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A Stochastic Maximum Principle for Optimal Control of Jump Diffusions and Applications to Finance *

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Abstract

An optimal control problem motivated by a portfolio and consumption choice problem in the financial market where the expected utility of the investor is assumed to be the Constant Relative Risk Aversion (CRRA) case is discussed. A local stochastic maximum principle is obtained in the jump-diffusion setting using classical variational method. The result is applied to make optimal portfolio and consumption choice strategy for the problem and the explicit optimal solution in the state feedback form is given.

Keywords: Stochastic maximum principle, optimal control, jump diffusions, portfolio and consumption choice, CRRA utility.

AMS Subject Classification: 49K45, 65K10, 91B28, 93E20.

§1. Introduction

Maximum principle for optimal control of stochastic systems has been studied for a long period, see [1]-[4]. But one of their former assumptions is that the functions in the cost functional satisfy the usual linear or quadratic growth conditions. Unfortunately, this requirement excludes at least one important case which arises from the portfolio and consumption choice problem in the financial market — the *constant relative risk aversion* (CRRA for short) case (see [5] for example).

Stochastic processes with random jumps have become increasing popular for modelling fluctuations in financial market, both for risk management and option pricing purposes (see [6]). The stochastic control problem with jump diffusions is encountered naturally in

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the financial market. For example, the analysis of price evolution does reveal sudden and rare breaks logically accounted for exogenous events on information. Such a behavior from probabilistic point of view is naturally modeled by jump-diffusion processes, that is, the processes governed by both Brownian motion and Poisson random measure. Stochastic maximum principles for optimal control of jump diffusions and their applications to finance are seen in [7], [8].

In this paper we discuss an optimal control problem motivated by a portfolio and consumption choice problem in the financial market with CRRA utility functional. A local stochastic maximum principle is obtained in the jump-diffusion setting using classical variational method. The result is applied to obtain the optimal portfolio and consumption choice strategy in the state feedback form explicitly.

§2. Stochastic Control Problem and Maximum Principle

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$ be a complete stochastic basis with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is generated by the following two mutually independent processes:

(i) A one-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$;

(ii) A Poisson random measure N on $\mathbf{E}\times[0,\infty)$, where $\mathbf{E}\subset\mathbf{R}/\{0\}$ is a nonempty open set equipped with its Borel field $\mathcal{B}(\mathbf{E})$, with compensator $\widehat{N}(\mathrm{d}e\mathrm{d}t)=\pi(\mathrm{d}e)\mathrm{d}t$, such that $\widetilde{N}(A\times[0,t])=(N-\widehat{N})(A\times[0,t])_{t\geq 0}$ is a martingale for all $A\in\mathcal{B}(\mathbf{E})$ satisfying $\pi(A)<\infty$. π is assumed to be a σ-finite measure on $(\mathbf{E},\mathcal{B}(\mathbf{E}))$ and is called the characteristic measure.

Let $\mathcal{H}$ be a finite-dimensional vector space and $T>0$ a fixed real number which is called time horizon. We denote by $\mathbf{L}^2(\Omega,\mathcal{F}_T;\mathcal{H})$ the space of $\mathcal{H}$-valued square-integrable $\mathcal{F}_T$-measurable random variables, by $\mathbf{L}^2_{\mathcal{F}}([0,T];\mathcal{H})$ the space of $\mathcal{H}$-valued square-integrable $\mathcal{F}_t$-adapted processes, by $\mathbf{L}^2_{\mathcal{F},p}([0,T];\mathcal{H})$ the space of $\mathcal{H}$-valued square-integrable $\mathcal{F}_t$-predictable processes, and by $\mathbf{F}^2_p([0,T];\mathcal{H})$ the space of $\mathcal{H}$-valued $\mathcal{F}_t$-predictable processes $f(\cdot,\cdot,\cdot)$ defined on $\Omega\times[0,T]\times\mathbf{E}$ such that $\mathsf{E}\displaystyle\int_{\mathbf{E}}\int_0^T|f(\cdot,t,e)|^2\pi(\mathrm{d}e)\mathrm{d}t<\infty$.

