应用概率统计　第二十七卷
第一期　2011年2月

Chinese Journal of Applied Probability
and Statistics Vol.27 No.1 Feb. 2011

The Feller Property for Generalized Branching Processes with Resurrection *

ZHANG XIUZHEN　　LI YANGRONG

(*School of Mathematics and Statistics, Southwest China University, Chongqing, 400715*)

Abstract

We first establish a criterion for the minimal Q-function to be a Feller transition function when Q is a quasi-monotone q-matrix. We then apply this result to generalized branching q-matrices and obtain the corresponding Feller criteria for generalized branching processes. In particular, it is shown that there always exists a separating point θ_0 with $1 \leq \theta_0 \leq 2$ or $\theta_0 < +\infty$ such that whether the generalized branching processes (with resurrection) are Feller processes or not according to $\theta < \theta_0$ or $\theta > \theta_0$, where θ is the nonlinear number given in the branching q-matrix.

Keywords: Continuous-time Markov chains, generalized branching processes, Feller processes, generalized branching q-matrices, q-function, q-resolvent function.

AMS Subject Classification: 60J27.

§1. Introduction

A *generalized branching processes* Z_t is a continuous-time Markov chain (CTMC) on the state space $\mathbb{Z}^+$ whose transition function $P(t)$ is the minimal Q-function (that is, $P'(0) = Q$ componentwise), where the q-matrix $Q = \{q_{ij}; i, j \in \mathbb{Z}^+\}$, is given by

$$q_{ij} = \begin{cases} h_j, & \text{if } i = 0; \\ i^\theta b_{j-i+1}, & \text{if } i \geq 1 \text{ and } j \geq i-1; \\ 0 & \text{otherwise}, \end{cases} \tag{1.1}$$

with

$$\begin{cases} -h_0 = \sum\limits_{j=1}^{\infty} h_j < +\infty \text{ and } h_j \geq 0 \text{ for } j \geq 1; \\ -b_1 = \sum\limits_{j\neq 1} b_j < +\infty \text{ and } b_j \geq 0 \text{ for } j \neq 1; \\ \theta > 0. \end{cases} \tag{1.2}$$

*This work was supported by the China Postdoctoral Science Foundation (2005038326).
Received September 17, 2009. Revised January 20, 2010.

In order to avoid discussing some trivial cases, we assume that

$$b_0 > 0 \qquad \text{and} \qquad \sum_{j=2}^{\infty} b_j > 0. \tag{1.3}$$

A q-matrix Q satisfying (1.1)–(1.3) is called a generalized branching q-matrix, Q is said to be *super-linear* if $\theta > 1$, and *sub-linear* if $0 < \theta \le 1$. Q is called absorbing (or *without resurrection*) if $h_0 = 0$ (and thus $h_j = 0$ for all $j \ge 0$), and is called *with resurrection* otherwise. All these notions will be applied to the corresponding processes, transition functions as well as resolvent functions.

Regularity, recurrence, ergodicity and extinction properties have been investigated by many works, e.g. R. Chen (1997), A. Chen (2002a, b), Zhang et al (2001), Chen et al (2005, 2006), Li (2008).

In present paper, we are concerned with the Feller property of generalized branching processes. Recall that a transition function $P(t) = (P_{ij}(t))$, $t \ge 0$ is called to be a Feller transition function if

$$P_{ij}(t) \to 0 \ \text{ as } \ i \to \infty \ \text{ for all } \ j \in \mathbb{Z}^+ \ \text{ and } \ t \ge 0. \tag{1.4}$$

This concept is introduced by Reuter and Riley (1972), with some developments and related discussions given by Zhang et al (1999, 2001), Chen (2001), Li (2003, 2006, 2007, 2009). Since Feller properties describe the asymptotic behavior at the remote states (i.e. $i \to \infty$), it is also called to have *asymptotic remoteness* in some literatures, e.g. van Doore and Zeifman (2005).

All ordinary branching processes ($\theta = 1$ and $h_0 = 0$) are Feller processes, this fact has been proved by Pakes (1993). The Feller property in the sub-linear case has been discussed by Zhang et al (2001) (but their proves needs some supplements). However, the general case (in particular, the super-linear case) may be more complex. To view this, we first consider a special example: the non-linear birth-death processes (i.e. $h_j = 0$ for $j \ge 2$ and $b_k = 0$ for $k \ge 3$).

Proposition 1.1 For a non-linear birth-death q-matrix Q, the minimal Q-function is Feller if and only if one of the following three conditions holds true.

(i) $b_2 > b_0$, (for all $\theta > 0$);

(ii) $b_2 < b_0$, $0 < \theta \le 1$;

(iii) $b_2 = b_0$, $0 < \theta \le 2$.

