $\mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})} \frac{|\mathsf{E}XY - \mathsf{E}X\mathsf{E}Y|}{(\mathsf{Var}\,X \mathsf{Var}\,Y)^{1/2}}. \tag{1.1}$$

Define the $\widetilde{\rho}$-mixing coefficients by

$$\widetilde{\rho}(k) = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T) : \text{ finite subsets } S, T \subset N, \text{ such that } \mathrm{dist}(S, T) \geq k\}, \qquad k \geq 0.$$

Obviously, $0 \leq \widetilde{\rho}(k+1) \leq \widetilde{\rho}(k) \leq 1$, and $\widetilde{\rho}(0) = 1$.

Definition 1.1 A random variable sequence $\{X_n, n \geq 1\}$ is said to be a $\widetilde{\rho}$-mixing random variable sequence if there exists $k \in N$ such that $\widetilde{\rho}(k) < 1$.

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$\widetilde{\rho}$-mixing random variables were introduced by Bradley (1992) and many applications have been found. $\widetilde{\rho}$-mixing is similar to ρ-mixing, but both are quite different. Many authors have studied this concept providing interesting results and applications. See for example, Bryc and Smolenski (1993), Yang (1998), Peligrad and Gut (1999), Wu (2001, 2002, 2008), Utev and Peligrad (2003), Gan (2004), Cai (2008), An and Yuan (2008). When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired.

The main purpose of this paper is to give some new results of $\widetilde{\rho}$-mixing sequence. We obtain the convergence properties for $\widetilde{\rho}$-mixing sequence, a class of strong limit theorems for the partial sums of $\widetilde{\rho}$-mixing sequence which generalize the corresponding results of independent sequence.

Throughout the paper, let $I(A)$ be the indicator function of the set A and $X^{(a)} = XI(|X| \le a)$ for some $a > 0$. C denotes a positive constant which may be different in various places.

The main results of this paper are depending on the following lemmas:

Lemma 1.1 (cf. Wu, 2008, Theorem 2) Let $\{X_n, n \ge 1\}$ be a $\widetilde{\rho}$-mixing random variable sequence. Assume that

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| > c) < \infty, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \mathsf{E}(X_n^{(c)}) \ \text{ converges}, \tag{1.3}$$

$$\sum_{n=1}^{\infty} \mathsf{Var}\,(X_n^{(c)}) < \infty, \tag{1.4}$$

then $\sum\limits_{n=1}^{\infty} X_n$ converges almost surely.

Lemma 1.2 Let $\{s_n, n \ge 1\}$ and $\{t_n, n \ge 1\}$ be nonnegative numbers satisfying $s_n \le t_n$ for each $n \ge 1$, $\{u_n, n \ge 1\}$ be real numbers. Then

$$\sum_{n=1}^{\infty} |u_n|^{s_n} < +\infty \Rightarrow \sum_{n=1}^{\infty} |u_n|^{t_n} < +\infty.$$

Proof If $\sum\limits_{n=1}^{\infty} |u_n|^{s_n} < +\infty$, then there exists a positive integer N_1 such that $|u_n|^{s_n} < 1$ for all $n \ge N_1$. Therefore, $|u_n|^{t_n} \le |u_n|^{s_n}$ for all $n \ge N_1$, and

$$\sum_{n=1}^{\infty} |u_n|^{t_n} \le \sum_{n=1}^{N_1} |u_n|^{t_n} + \sum_{n=N_1}^{\infty} |u_n|^{s_n} < \infty.$$

The proof is completed. $\square$

§2. Strong Law of Large Numbers for $\widetilde{\rho}$-Mixing Sequence

Theorem 2.1 Let $\{X_n, n \ge 1\}$ be a $\widetilde{\rho}$-mixing sequence and $\{a_n, n \ge 1\}$ be a positive number sequence. Let $\{g_n(x), n \ge 1\}$ be a sequence of even functions defined on R, positive and non-decreasing on the half-line $x > 0$. One or the other of the following conditions is satisfied for every $n \ge 1$:

(i) In the interval $(0,1]$, there exists a $\delta > 0$ such that $g_n(x) \ge \delta x$;

(ii) In the interval $(0,1]$, there exist $\beta \in (1,2]$ and $\delta > 0$ such that $g_n(x) \ge \delta x^\beta$ and in the interval $(1,+\infty)$, there exists a $\delta > 0$ such that $g_n(x) \ge \delta x$. $\mathsf{E}X_n = 0$ for all $n \ge 1$.