Suppose we have a financial market consisting of two investment possibilities:

(i) A risk-free security (e.g. a bond), whose price $S_0(t)$ at time t is given by

$$\mathrm{d}S_0(t)=\rho_t S_0(t)\mathrm{d}t,\qquad S_0(0)>0,\tag{2.1}$$

where ρ_t is a bounded deterministic function;

(ii) A risky security (e.g. a stock), whose price $S_1(t)$ at time t is given by

$$\mathrm{d}S_1(t)=S_1(t-)\Big[\mu_t\mathrm{d}t+\sigma_t\mathrm{d}B(t)+\int_{\mathbf{E}}\eta_t(e)\widetilde{N}(\mathrm{d}e\mathrm{d}t)\Big],\qquad S_1(0)>0,\tag{2.2}$$

where $\mu_t, \sigma_t \neq 0, \eta_t(e)$ are bounded deterministic functions and $\mu_t > \rho_t$. To ensure that $S_1(t) > 0$ for all t we assume that $\eta_t(e) > -1$, $\forall\, e \in \mathbf{E}$ and in addition we assume that $\int_{\mathbf{E}} \eta_\cdot^2(e)\pi(\mathrm{d}e)$ is a bounded function.

Let $v(t) \doteq \theta_1(t)S_1(t)$ denote the amount invested in the risky security at time t which we called portfolio. We shall also allow the investor to withdraw consumption from his or her wealth with a consumption rate process $c(t)$ at time t. Given the initial wealth $x(0) = x_0 > 0$, combining (2.1) and (2.2) we can get the wealth dynamics

$$\begin{cases} \mathrm{d}x(t) = [\rho_t x(t) + (\mu_t - \rho_t)v(t) - c(t)]\mathrm{d}t + \sigma_t v(t)\mathrm{d}B(t) \\ \qquad\qquad + \displaystyle\int_{\mathbf{E}} \eta_t(e)v(t-)\widetilde{N}(\mathrm{d}e\mathrm{d}t), \\ x(0) = x_0. \end{cases} \tag{2.3}$$

The investor wants to maximize his/her expected utility

$$J(v(\cdot), c(\cdot)) \doteq \mathsf{E}\Big[\int_0^T g(c(t), t)\mathrm{d}t + h(x(T), T)\Big] \tag{2.4}$$

with

$$g(c,t) \doteq Le^{-\beta t}\frac{c^{1-R}}{1-R}, \qquad h(x,T) \doteq K\frac{x^{1-R}}{1-R}, \tag{2.5}$$

by choosing an appropriate portfolio-consumption pair $(v^*(\cdot), c^*(\cdot))$ over some admissible portfolio-consumption pairs set $\mathbf{U}_{ad}$. In the above $L, K, \beta > 0$ and $R \in (0,1)$ which is called the Arrow-Pratt *measure of risk aversion* (see [9]).

In this paper we study the following optimal control problem which is a generalization of the above problem

$$\begin{cases} \mathrm{d}x(t) = b(t, x(t), v(t), c(t))\mathrm{d}t + \sigma(t, x(t), v(t), c(t))\mathrm{d}B(t) \\ \qquad\qquad + \displaystyle\int_{\mathbf{E}} \gamma(t, x(t-), v(t), c(t), e)\widetilde{N}(\mathrm{d}e\mathrm{d}t), \\ x(0) = x_0, \end{cases} \tag{2.6}$$

where $b : [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R} \to \mathbf{R}$, $\sigma : [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R} \to \mathbf{R}$, $\gamma : [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{E} \to \mathbf{R}$.

We notice that the functions g and h in the expected utility (2.4) do not satisfy the linear or quadratic growth conditions. We treat such a case using classical variational method and obtain a local maximum principle. We apply the maximum principle obtained to problem (2.3)-(2.4) to get the explicit optimal portfolio and consumption choice strategy in the state feedback form.