It is interesting to notice that there is a separating point θ_0 with $\theta_0 = +\infty$ in (i), $\theta_0 = 1$ in (ii) and $\theta_0 = 2$ in (iii). A similar phenomenon will occur in the general case as proved in the Section 3.

The above proposition can be proved by using a Feller criterion for monotone q-matrices obtained by Li (2006). However, in the general case, a difficulty appears since Q is not necessarily monotone, while a Feller criterion in the non-monotone cases remains open until now.

To overcome this difficulty, we notice that Q has a decomposition: $Q = \widetilde{Q} + A$ such that $\widetilde{Q}$ is monotone and A satisfying other conditions (such as row-sum bounded). Such a q-matrix Q will be called to be a *quasi-monotone* q-matrix (a formal definition will be given in the next section).

In Section 2, we'll establish a Feller criterion in the quasi-monotone case by using functional analysis methods (see Theorem 2.1). The Feller criterion is stated as: Q is either Feller and strong zero-entrance, or nonzero-exit.

Applying the above Feller criterion to generalized branching q-matrices, we obtain some Feller criteria for the generalized branching processes. In particular, there is always a separating point θ_0 with $\theta_0 \in [1,2] \cup \{\infty\}$ whether the process is Feller process or not according to $\theta < \theta_0$ or $\theta > \theta_0$ respectively (see Theorem 3.1).

§2. Feller Criteria in the Quasi-Monotone Cases

In this section, $Q = (q_{ij}; i, j \in \mathbb{Z}^+)$ will denote a (stable) q-matrix, that is,

$$q_{ij} \geq 0 \ \ (i \neq j) \quad \text{and} \quad \sum_{j \neq i} q_{ij} \leq -q_{ii} := q_i < +\infty, \qquad \text{for all } \ i \in \mathbb{Z}^+. \tag{2.1}$$

It is well-known that there always exists a minimal Q-function $P(t)$ with $P'(0) = Q$ componentwise. For the details, we refer to Anderson (1991).

We also introduce a notion of a quasi-monotone q-matrix.

Definition 2.1 A q-matrix $Q = (q_{ij})$ is called to be quasi-monotone if Q has a decomposition: $Q = \widetilde{Q} + A$ such that $\widetilde{Q} = (\widetilde{q}_{ij})$ is a monotone q-matrix, that is,

$$\sum_{k=j}^{\infty} \widetilde{q}_{ik} \leq \sum_{k=j}^{\infty} \widetilde{q}_{i+1,k}, \qquad \text{for all } \ i, j \in \mathbb{Z}^+ \ \text{ such that } \ j \neq i+1, \tag{2.2}$$

and $A = (a_{ij})$ is a infinite matrix (not necessarily a q-matrix) satisfying

(i) A is Feller, i.e. $a_{ij} \to 0$ as $i \to \infty$ for all $j \in \mathbb{Z}^+$; and

(ii) A is row-sum bounded, that is

$$\|A\| = \sup_{i \in \mathbb{Z}^+} \sum_{j=0}^{\infty} |a_{ij}| < +\infty. \tag{2.3}$$

We also recall that a q-matrix $Q = (q_{ij})$ is *zero-exit* if $l^+_\infty(\lambda) = 0$ (or equivalently if $l_\infty(\lambda) = 0$, by Anderson, 1991, Theorem 2.2.7), Q is *zero-entrance* if $l^+_1(\lambda) = 0$, and *strong zero-entrance* if $l_1(\lambda) = 0$, where

$$\begin{cases} l_\infty(\lambda) = \{x \in l_\infty | (\lambda I - Q)x = 0\}, & l^+_\infty(\lambda) = \{x \in l_\infty(\lambda) | x \geq 0\}; \\ l_1(\lambda) = \{y \in l_1 | y(\lambda I - Q) = 0\}, & l^+_1(\lambda) = \{y \in l_1(\lambda) | y \geq 0\}. \end{cases} \tag{2.4}$$

Q is *regular* if Q is conservative and zero-exit. Equivalence problem (raised by Reute and Riley, 1972) of strong zero-entrance and zero-entrance remains open.

Our main interest is to give an Feller criterion for quasi-monotone q-matrices.

Theorem 2.1 For a given quasi-monotone q-matrix Q, the minimal Q-function is a Feller transition function if and only if either

(i) Q is Feller and strong zero-entrance; or

(ii) Q is nonzero-exit.

To prove this theorem, we need three lemmas whose proves require a well-known result in functional analysis (see Yosida, 1978).

Banach's Theorem Let T be a bounded linear operator on a Banach space X such that $\|T\| < 1$. Then $I - T$ has a bounded inverse operator $(I - T)^{-1}$ with

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n. \tag{2.5}$$

Lemma 2.1 Suppose a q-matrix $\widetilde{Q}$ has a decomposition: $\widetilde{Q} = Q + A$ such that Q is also a q-matrix and A is row-sum bounded. Then $\widetilde{Q}$ is zero-exit if and only if Q is.