For some $M > 0$, we assume that

$$\sum_{n=1}^{\infty} \mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty, \tag{2.1}$$

then $\sum\limits_{n=1}^{\infty} X_n/a_n$ converges almost surely. Furthermore, if $0 < a_n \uparrow \infty$, then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = 0, \qquad \text{a.s..} \tag{2.2}$$

Proof Let $X_n^{(Ma_n)} \doteq X_n I(|X_n| \le Ma_n)$, then

$$\frac{X_n^{(Ma_n)}}{Ma_n} = \frac{X_n}{Ma_n} I\Big(\Big|\frac{X_n}{Ma_n}\Big| \le 1\Big) \doteq \Big(\frac{X_n}{Ma_n}\Big)^{(1)}.$$

By the definition of $\widetilde{\rho}$-mixing sequence, we can see that $\{X_n/Ma_n, n \ge 1\}$ is also a $\widetilde{\rho}$-mixing sequence. Therefore, we only need to test (1.2), (1.3) and (1.4), where $c = 1$.

Firstly, if the function $g_n(x)$ satisfies condition (i), when $|X_n| > Ma_n > 0$, we have

$$\frac{1}{\delta} g_n\Big(\frac{X_n}{Ma_n}\Big) \ge 1.$$

Therefore

$$\mathsf{P}(|X_n| > Ma_n) = \mathsf{E}(I(|X_n| > Ma_n)) \le \frac{1}{\delta}\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big).$$

By (2.1)

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| > Ma_n) \le \frac{1}{\delta} \sum_{n=1}^{\infty} \mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty. \tag{2.3}$$

If the function $g_n(x)$ satisfies condition (ii), we also have (2.3).

Secondly, if the function $g_n(x)$ satisfies condition (i), we have

$$\begin{aligned}
|\mathsf{E}X_n^{(Ma_n)}| &\le \mathsf{E}(|X_n| I(|X_n| \le Ma_n)) = Ma_n \mathsf{E}\Big(\frac{|X_n|}{Ma_n} I(|X_n| \le Ma_n)\Big) \\
&\le \frac{1}{\delta} Ma_n \mathsf{E}g_n\Big(\frac{|X_n|}{Ma_n} I(|X_n| \le Ma_n)\Big) \le \frac{1}{\delta} Ma_n \mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big).
\end{aligned}$$

If the function $g_n(x)$ satisfies condition (ii), we can get

$$\begin{aligned}|\mathsf{E}X_n^{(Ma_n)}| &= |\mathsf{E}(X_nI(|X_n|\le Ma_n))| = |\mathsf{E}(X_nI(|X_n|>Ma_n))|\\ &\le \mathsf{E}(|X_n|I(|X_n|>Ma_n)) \le \frac{1}{\delta}Ma_n\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big).\end{aligned}$$

Therefore, whether $g_n(x)$ satisfies condition (i) or condition (ii), we can obtain

$$\sum_{n=1}^{\infty}\frac{|\mathsf{E}X_n^{(Ma_n)}|}{Ma_n} \le \frac{1}{\delta}\sum_{n=1}^{\infty}\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty. \tag{2.4}$$