For convenience we rewrite the CRRA type cost functional as

$$J(v(\cdot), c(\cdot)) \doteq \mathsf{E}\Big[\int_0^T Le^{-\beta t}\frac{c(t)^{1-R}}{1-R}\mathrm{d}t + K\frac{x(T)^{1-R}}{1-R}\Big]. \tag{2.7}$$

We assume that

$$\begin{cases} b,\sigma,\gamma \text{ are continuously differentiable with respcet to } (x,v,c) \\ \text{and their derivatives are bounded.} \end{cases} \tag{H2.1}$$

Let $\mathbf{U}=\mathbf{U}_1\times\mathbf{U}_2$ be a nonempty convex subset of $\mathbf{R}^2$. We define the admissible control (portfolio and consumption choice strategy) set

$$\mathbf{U}_{ad}=\big\{(v(\cdot),c(\cdot))\in\mathbf{L}^2_{\mathcal{F},p}([0,T];\mathbf{U}_1)\times\mathbf{L}^2_{\mathcal{F},p}([0,T];\mathbf{U}_2);(v(t),c(t))\in\mathbf{U},\text{a.e.},\text{a.s.}\big\}.$$

An admissible control $(v^*(\cdot),c^*(\cdot))$ is called optimal if it attains the maximum of $J(v(\cdot),c(\cdot))$ in (2.7). Under assumption (H2.1), we know equation (2.6) admits a unique solution $x(\cdot)\in L^2_{\mathcal{F}}([0,T];\mathbf{R})$ for the given $(x_0,(v(\cdot),c(\cdot)))\in\mathbf{R}\times\mathbf{U}_{ad}$ (see [10]). We call such $x(\cdot)$ the corresponding trajectory.

In order to derive the variational inequality, we need

$$c(t)>0,\text{a.e.},\text{a.s.};\quad x(T)>0,\text{a.s.};\quad \mathsf{E}\Big[\int_0^T c(t)^{-2R}\mathrm{d}t\Big]<\infty;\quad \mathsf{E}[x(T)^{-2R}]<\infty. \tag{H2.2}$$

Let $(v^*(\cdot),c^*(\cdot))$ be an optimal control for (2.6)-(2.7), and $x^*(\cdot)$ the corresponding optimal trajectory. Let $(v(\cdot),c(\cdot))\in\mathbf{L}^2_{\mathcal{F},p}([0,T];\mathbf{U}_1)\times\mathbf{L}^2_{\mathcal{F},p}([0,T];\mathbf{U}_2)$ be given such that $(v^*(\cdot)+v(\cdot),c^*(\cdot)+c(\cdot))\in\mathbf{U}_{ad}$. We take $v^\rho(\cdot)=v^*(\cdot)+\rho v(\cdot)$, $c^\rho(\cdot)=c^*(\cdot)+\rho c(\cdot)$, $0\le\rho\le1$. Since $\mathbf{U}_{ad}$ is convex, then $(v^\rho(\cdot),c^\rho(\cdot))\in\mathbf{U}_{ad}$. We denote by $x^\rho(\cdot)$ the trajectory of the control system (2.6) corresponding to $(v^\rho(\cdot),c^\rho(\cdot))$.

We introduce the following variational equation

$$\begin{cases} \mathrm{d}x^{1,\rho}(t)=[b_x(t,x^*(t),v^*(t),c^*(t))x^{1,\rho}(t)+b_v(t,x^*(t),v^*(t),c^*(t))v(t) \\ \qquad\qquad +b_c(t,x^*(t),v^*(t),c^*(t))c(t)]\mathrm{d}t \\ \qquad\qquad +[\sigma_x(t,x^*(t),v^*(t),c^*(t))x^{1,\rho}(t)+\sigma_v(t,x^*(t),v^*(t),c^*(t))v(t) \\ \qquad\qquad +\sigma_c(t,x^*(t),v^*(t),c^*(t))c(t)]\mathrm{d}B(t) \\ \qquad\qquad +\displaystyle\int_{\mathbf{E}}[\gamma_x(t,x^*(t),v^*(t),c^*(t),e)x^{1,\rho}(t-)+\gamma_v(t,x^*(t),v^*(t), \\ \qquad\qquad c^*(t),e)v(t)+\gamma_c(t,x^*(t),v^*(t),c^*(t),e)c(t)]\widetilde{N}(\mathrm{d}e\mathrm{d}t), \\ x^{1,\rho}(0)=0. \end{cases} \tag{2.8}$$