Proof we first notice that Q and $\widetilde{Q}$ induce two (unbounded) operators denoted by Q_∞ and $\widetilde{Q}_\infty$ respectively on the Banach space l_∞, and defined by

$$\begin{cases} Q_\infty x = Qx, & x \in \mathrm{Dom}(Q_\infty) = \{x \in l_\infty | Qx \in l_\infty\}; \\ \widetilde{Q}_\infty x = \widetilde{Q}x, & x \in \mathrm{Dom}(\widetilde{Q}_\infty) = \{x \in l_\infty | \widetilde{Q}x \in l_\infty\}. \end{cases} \tag{2.6}$$

By the row-sum bounded assumption of A, A induces a bounded operator on l_∞ (denoted by A yet) with $\|A\|$ given by (2.3) (see Anderson, 1991, Lemma 1.4.4). Then $\widetilde{Q}_\infty = Q_\infty + A$ with $\mathrm{Dom}(\widetilde{Q}_\infty) = \mathrm{Dom}(Q_\infty)$. We also note that the minimal Q-resolvent $\phi(\lambda)$ and $\widetilde{Q}$-resolvent $\widetilde{\phi}(\lambda)$ induce bounded operators on l_∞ with

$$\|\phi(\lambda)\| \leq \frac{1}{\lambda} \quad \text{and} \quad \|\widetilde{\phi}(\lambda)\| \leq \frac{1}{\lambda}. \tag{2.7}$$

Suppose now Q is zero-exit, we claim that

$$(\lambda I - Q_\infty)^{-1} = \phi(\lambda). \tag{2.8}$$

Indeed, Anderson (1991, Lemma 4.1.3) has proved that if x is a column vector such that $\phi(\lambda)x$ is well-defined, then $(\lambda I-Q)\phi(\lambda)x$ is well-defined with

$$(\lambda I-Q)\phi(\lambda)x=x. \tag{2.9}$$

In particular, if $x\in l_\infty$, then $z=\phi(\lambda)x\in l_\infty$, and thus (2.9) implies that $z\in \mathrm{Dom}(\lambda I-Q_\infty)=\mathrm{Dom}(Q_\infty)$ with $(\lambda I-Q_\infty)z=x$, which proved the operator $\lambda I-Q_\infty$ is surjective, but since Q is zero-exit, $\lambda I-Q_\infty$ is also injective. Thus $(\lambda I-Q_\infty)^{-1}$ exists with $(\lambda I-Q_\infty)^{-1}x=z=\phi(\lambda)x$, which proved (2.8).

Using (2.7), we can choose a large $\lambda_0>0$ (say $\lambda_0>\|A\|$) such that $\|\phi(\lambda_0)A\|<1$, then by Banach's Theorem $I-\phi(\lambda_0)A$ has a bounded inverse operator with

$$(I-\phi(\lambda_0)A)^{-1}=\sum_{n=0}^{\infty}(\phi(\lambda_0)A)^n. \tag{2.10}$$

Using (2.9) and (2.10) we have, for all $x\in\mathrm{Dom}(Q_\infty)=\mathrm{Dom}(\widetilde{Q}_\infty)$

$$\begin{aligned}&(I-\phi(\lambda_0)A)^{-1}\phi(\lambda_0)(\lambda_0 I-Q_\infty-A)x\\ =\;&(I-\phi(\lambda_0)A)^{-1}x-(I-\phi(\lambda_0)A)^{-1}(\phi(\lambda_0)A)x\\ =\;&\sum_{n=0}^{\infty}(\phi(\lambda_0)A)^n x-\sum_{n=0}^{\infty}(\phi(\lambda_0)A)^{n+1}x=x.\end{aligned}$$

Therefore, if $(\lambda_0 I-\widetilde{Q}_\infty)x=0$, then the above formulation implies that $x=0$, which shows that $\widetilde{Q}$ is zero-exit.

Conversely, considering the decomposition $Q=\widetilde{Q}+(-A)$, where $-A$ is also row-bounded, we obtain the converse conclusion by the same method as above. □

Remark 1 In a very special case when A has nonzero elements in the first row only (and other conditions), Lemma 2.1 has been proved in Chen (2002b, Lemma 2.4) by using Resolvent decomposition Theorem. Thus it may be not easy to prove Lemma 2.1 by using ordinary methods.

Lemma 2.2 Suppose Q, $\widetilde{Q}$, A are defined as in Lemma 2.1, then Q is strong zero-entrance if and only if $\widetilde{Q}$ is.