Finally, if the function $g_n(x)$ satisfies condition (i), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathsf{E}(X_n^{(Ma_n)})^2}{a_n^2} &= \sum_{n=1}^{\infty}\frac{\mathsf{E}(X_n^2I(|X_n|\le Ma_n))}{a_n^2}\\ &= M^2\sum_{n=1}^{\infty}\mathsf{E}\Big(\Big(\frac{X_n}{Ma_n}\Big)^2I(|X_n|\le Ma_n)\Big)\\ &\le M^2\sum_{n=1}^{\infty}\mathsf{E}\Big(\frac{|X_n|}{Ma_n}I(|X_n|\le Ma_n)\Big)\\ &\le \frac{M^2}{\delta}\sum_{n=1}^{\infty}\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty.\end{aligned}$$

If the function $g_n(x)$ satisfies condition (ii), we can get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathsf{E}(X_n^{(Ma_n)})^2}{a_n^2} &= M^2\sum_{n=1}^{\infty}\mathsf{E}\Big(\Big(\frac{X_n}{Ma_n}\Big)^2I(|X_n|\le Ma_n)\Big)\\ &\le M^2\sum_{n=1}^{\infty}\mathsf{E}\Big(\Big|\frac{X_n}{Ma_n}\Big|^{\beta}I(|X_n|\le Ma_n)\Big)\\ &\le \frac{M^2}{\delta}\sum_{n=1}^{\infty}\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}I(|X_n|\le Ma_n)\Big)\\ &\le \frac{M^2}{\delta}\sum_{n=1}^{\infty}\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty.\end{aligned}$$

Therefore, whether $g_n(x)$ satisfies condition (i) or condition (ii), we can obtain

$$\sum_{n=1}^{\infty}\frac{\mathsf{Var}\,(X_n^{(Ma_n)})}{(Ma_n)^2} \le \sum_{n=1}^{\infty}\frac{\mathsf{E}(X_n^{(Ma_n)})^2}{(Ma_n)^2} \le \frac{1}{\delta}\sum_{n=1}^{\infty}\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty. \tag{2.5}$$

Thus, $\sum\limits_{n=1}^{\infty}X_n/a_n$ converges almost surely following from Lemma 1.1, (2.3), (2.4) and (2.5). We can easily get (2.2) by Kronecker's Lemma. The proof of the theorem is completed. □

Remark 1 In Theorem 2.1, if the even function $g_n(x)$ is positive and non-decreasing on the half-line $x>0$ satisfying (i) and (2.1), then $\sum\limits_{n=1}^{\infty}X_n/a_n$ converges almost surely for arbitrary random variable sequence $\{X_n, n\ge 1\}$.

Indeed, by (2.1) and Monotone Convergence Theorem, we have

$$\mathsf{E}\Big(\sum_{n=1}^{\infty} g_n\Big(\frac{X_n}{Ma_n}\Big)\Big) = \sum_{n=1}^{\infty} \mathsf{E} g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty,$$

hence

$$\sum_{n=1}^{\infty} g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty \qquad \text{a.s..}$$

Denote

$$A = \Big\{\omega : \sum_{n=1}^{\infty} g_n\Big(\frac{X_n(\omega)}{Ma_n}\Big) < \infty\Big\},$$

then $\mathsf{P}(A) = 1$. $\forall \omega \in A$,

$$\sum_{n=1}^{\infty} g_n\Big(\frac{X_n(\omega)}{Ma_n}\Big) < \infty, \tag{2.6}$$

hence

$$\lim_{n\to\infty} g_n\Big(\frac{X_n(\omega)}{Ma_n}\Big) = 0, \qquad \omega \in A. \tag{2.7}$$

Since $\{g_n(x), n \geq 1\}$ is even and non-decreasing on the half-line $x > 0$ and $g_n(x) \geq \delta x$ for $x \in (0,1]$, there exists a positive integer $N_0(\omega)$ such that $|X_n(\omega)/(Ma_n)| \leq 1$ when $n > N_0(\omega)$ (otherwise, there exist n_i, $i \geq 1$ such that $|X_{n_i}(\omega)/(Ma_{n_i})| > 1$, thus, $g_{n_i}(X_{n_i}(\omega)/(Ma_{n_i})) \geq g_{n_i}(1) \geq \delta > 0$, which is contrary to (2.7)). Using $g_n(x) \geq \delta x$ for $x \in (0,1]$ again, we have

$$\Big|\frac{X_n(\omega)}{Ma_n}\Big| \leq \frac{1}{\delta} g_n\Big(\frac{X_n(\omega)}{Ma_n}\Big), \qquad \omega \in A,\ n > N_0(\omega). \tag{2.8}$$