Let $\widetilde{x}^\rho(t)=\rho^{-1}(x^\rho(t)-x^*(t))-x^{1,\rho}(t)$. Under (H2.1) we can derive the following estimate

$$\lim_{\rho\to0}\sup_{0\le t\le T}\mathsf{E}|\widetilde{x}^\rho(t)|^2=0, \tag{2.9}$$

whose proof is similar to [11] and omitted here. Consequently we have

Lemma 2.1 We assume (H2.1), (H2.2) hold. Then we have the following variational inequality

$$\mathsf{E}\Big[\int_0^T Le^{-\beta t}c^*(t)^{1-R}c(t)\mathrm{d}t + Kx^*(T)^{-R}x(T)\Big] \le 0. \tag{2.10}$$

Proof Because $J(v^\rho(\cdot), c^\rho(\cdot)) - J(v^*(\cdot), c^*(\cdot)) \le 0$, we have

$$\mathsf{E}\Big\{\int_0^T \frac{L}{1-R}e^{-\beta t}[c^\rho(t)^{1-R} - c^*(t)^{1-R}]\mathrm{d}t + \frac{K}{1-R}[x^\rho(T)^{-R} - x^*(T)^{-R}]\Big\} \le 0. \tag{2.11}$$

We first manipulate the first term of (2.11). Denoting $A_t \doteq \{(t,\omega) : c(t) \ge 0\}$ and I_A the characteristic function of set A we have

$$\mathsf{E}\int_0^T \frac{L}{1-R}e^{-\beta t}[c^\rho(t)^{1-R} - c^*(t)^{1-R}]\mathrm{d}t = I_1 + I_2, \tag{2.12}$$

with

$$\begin{aligned} I_1 &= \mathsf{E}\int_0^T I_{A_t}\frac{L}{1-R}e^{-\beta t}[c^\rho(t)^{1-R} - c^*(t)^{1-R}]\mathrm{d}t, \\ I_2 &= \mathsf{E}\int_0^T I_{A_t^c}\frac{L}{1-R}e^{-\beta t}[c^\rho(t)^{1-R} - c^*(t)^{1-R}]\mathrm{d}t. \end{aligned}$$

For I_1, by the Taylor formula, we have

$$\begin{aligned} I_1 \quad = \quad & \rho\mathsf{E}\int_0^T I_{A_t}Le^{-\beta t}c^*(t)^{-R}c(t)\mathrm{d}t \\ & + \rho\mathsf{E}\int_0^T I_{A_t}Le^{-\beta t}[(c^*(t) + \theta\rho c(t))^{-R} - c^*(t)^{-R}]c(t)\mathrm{d}t \end{aligned} \tag{2.13}$$

with some $\theta \in (0,1)$. Since

$$\begin{aligned} &\lim_{\rho\to 0} I_{A_t}L^2e^{-2\beta t}[(c^*(t) + \theta\rho c(t))^{-R} - c^*(t)^{-R}]^2 = 0, \\ &|I_{A_t}L^2e^{-2\beta t}[(c^*(t) + \theta\rho c(t))^{-R} - c^*(t)^{-R}]^2| \le 4L^2c^*(t)^{-2R}, \end{aligned}$$

it follows from the Lebesgue controlled convergence theorem that

$$\lim_{\rho\to 0}\mathsf{E}\int_0^T I_{A_t}Le^{-\beta t}[(c^*(t) + \theta\rho c(t))^{-R} - c^*(t)^{-R}]c(t)\mathrm{d}t = 0.$$

Thus from (2.13), we have that

$$I_1 = \rho\mathsf{E}\int_0^T I_{A_t}Le^{-\beta t}c^*(t)^{-R}c(t)\mathrm{d}t + o(\rho). \tag{2.14}$$

For I_2, we can derive that

$$\begin{aligned} I_2 &= \mathsf{E}\int_0^T I_{A_t^c}\frac{L}{1-R}e^{-\beta t}c^*(t)^{1-R}\Big[\Big(1+\rho\frac{c(t)}{c^*(t)}\Big)^{1-R}-1\Big]\mathrm{d}t \\ &= \rho\mathsf{E}\int_0^T I_{A_t^c}Le^{-\beta t}c^*(t)^{-R}c(t)\mathrm{d}t+\rho^2 I(\rho), \end{aligned} \tag{2.15}$$