Proof We use the similar method as in Lemma 2.1. Let Q_1 be the induced operator on l_1 by $y\longmapsto yQ$, where y is a row vector. If Q is strong zero-entrance, we claim that

$$(\lambda I-Q_1)^{-1}=\phi(\lambda), \tag{2.11}$$

where the minimal Q-resolvent $\phi(\lambda)$ induces a bounded operator on l_1 with the same norm as in (2.7). Indeed, let Ω be the generator of the continuous contractive semigroup $P(t)$

induced by the minimal Q-function, then, by Anderson (1991, Prop. 1.4.6), $\Omega \subset Q_1$, that is, Ω is a restriction of Q_1. Thus, $\lambda I - \Omega \subset \lambda I - Q_1$. By the strong zero-entrance assumption, $\lambda I - Q_1$ is injective, which implies that $\lambda I - \Omega = \lambda I - Q_1$. Thus (2.11) holds.

We now choose $\lambda_0 > 0$ such that $\|A\phi(\lambda_0)\| < 1$, then by Banach's Theorem,

$$(I - A\phi(\lambda_0))^{-1} = \sum_{n=0}^{\infty} (A\phi(\lambda_0))^n,$$

which, together with (2.11), implies that, for $y \in \mathrm{Dom}(Q_1)$,

$$y(\lambda_0 I - (Q_1 + A))(\phi(\lambda_0)(I - A\phi(\lambda_0))^{-1}) = y.$$

Thus $\lambda_0 I - \widetilde{Q}_1$ is injective, that is, $\widetilde{Q}$ is strong zero-entrance. $\square$

Lemma 2.3 Let two q-matrix Q and $\widetilde{Q}$ satisfy $\widetilde{Q} = Q + A$, where A is a Feller and row-sum bounded matrix (not necessary q-matrix). Then the minimal Q-function $P(t)$ is Feller transition function if and only if the minimal $\widetilde{Q}$-function $\widetilde{P}(t)$ is.

Proof Suppose $P(t)$ is Feller, then, by a result in Reuter and Riley (1972), the minimal Q-resolvent $\phi(\lambda)$ is also Feller. Thus $\phi(\lambda)$ induces a bounded operator on c_0. Choose λ_0 such that $\|A\phi(\lambda_0)\| < 1$. Then by Banach's Theorem, $I - A\phi(\lambda_0)$ has a bounded inverse operator on c_0 with $(I - A\phi(\lambda_0))^{-1} = \sum\limits_{n=0}^{\infty} (A\phi(\lambda_0))^n$, which implies that, for all $x \in c_0$, $(\lambda_0 I - (Q + A))(\phi(\lambda_0)(I - A\phi(\lambda_0))^{-1})x = x$. Thus $\lambda_0 I - \widetilde{Q}$ is surjective on c_0. But Li (2003, Theorem 6.1) has proved that, for a general q-matrix Q, the minimal Q-function is Feller provided $\lambda I - Q$ is surjective on c_0. Using this result, we know that the minimal $\widetilde{Q}$-function $\widetilde{P}(t)$ is Feller. The converse is similarly proved by considering the decomposition $Q = \widetilde{Q} + (-A)$. $\square$

Proof of Theorem 2.1 Let Q be a quasi-monotone q-matrix, then $Q = \widetilde{Q} + A$ such that $\widetilde{Q}$ is a monotone q-matrix, and A is Feller and row-sum bounded. By the Feller criteria for monotone q-matrices (Li, 2006, Theorem 4.3), the minimal $\widetilde{Q}$-function is Feller if and only if $\widetilde{Q}$ is either Feller and strong zero-entrance or nonzero-exit. Thus the desired conclusion follows from Lemma 2.1–2.3. $\square$

§3. The Feller Property for Generalized Branching Processes

We turn to the generalized branching q-matrix Q defined by (1.1)–(1.3). Apply Theorem 2.1 to this q-matrix, we have

Proposition 3.1 A generalized branching process is a Feller process (which means that its transition function is Feller) if and only if either Q is zero-entrance or Q is not regular.

Proof Q has a decomposition $Q = \widetilde{Q} + A$ such that $\widetilde{Q}$ is the corresponding absorbing q-matrix (i.e. $\widetilde{q}_{0j} = 0$ and $\widetilde{q}_{ij} = q_{ij}$ for $i \geq 1$ and all $j \in \mathbb{Z}^+$). It is easy to verify that $\widetilde{Q}$ is monotone (i.e. satisfies (2.2)), and A is obviously Feller and row-sum bounded. Thus Q is quasi-monotone. Therefore the required conclusion follows from Theorem 2.1. $\square$

Furthermore, We need to check two conditions in Proposition 3.1. Some results on regularity and zero-entrance have been obtained by Chen (1997), Chen (2002a, b), Zhang et al (2001), Chen et al (2005, 2006), in virtue of the following generating function of the sequence $\{b_j\}$ in (1.2).

$$B(s) = \sum_{j=0}^{\infty} b_j s^j, \qquad 0 \leq s \leq 1. \tag{3.1}$$

Here we gather their results as follows.