By (2.6) and (2.8), it follows that

$$\sum_{n=1}^{\infty} \Big|\frac{X_n(\omega)}{Ma_n}\Big| < \infty, \qquad \omega \in A.$$

Hence, $\sum\limits_{n=1}^{\infty} X_n(\omega)/a_n$ converges, $\omega \in A$, which implies that $\sum\limits_{n=1}^{\infty} X_n/a_n$ converges a.s..

Corollary 2.1 Let $\{X_n, n \geq 1\}$ be a $\widetilde{\rho}$-mixing random variable sequence and $\{a_n, n \geq 1\}$ be a positive number sequence satisfying $0 < a_n \uparrow \infty$. For some $M > 0$, one or the other of the following conditions is satisfied:

(i) $\sum\limits_{n=1}^{\infty} \mathsf{E}[|X_n|^{\beta}/(|Ma_n|^{\beta} + |X_n|^{\beta})] < \infty$, $\exists\, \beta \in (0,1]$;

(ii) $\sum\limits_{n=1}^{\infty} \mathsf{E}[|X_n|^{\beta}/(Ma_n|X_n|^{\beta-1} + |Ma_n|^{\beta})] < \infty$, $\exists\, \beta \in (1,2]$ and $\mathsf{E}X_n = 0$ for all $n \geq 1$.

Then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = 0, \qquad \text{a.s..} \tag{2.9}$$

Proof In Theorem 2.1, we take

$$g_n(x)=\frac{|x|^\beta}{1+|x|^\beta}\quad(0<\beta\le 1),\qquad \varphi_n(x)=\frac{|x|^\beta}{1+|x|^{\beta-1}}\quad(1<\beta\le 2)$$

in the conditions (i) and (ii) respectively. It is easy to see that $g_n(x)$ and $\varphi_n(x)$ are both even functions, positive and non-decreasing on the half-line $x>0$. And

$$\begin{aligned}&g_n(x)\ge\frac{1}{2}x^\beta\ge\frac{1}{2}x,\qquad 0<x\le 1,\ 0<\beta\le 1;\\&\varphi_n(x)\ge\frac{1}{2}x^\beta,\quad 0<x\le 1,\ 1<\beta\le 2\qquad\text{and}\qquad \varphi_n(x)\ge\frac{1}{2}x,\quad 1<x<+\infty.\end{aligned}$$

Therefore, by Theorem 2.1, we can easily get (2.9). □

Furthermore, by Corollary 2.1, we can obtain the following important Chung-type strong law of large numbers.

Corollary 2.2 Let $\{X_n,n\ge 1\}$ be a $\widetilde{\rho}$-mixing random variable sequence and $\{a_n,n\ge 1\}$ be a positive number sequence satisfying $0<a_n\uparrow\infty$. There exists some $\beta\in(0,2]$ such that

$$\sum_{n=1}^{\infty}\frac{\mathsf{E}|X_n|^\beta}{a_n^\beta}<\infty.$$

If $\beta\in(1,2]$, we further assume that $\mathsf{E}X_n=0$. Then

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=1}^{n}X_i=0,\qquad\text{a.s..}\tag{2.10}$$

Theorem 2.2 Let $\{X_n,n\ge 1\}$ be a $\widetilde{\rho}$-mixing sequence and $\{a_n,n\ge 1\}$ be a positive number sequence. Let $\{g_n(x),n\ge 1\}$ be a sequence of even functions defined on R, positive and non-decreasing on the half-line $x>0$. There exists a $\beta\in[2,\infty)$ such that $g_n(x)\ge\delta x^\beta$, $x>0$ for all $n\ge 1$. For some $M>0$, if

$$\sum_{n=1}^{\infty}\Big(\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big)\Big)^{1/\beta}<\infty,\tag{2.11}$$

then $\sum\limits_{n=1}^{\infty}X_n/a_n$ converges almost surely. Furthermore, if $0<a_n\uparrow\infty$, then

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=1}^{n}X_i=0,\qquad\text{a.s..}\tag{2.12}$$

Proof Since $\beta\ge 2$, by Lemma 1.2 and (2.11), we have

$$\sum_{n=1}^{\infty}\Big(\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big)\Big)^{2/\beta}<\infty;\qquad \sum_{n=1}^{\infty}\mathsf{E}g_n\Big(\frac{X_n}{Ma_n}\Big)<\infty.\tag{2.13}$$

By (2.13), similar to the proof of (2.3), we can get

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| > Ma_n) = \sum_{n=1}^{\infty} \mathsf{E}(I(|X_n| > Ma_n)) \leq \frac{1}{\delta} \sum_{n=1}^{\infty} \mathsf{E} g_n\Big(\frac{X_n}{Ma_n}\Big) < \infty. \tag{2.14}$$

By Hölder's inequality and the assumption of the function $g_n(x)$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\mathsf{E} X_n^{(Ma_n)}|}{Ma_n} &\leq \sum_{n=1}^{\infty} \mathsf{E}\Big(\frac{|X_n|}{Ma_n} I(|X_n| \leq Ma_n)\Big) \\ &\leq \sum_{n=1}^{\infty} \Big(\mathsf{E}\Big(\frac{|X_n|}{Ma_n}\Big)^{\beta} I(|X_n| \leq Ma_n)\Big)^{1/\beta} \\ &\leq \Big(\frac{1}{\delta}\Big)^{1/\beta} \sum_{n=1}^{\infty} \Big(\mathsf{E} g_n\Big(\frac{X_n}{Ma_n}\Big)\Big)^{1/\beta} < \infty. \end{aligned} \tag{2.15}$$

Since $(\mathsf{E}(|X|^r))^{1/r}$ is increasing for $r > 0$, by $\beta \geq 2$ and (2.13),

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\mathsf{E}(X_n^{(Ma_n)})^2}{(Ma_n)^2} &= \sum_{n=1}^{\infty} \mathsf{E}\Big(\Big(\frac{X_n}{Ma_n}\Big)^2 I(|X_n| \leq Ma_n)\Big) \\ &\leq \sum_{n=1}^{\infty} \Big(\mathsf{E}\Big(\frac{|X_n|}{Ma_n}\Big)^{\beta} I(|X_n| \leq Ma_n)\Big)^{2/\beta} \\ &\leq \Big(\frac{1}{\delta}\Big)^{2/\beta} \sum_{n=1}^{\infty} \Big(\mathsf{E} g_n\Big(\frac{X_n}{Ma_n}\Big)\Big)^{2/\beta} < \infty. \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{\mathsf{Var}\,(X_n^{(Ma_n)})}{(Ma_n)^2} \leq \sum_{n=1}^{\infty} \frac{\mathsf{E}(X_n^{(Ma_n)})^2}{(Ma_n)^2} < \infty. \tag{2.16}$$

Thus, $\sum\limits_{n=1}^{\infty} X_n/a_n$ converges almost surely by (2.14), (2.15), (2.16) and Lemma 1.1. By Kronecker's Lemma, we get (2.12) immediately. The proof is completed. □

If taking $g_n(x) \equiv |x|^{\beta}$, $\beta \geq 2$, we can get the following corollary:

Corollary 2.3 Let $\{X_n, n \geq 1\}$ be a $\widetilde{\rho}$-mixing sequence, $\{a_n, n \geq 1\}$ be a positive number sequence satisfying $0 < a_n \uparrow \infty$. If there exists some $\beta \in [2, +\infty)$ such that

$$\sum_{n=1}^{\infty} \frac{(\mathsf{E}|X_n|^{\beta})^{1/\beta}}{a_n} < \infty, \tag{2.17}$$

then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = 0, \qquad \text{a.s..} \tag{2.18}$$

§3. Strong Stability for Weighted Sums of $\widetilde{\rho}$-Mixing Sequence

Firstly, we will give some definitions as follows:

Definition 3.1 A random variable sequence $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant C such that

$$\mathsf{P}(|X_n| > x) \leq C\mathsf{P}(|X| > x) \tag{3.1}$$

for all $x \geq 0$ and $n \geq 1$.