Lemma 3.1 Suppose Q is a generalized branching q-matrix given by (1.1)–(1.3). We have

(i) If $\theta > 1$, then Q is regular if and only if $B'(1) \leq 0$;

(ii) If $\theta \leq 1$, then Q is regular provided $B'(1) \leq 0$;

(iii) If $\theta \leq 1$, then Q is always zero-entrance;

(iv) If $\theta > 1$ and $B'(1) > 0$, then the absorbing q-matrix ($h_0 = 0$) is zero-entrance.

We'll make zero-entrance more clear. Let us consider another generating function (instead of $B(s)$)

$$C(s) = \sum_{n=1}^{\infty} c_n s^n, \qquad 0 \leq s \leq 1, \tag{3.2}$$

where

$$c_n = -\frac{1}{b_0} \sum_{j=0}^{n} b_j, \qquad n \geq 1. \tag{3.3}$$

By (1.2) and (1.3), we know that $0 \leq c_n \downarrow 0$ and $c_1 > 0$. Thus $C(s)$ is a nonnegative increasing function and has radius r of convergence with $r \geq 1$. It is easy to verify that

$$B(s) = b_0(1-s)(1-C(s)) \ \text{ or equivelently } \ C(s) = 1 - \frac{B(s)}{b_0(1-s)}, \qquad 0 \leq s < 1. \tag{3.4}$$

Thus the key condition $B'(1) < 0, = 0, > 0, = +\infty$ are equivalent to $C(1) < 1, = 1, > 1, = +\infty$ respectively (we see that the derivative disappears).

Proposition 3.2 A generalized branching process is a Feller process if and only if its q-matrix Q given by (1.1)–(1.3) is zero-entrance, which is also equivalent to

$$R=\sum_{n=1}^{\infty}R_n=+\infty, \tag{3.5}$$

where R_n is defined by $R_0=1$ and

$$R_n=\frac{1}{(n+1)^{\theta}}\Big(1+\sum_{m=0}^{n-1}(m+1)^{\theta}c_{n-m}R_m\Big),\qquad n\geq 1. \tag{3.6}$$

Furthermore, we have

(i) if $\theta\leq 1$, then the process is a Feller process;

(ii) if $\theta>1$, then whether the process is a Feller process or not according to either $C(1)<1$ or $C(1)>1$.

Proof We first show that Q is zero-entrance if and only if $R=+\infty$. Indeed, let $\widetilde{Q}$ be the corresponding absorbing q-matrix. Then it follows from Lemma 2.2 that Q is zero-entrance if and only if $\widetilde{Q}$ is. By Chen et al (2005, Theorem 3), a downwards skip-free q-matrix is zero-entrance if and only if $R=\sum\limits_{n=1}^{\infty}R_n=+\infty$, with $R_0=R_{-1}=1$ and

$$R_n=\frac{1}{q_{n+1,n}}\Big(1+\sum_{m=0}^{n}\sum_{k=n+1}^{\infty}q_{mk}R_{m-1}\Big),\qquad n\geq 1. \tag{3.7}$$

Applying this result to $\widetilde{Q}$ (not Q) and by (3.3), we can rewrite (3.7) as follows

$$R_n=\frac{1}{(n+1)^{\theta}}\Big(\frac{1}{b_0}+\sum_{m=1}^{n}m^{\theta}c_{n-m+1}R_{m-1}\Big),\qquad n\geq 1. \tag{3.8}$$

Here b_0 can be taken value 1, the reason is that $\widetilde{Q}$ and $(1/b_0)\cdot\widetilde{Q}$ have the same zero-entrance property and have the same sequences $\{c_n\}$. Then R_n has the formulation in (3.6). We have shown that $\widetilde{Q}$ and thus Q is zero-entrance if and only if $\sum\limits_{n=1}^{\infty}R_n=+\infty$ where R_n are given by (3.6).

If now $\theta\leq 1$, by Lemma 3.1 (iii), Q is zero-entrance. This is also from (3.5)–(3.6).

If $\theta>1$ and $C(1)>1$, then by Lemma 3.1 (iv), the absorbing q-matrix $\widetilde{Q}$ is zero-entrance, and thus Q is zero-entrance by the above proof.

If $\theta>1$ and $C(1)<1$, we claim that Q is nonzero-entrance. Indeed, let

$$R(s)=\sum_{n=1}^{\infty}R_ns^n\qquad\text{and}\qquad A(s)=\sum_{n=1}^{\infty}\frac{1}{(n+1)^{\theta}}s^n. \tag{3.9}$$

Then by (3.6) it is easy to show that $R(s)\leq A(s)+C(s)(R(s)+1)$, that is, $(1-C(s))R(s)\leq A(s)+C(s)$. Noting that $1-C(s)>0$ for $0\leq s\leq 1$ (as $C(1)<1$) and $A(1)<+\infty$ (as $\theta>1$), we see that $R=R(1)\leq(A(1)+C(1))/(1-C(1))<+\infty$.

Finally, we notice that if $C(1)>1$ then Q is always zero-entrance. Thus, by Lemma 3.1 (i) (ii), it is impossible that Q is neither regular nor zero-entrance. This, together with Proposition 3.1, implies that the process is Feller if and only if Q is zero-entrance. □

Remark 2 By the above proof, we see that Q is either regular or zero-entrance. Thus the minimal Q-function is in fact the unique Q-function satisfying the forward equations: $P'(t)=P(t)Q$. This fact has been proved by Chen (2002a) only in the absorbing cases.

There is also a remainder case with $C(1)=1$, $\theta>1$ in Proposition 3.2. To make it clearly, we establish a criterion in terms of integrals.

Proposition 3.3 Suppose $C(1)\le 1$, then the process has the Feller property if and only if

$$I(\theta)=\int_0^1\frac{(1-s)^{\theta-2}}{1-C(s)}\,\mathrm{d}s=+\infty. \tag{3.10}$$

Proof By Proposition 3.2, we have only to prove that the series (3.5) and the integral $I(\theta)$ have the same convergence. Let $T_n=(n+1)^\theta R_n$, $n\ge 0$. Then (3.6) can be read as

$$T_n=1+\sum_{m=0}^{n-1}c_{n-m}T_m,\qquad n\ge 1. \tag{3.11}$$

Let $T(s)=\sum\limits_{n=1}^{\infty}T_ns^n$, $0\le s<1$, Then by (3.11) we have $T(s)=1/(1-s)-1+C(s)(1+T(s))$. Noting that $1-C(s)>0$ for $0\le s<1$ (as $C(1)\le 1$), we have

$$\sum_{n=1}^{\infty}(n+1)^\theta R_ns^n=T(s)=\frac{1}{(1-s)(1-C(s))}-1,\qquad 0\le s<1. \tag{3.12}$$

Multiplying $(1-s)^{\theta-1}$ to the both sides of (3.12) and integrating it from $s=0$ to $s=1$, we find

$$\sum_{n=1}^{\infty}(n+1)^\theta R_n\int_0^1(1-s)^{\theta-1}s^n\mathrm{d}s=\int_0^1\frac{(1-s)^{\theta-2}}{1-C(s)}\,\mathrm{d}s-\int_0^1(1-s)^{\theta-1}\mathrm{d}s. \tag{3.13}$$

Noting that

$$\int_0^1(1-s)^{\theta-1}s^n\mathrm{d}s=B(\theta,n+1)=\frac{\Gamma(\theta)\Gamma(n+1)}{\Gamma(\theta+n+1)}, \tag{3.14}$$

where $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ denote the Beta and Gamma functions respectively, we can rewrite (3.13) as follows

$$\sum_{n=1}^{\infty}\frac{(n+1)^\theta\Gamma(n+1)}{\Gamma(\theta+n+1)}R_n=\frac{1}{\Gamma(\theta)}\int_0^1\frac{(1-s)^{\theta-2}}{1-C(s)}\,\mathrm{d}s-\frac{1}{\Gamma(\theta+1)}. \tag{3.15}$$

We need to prove that

$$\lim_{n\to\infty}\frac{n^\theta\Gamma(n)}{\Gamma(\theta+n)}=1,\qquad \text{for every fixed }\ \theta>0. \tag{3.16}$$

Indeed, first, (3.16) holds obviously for $\theta=m$ (any integer). We then prove (3.16) when $\theta\ge 2$. To this end, we let

$$f(\theta):=\frac{\Gamma(\theta+n)}{n^\theta}=\frac{1}{n^\theta}\int_0^\infty x^{\theta+n-1}e^{-x}\mathrm{d}x=n^n\int_0^\infty y^{\theta+n-1}e^{-ny}\mathrm{d}y. \tag{3.17}$$

Differentiating it twice at θ with $\theta>0$, we have

$$f''(\theta)=n^n\int_0^\infty y^{\theta+n-1}e^{-ny}(\log y)^2\mathrm{d}y>0.$$

Thus $f'(\theta)$ is increasing. But obviously $f(1)<f(2)$, then by the mean value theorem, there is a $\theta_0\in(1,2)$ such that $f'(\theta_0)>0$ and thus $f'(\theta)>0$ for $\theta\ge 2$ at least. Therefore $f(\theta)$ is increasing in $[2,+\infty)$. Given now a $\theta\ge 2$, we can choose an integer m such that $m\ge\theta$. Then

$$\frac{n^m\Gamma(n)}{\Gamma(n+m)}\le\frac{n^\theta\Gamma(n)}{\Gamma(\theta+n)}\le\frac{n^2\Gamma(n)}{\Gamma(2+n)},$$

which proved that (3.16) holds for all $\theta\ge 2$ If $0<\theta<2$, then

$$\lim_{n\to\infty}\frac{n^\theta\Gamma(n)}{\Gamma(\theta+n)}=\lim_{n\to\infty}\frac{n^{\theta+2}\Gamma(n)}{\Gamma(\theta+2+n)}\cdot\frac{(\theta+1+n)(\theta+n)}{n^2}=1.$$