Definition 3.2 A random variable sequence $\{Y_n, n \geq 1\}$ is said to be strongly stable if there exist two constant sequences $\{b_n, n \geq 1\}$ and $\{d_n, n \geq 1\}$ with $0 < b_n \uparrow \infty$ such that

$$b_n^{-1} Y_n - d_n \to 0 \qquad \text{a.s..} \tag{3.2}$$

The following lemma is useful.

Lemma 3.1 Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following statement holds:

$$\mathsf{E}|X_n|^\alpha I(|X_n| \leq b) \leq C\{\mathsf{E}|X|^\alpha I(|X| \leq b) + b^\alpha \mathsf{P}(|X| > b)\},$$

where C is a positive constant.

Proof It is easy to see that

$$\alpha \int_0^b s^{\alpha-1} \mathsf{P}(|X_n| > s) \mathrm{d}s = b^\alpha \mathsf{P}(|X_n| > b) + \mathsf{E}|X_n|^\alpha I(|X_n| \leq b).$$

It follows that

$$\begin{aligned} \mathsf{E}|X_n|^\alpha I(|X_n| \leq b) &\leq \alpha \int_0^b s^{\alpha-1} \mathsf{P}(|X_n| > s) \mathrm{d}s \\ &\leq C\alpha \int_0^b s^{\alpha-1} \mathsf{P}(|X| > s) \mathrm{d}s \\ &\leq C\{\mathsf{E}|X|^\alpha I(|X| \leq b) + b^\alpha \mathsf{P}(|X| > b)\}. \qquad \square \end{aligned}$$

Theorem 3.1 Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences of positive numbers with $c_n = b_n / a_n$ and $b_n \uparrow \infty$. Let $\{X_n, n \geq 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables which is stochastically dominated by a random variable X. Define $N(x) = \mathrm{Card}\{n : c_n \leq x\}$, $R(x) = \displaystyle\int_x^\infty N(y) y^{-3} \mathrm{d}y$, $x > 0$. If the following conditions are satisfied:

(i) $N(x) < \infty$ for any $x > 0$,

(ii) $R(1) = \displaystyle\int_1^\infty N(y) y^{-3} \mathrm{d}y < \infty$,

(iii) $\mathsf{E}X^2 R(|X|) < \infty$,

then there exist $d_n \in R$, $n = 1, 2, \cdots$ such that

$$b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \to 0 \qquad \text{a.s..} \tag{3.3}$$

Proof Since $N(x)$ is nondecreasing, then for any $x > 0$

$$R(x) \geq N(x) \int_x^{\infty} y^{-3} \mathrm{d}y = \frac{1}{2} x^{-2} N(x), \tag{3.4}$$

which implies that $\mathsf{E}N(|X|) \leq 2\mathsf{E}X^2 R(|X|) < \infty$. Therefore

$$\sum_{i=1}^{\infty} \mathsf{P}(X_i \neq X_i^{(c_i)}) = \sum_{i=1}^{\infty} \mathsf{P}(|X_i| > c_i) \leq C \sum_{i=1}^{\infty} \mathsf{P}(|X| > c_i) \leq C\mathsf{E}N(|X|) < \infty. \tag{3.5}$$

By Borel-Cantelli lemma for any sequence $\{d_n, n \geq 1\} \subset R$, the sequences $\Big\{b_n^{-1} \sum\limits_{i=1}^{n} a_i X_i - d_n\Big\}$ and $\Big\{b_n^{-1} \sum\limits_{i=1}^{n} a_i X_i^{(c_i)} - d_n\Big\}$ converge on the same set and to the same limit. We will show that $b_n^{-1} \sum\limits_{i=1}^{n} a_i (X_i^{(c_i)} - \mathsf{E}X_i^{(c_i)}) \to 0$ a.s., which gives the theorem with $d_n = b_n^{-1} \sum\limits_{i=1}^{n} a_i \mathsf{E}X_i^{(c_i)}$. It follows from Lemma 3.1 that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mathsf{Var}\,(a_n X_n^{(c_n)})}{b_n^2} &\leq \sum_{n=1}^{\infty} c_n^{-2} \mathsf{E}(X_n^{(c_n)})^2 = \sum_{n=1}^{\infty} c_n^{-2} \mathsf{E}X_n^2 I(|X_n| \leq c_n) \\
&\leq C \sum_{n=1}^{\infty} c_n^{-2} [c_n^2 \mathsf{P}(|X| > c_n) + \mathsf{E}X^2 I(|X| \leq c_n)] \\
&\leq C\mathsf{E}N(|X|) + C \sum_{n=1}^{\infty} c_n^{-2} \mathsf{E}X^2 I(|X| \leq c_n).
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n^{-2} \mathsf{E}X^2 I(|X| \leq c_n) &= \sum_{n: c_n \leq 1} c_n^{-2} \mathsf{E}X^2 I(|X| \leq c_n) + \sum_{n: c_n > 1} c_n^{-2} \mathsf{E}X^2 I(|X| \leq c_n) \\
&\doteq I_1 + I_2.
\end{aligned} \tag{3.7}$$

Since $N(1) = \mathrm{Card}\{n : c_n \leq 1\} \leq 2R(1) < \infty$ from (3.4) and condition (ii), then $I_1 < \infty$.

$$\begin{aligned}
I_2 &= \sum_{n: c_n > 1} c_n^{-2} \mathsf{E}X^2 I(|X| \leq c_n) \\
&= \sum_{k=2}^{\infty} \sum_{k-1 < c_n \leq k} c_n^{-2} \mathsf{E}X^2 I(|X| \leq c_n) \\
&\leq \sum_{k=2}^{\infty} (N(k) - N(k-1))(k-1)^{-2} \mathsf{E}X^2 I(|X| \leq k) \\
&\leq \sum_{k=2}^{\infty} (N(k) - N(k-1))(k-1)^{-2} \mathsf{E}X^2 I(|X| \leq 1) \\
&\quad + \sum_{k=2}^{\infty} (N(k) - N(k-1))(k-1)^{-2} \mathsf{E}X^2 I(1 < |X| \leq k) \\
&\doteq I_{21} + I_{22}.
\end{aligned}$$

$$\begin{aligned}
I_{21} &\le C\sum_{k=2}^{\infty}(N(k)-N(k-1))\sum_{j=k-1}^{\infty} j^{-3}\\
&= C\sum_{j=1}^{\infty} j^{-3}\sum_{k=2}^{j+1}(N(k)-N(k-1))\\
&\le C\sum_{j=1}^{\infty}(j+1)^{-3}N(j+1)\\
&\le C\int_1^{\infty} y^{-3}N(y)\mathrm{d}y<\infty.
\end{aligned}$$