Thus we have proved (3.16). Using (3.16) we see from (3.15) that the series $\sum R_n$ and the integral $I(\theta)$ in (3.10) have the same convergence as required. □

As a corollary, we can obtain the strong ergodicity criterion. The criterion (iv) in the following result has been obtained in Chen (2002b) by using the probabilistic method.

Corollary 3.1 Let Q be a generalized branching q-matrix with resurrection ($h_0\ne 0$) and assume $C(1)\le 1$. Then the following statements are equivalent to each other.

(i) The process is strongly ergodic;

(ii) The process is not a Feller process;

(iii) $R=\sum\limits_{n=1}^{\infty}R_n<+\infty$, where R_n is defined by (3.6);

(iv) $I(\theta)<+\infty$, where $I(\theta)$ is the integral given by (3.10).

Proof Considering a decomposition $Q=\widetilde{Q}+A$ such that $\widetilde{Q}$ is also a q-matrix whose first row is $(h_0,-h_0,0,0,\cdots)$ and whose other rows preserve the same elements as Q, then $A=Q-\widetilde{Q}$ is obviously Feller and row-sum bounded matrix (not necessarily q-matrix). It is easy to verify that $\widetilde{Q}$ is monotone (i.e. satisfies (2.2)) and regular (by the assumption

$C(1) \le 1$). Thus the minimal $\widetilde{Q}$-function $\widetilde{P}(t)$ is monotone. By Chen (2002b), $\widetilde{P}(t)$ is also (ordinary) ergodic (since $\widetilde{Q}$ is of finite range). By Zhang et al (2001, Theorem 2.2), the monotone and ergodic $\widetilde{P}(t)$ is strongly ergodic if and only if $\widetilde{P}(t)$ is not Feller, which is, by Lemma 2.3, equivalent to that the minimal Q-function $P(t)$ is not Feller. But the strong ergodicity does not depend on the sequence $\{h_j\}$ as pointed out in Chen (2002b), that is, $P(t)$ is strongly ergodic if and only if $\widetilde{P}(t)$ is. Thus we have proved (i) $\Leftrightarrow$ (ii). The other conclusions follows from Proposition 3.2 and 3.3. □

Using the above two propositions, we get our final result stated as follows.

Theorem 3.1 Let Q be a generalized branching q-matrix defined by (1.1)–(1.3). We have

(i) if $C(1) > 1$, then the process is Feller;

(ii) if $C(1) < 1$, then the process is Feller if and only if $0 < \theta \le 1$;

(iii) if $C(1) = 1$ and $C'(1) < +\infty$, then the process is Feller if and only if $0 < \theta \le 2$;

(iv) if $C(1) = 1$ and $C'(1) = +\infty$, then there is also a separating point $\theta_0 \in [1, 2]$ whether the process has the Feller property or not according to $\theta > \theta_0$ or $\theta < \theta_0$.

Proof (i) and (ii) follows from Proposition 3.2, (iii) follows easily from Proposition 3.3. We now prove (iv). It is easy to see that the integral $I(\theta)$ given by (3.10) is a decreasing function if $\theta > 0$. Thus a separating point θ_0 indeed exist. But $I(\theta) = +\infty$ if $\theta \le 1$, while $I(\theta) < +\infty$ if $\theta > 2$. Therefore the separating point $\theta_0 \in [1, 2]$ as required. □

Finally we give two examples to illustrate that the separating point θ_0 can be taken any values in $[1, 2]$.

Example 1 For $0 < a < 1$, let

$$c_n = \frac{(-1)^{n+1}}{n!}\, a(a-1)\cdots(a-n+1), \qquad n \ge 1.$$

Then $C(s) = 1 - (1-s)^a$, and thus $0 < c_n \downarrow 0$, $C(1) = 1$, and $C'(1) < +\infty$. The integral $I(\theta)$ is

$$I(\theta) = \int_0^1 (1-s)^{\theta-2-a} \mathrm{d}s.$$

Then $I(\theta) = +\infty$ if $\theta \le 1 + a$, while $I(\theta) < +\infty$ if $\theta > 1 + a$. Therefore the separating point $\theta_0 = 1 + a \in (1, 2)$.