Since $N(x)$ is nondecreasing and $R(x)$ is nonincreasing, then

$$\begin{aligned}
I_{22} &= \sum_{k=2}^{\infty}(N(k)-N(k-1))(k-1)^{-2}\mathsf{E}X^2 I(1<|X|\le k)\\
&= \sum_{k=2}^{\infty}(N(k)-N(k-1))(k-1)^{-2}\sum_{m=2}^{k}\mathsf{E}X^2 I(m-1<|X|\le m)\\
&= \sum_{m=2}^{\infty}\mathsf{E}X^2 I(m-1<|X|\le m)\sum_{k=m}^{\infty}(N(k)-N(k-1))(k-1)^{-2}\\
&\le \sum_{m=2}^{\infty}\mathsf{E}X^2 I(m-1<|X|\le m)\sum_{k=m}^{\infty}N(k)((k-1)^{-2}-k^{-2})\\
&\le C\sum_{m=2}^{\infty}\mathsf{E}X^2 I(m-1<|X|\le m)\sum_{k=m}^{\infty}\int_k^{k+1}N(x)x^{-3}\mathrm{d}x\\
&= C\sum_{m=2}^{\infty}R(m)\mathsf{E}X^2 I(m-1<|X|\le m)\\
&\le C\sum_{m=2}^{\infty}\mathsf{E}X^2 R(|X|)I(m-1<|X|\le m)\\
&\le C\mathsf{E}X^2R(|X|)<\infty.
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\mathsf{Var}\,(a_nX_n^{(c_n)})}{b_n^2}<\infty \tag{3.8}$$

following from the above statements. By Theorem 1 in Wu and Jiang (2008) and Kronecker's Lemma, it follows that

$$b_n^{-1}\sum_{i=1}^{n}a_i(X_i^{(c_i)}-\mathsf{E}X_i^{(c_i)})\to 0 \qquad \text{a.s..} \tag{3.9}$$

Taking $d_n=b_n^{-1}\sum_{i=1}^{n}a_i\mathsf{E}X_i^{(c_i)}$, $n\ge 1$, then $b_n^{-1}\sum_{i=1}^{n}a_iX_i^{(c_i)}-d_n\to 0$ a.s.. We complete the proof of the theorem. □

Corollary 3.1 Let the conditions of Theorem 3.1 be satisfied. If $\mathsf{E}X_n=0$, $n\ge 1$ and $\int_1^{\infty}\mathsf{E}N(|X|/s)\mathrm{d}s<\infty$, then $b_n^{-1}\sum_{i=1}^{n}a_iX_i\to 0$ a.s..

Proof By Theorem 3.1, we only need to prove

$$b_n^{-1}\sum_{i=1}^{n} a_i \mathsf{E}X_i^{(c_i)} \to 0 \qquad \text{a.s..} \tag{3.10}$$

In fact

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{a_i|\mathsf{E}X_i^{(c_i)}|}{b_i} &= \sum_{i=1}^{\infty} c_i^{-1}|\mathsf{E}X_i I(|X_i|\le c_i)| \\
&\le \sum_{i=1}^{\infty} c_i^{-1}\mathsf{E}|X_i|I(|X_i|>c_i) \\
&= \sum_{i=1}^{\infty} c_i^{-1}\Big(c_i\mathsf{P}(|X_i|>c_i)+\int_{c_i}^{\infty}\mathsf{P}(|X_i|>t)\mathrm{d}t\Big) \\
&\le C\sum_{i=1}^{\infty}\mathsf{P}(|X|>c_i)+C\sum_{i=1}^{\infty}\int_{1}^{\infty}\mathsf{P}(|X|>sc_i)\mathrm{d}s \\
&\le C\mathsf{E}N(|X|)+C\int_{1}^{\infty}\mathsf{E}N(|X|/s)\mathrm{d}s<\infty,
\end{aligned}$$

which implies (3.10) by Kronecker's Lemma. We complete the proof of the corollary. □

Remark 2 It is easily seen that $\tilde{\rho}$-mixing sequence contains independent sequence as a special case. Thus the main results of this paper hold for independent sequence.

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弱相依随机序列的若干新结果

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设$\{X_n, n \geq 1\}$为$\widetilde{\rho}$-混合序列. 利用随机变量的截尾方法和$\widetilde{\rho}$-混合序列的三级数定理, 讨论了$\widetilde{\rho}$-混合序列的收敛性质, 并且得到了$\widetilde{\rho}$-混合序列的一类强极限定理, 这些结果推广了独立序列的相应结果. 最后研究了$\widetilde{\rho}$-混合序列加权和的强稳定性.

关键词: 强极限定理, $\widetilde{\rho}$-混合序列, 收敛性质, 强稳定性, 加权和.

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