Example 2 Even in the case (iv), let θ_0 take the value 2. The example is given by

$$c_n = \frac{1}{n(n+1)}, \qquad n \ge 1.$$

Then $C(1) = 1$ and $C'(1) = +\infty$. It is easy to verify that

$$1 - C(s) = 1 - \sum_{n=1}^{\infty} \frac{1}{n} s^n + \sum_{n=1}^{\infty} \frac{1}{n+1} s^n = \frac{(s-1)\log(1-s)}{s}, \qquad 0 < s < 1.$$

which implies that the integrals at $\theta = 2$

$$I(2) = \int_0^1 \frac{\mathrm{d}s}{1 - C(s)} \geq \frac{1}{2} \int_{1/2}^1 \frac{\mathrm{d}s}{(s-1)\log(1-s)} = -\frac{1}{2} \int_0^{1/2} \frac{\mathrm{d}s}{s \log s} = +\infty.$$

Remark 3 Although the generated branching processes is not necessary a Feller process, the process is always column continuous in the sense of Li (2007), that is, $P_{ij}(t) \to \delta_{ij}$ uniformly in i, as $t \downarrow 0$.

References

[1] Chen, R.R., An extended class of time-continuous branching processes, *J. Appl. Prob.*, **34**(1997), 14–23.

[2] Chen, A.Y., Uniqueness and extinction properties of generalized Markov branching processes, *J. Math. Anal. Appl.*, **274**(2002a), 482–494.

[3] Chen, A.Y., Ergodicity and stability of generalized Markov branching processes with resurrection, *J. Appl. Prob.*, **39**(2002b), 786–803.

[4] Zhang, H.J., Chen, A.Y., Lin, X. and Hou, Z.T., Strong ergodicity of monotone transition functions, *Statist. Prob. Letters.*, **55**(2001), 63–69.

[5] Chen, A.Y., Pollett, P., Zhang, H.J. and Cairns, B., Uniqueness criteria for continuous-time Markov chains with general transition structures, *Adv. Appl. Prob.*, **37**(2005), 1056–1074.

[6] Chen, A.Y., Li, J.P. and Ramesh, N.I., General Harris regularity criterion for nonlinear Markov branching processes, *Statist. Prob. Letters.*, **76**(2006), 446–452.

[7] Reuter, G.E.H. and Riley, P.W., The Feller property for Markov semigroups on a countable state space, *J. London Math. Soc.*, **5**(1972), 267–275.

[8] Zhang, H.J. and Chen A.Y., Stochastic comparability and dual Q-functions, *J. Math. Anal. Appl.*, **234**(1999), 482–499.

[9] Chen, A.Y., Applications of Feller-Reuter-Riley transition functions, *J. Math. Anal. Appl.*, **260**(2001), 439–456.

[10] Li, Y.R., Contraction integrated semigroups and their application to continuous-time Markov chains, *Acta Math. Sinica.*, **19**(2003), 605–618.

[11] Li, Y.R., Dual and Feller-Reuter-Riley transition functions, *J. Math. Anal. Appl.*, **313**(2006), 461–474.

[12] Li, Y.R., Column continuous transition functions, *J. Math. Anal. Appl.*, **328**(2007), 640–654.

[13] Li, Y.R., Pakes, A.G., Li, J. and Gu, A.H., The limit behavior of dual Markov branching processes, *J. Appl. Prob.*, **45**(2008), 176–189.

[14] Li, Y.R. and Li, J., Criteria for Feller transiton functions, *J. Math. Anal. Appl.*, **359**(2009), 653–665.

[15] Van Doorn, E.A. and Zeifman, A.I., Extinction probability in a birth–death process with killing, *J. Appl. Prob.*, **42**(2005), 185–198.

[16] Pakes, A.G., Explosive Markov branching processes: entrance laws and limiting behavior, *Adv. Appl. Prob.*, **25**(1993), 737–756.

[17] Anderson, W.J., *Continuous-Time Markov Chains*, Springer-Verlag, 1991.

[18] Yosida, K., *Function Analysis*, Springer-Verlag, 1978.

[19] Athreya, K.B. and Japers, P., *Classical and Modern Branching Processes*, Springer, 1996.

有恢复的生成分支过程的Feller性质

张秀珍　　李扬荣

(西南大学数学与统计学院, 重庆, 400715)

首先, 当Q是一个拟单调的q矩阵的时候, 我们找出最小的Q函数是一个Feller的转移函数的准则. 然后我们把这个结论应用于生成分支q矩阵并得到相应的生成分支过程的Feller准则. 特别地, 设θ是分支q矩阵中的非线性数, 总是存在一个分点θ_0满足$1 \leq \theta_0 \leq 2$或$\theta_0 < +\infty$使得生成分支过程是否是Feller的要依据$\theta < \theta_0$或者$\theta > \theta_0$.

关键词: 连续时间马尔科夫链, 生成分支过程, Feller过程, 生成分支q矩阵, q函数, q豫解函数.

学科分类号: O211.6.