with

$$\begin{aligned} I(\rho) &= \mathsf{E}\int_0^T I_{A_t^c}\frac{L}{1-R}e^{-\beta t}c^*(t)^{1-R}\Big[\frac{\alpha(\alpha-1)}{2!}\Big(\frac{c(t)}{c^*(t)}\Big)^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!} \\ &\quad \Big(\frac{c(t)}{c^*(t)}\Big)^3\rho+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\Big(\frac{c(t)}{c^*(t)}\Big)^n\rho^{n-2}\Big]\mathrm{d}t \end{aligned}$$

and $\alpha\doteq 1-R>0$. Since

$$\mathsf{E}\int_0^T |c^*(t)|^{1-R}\mathrm{d}t\le C\Big[\mathsf{E}\int_0^T |c^*(t)|^2\mathrm{d}t\Big]^{(1-R)/2}<+\infty,$$
$$\mathsf{E}\int_0^T |c^1(t)|^{1-R}\mathrm{d}t\le C\Big[\mathsf{E}\int_0^T |c^1(t)|^2\mathrm{d}t\Big]^{(1-R)/2}<+\infty,$$

and noticing that

$$I_{A_t^c}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\Big(\frac{c(t)}{c^*(t)}\Big)^n\rho^{n-2}\le 0,\qquad n=2,3,\cdots,$$

we have that for small ρ,

$$\begin{aligned} 0\le -I(\rho)\le -I(1) &= \mathsf{E}\int_0^T I_{A_t^c}Le^{-\beta t}c^*(t)^{-R}c(t)\mathrm{d}t \\ &\quad +\mathsf{E}\int_0^T I_{A_t^c}\frac{L}{1-R}e^{-\beta t}[c^*(t)^{1-R}-c^1(t)^{1-R}]\mathrm{d}t<+\infty. \end{aligned}$$

Thus, it follows from (2.15) that

$$I_2=\rho\mathsf{E}\int_0^T I_{A_t^c}Le^{-\beta t}c^*(t)^{-R}c(t)\mathrm{d}t+o(\rho). \tag{2.16}$$

Integrating (2.12), (2.14) with (2.16), it follows that

$$\mathsf{E}\int_0^T \frac{L}{1-R}e^{-\beta t}[c^\rho(t)^{1-R}-c^*(t)^{1-R}]\mathrm{d}t=\rho\mathsf{E}\int_0^T Le^{-\beta t}c^*(t)^{-R}c(t)\mathrm{d}t+o(\rho). \tag{2.17}$$

Thanks to (2.9), similarly we can manipulate the second term of (2.10) and have

$$\frac{K}{1-R}\mathsf{E}[x^\rho(T)^{1-R}-x^*(T)^{1-R}]=\rho\mathsf{E}[Kx^*(T)^{-R}x(T)]+o(\rho). \tag{2.18}$$

Therefore, (2.11), (2.17) and (2.18) yield

$$\rho\mathsf{E}\Big[\int_0^T Le^{-\beta t}c^*(t)^{-R}c(t)\mathrm{d}t+Kx^*(T)^{-R}x(T)\Big]+o(\rho)\le 0. \tag{2.19}$$

Taking $\rho \to 0$ in (2.19), then (2.10) holds. $\square$

We introduce the following adjoint equation

$$\begin{cases} -\mathrm{d}p^*(t) = \Big[b_x(t,x^*(t),v^*(t),c^*(t))p^*(t) + \sigma_x(t,x^*(t),v^*(t),c^*(t))q^*(t) \\ \qquad\qquad + \displaystyle\int_{\mathbf{E}} \gamma_x(t,x^*(t),v^*(t),c^*(t),e)k^*(t,e)\pi(\mathrm{d}e)\Big]\mathrm{d}t \\ \qquad\qquad - q^*(t)\mathrm{d}B(t) - \displaystyle\int_{\mathbf{E}} k^*(t,e)\widetilde{N}(\mathrm{d}e\mathrm{d}t), \\ p^*(T) = Kx^*(T)^{-R}. \end{cases} \tag{2.20}$$

Under (H2.1), (H2.2) we know that there exists a unique triple $(p^*(\cdot),q^*(\cdot),k^*(\cdot,\cdot)) \in \mathbf{L}^2_{\mathcal{F}}([0,T];\mathbf{R}) \times \mathbf{L}^2_{\mathcal{F},p}([0,T];\mathbf{R}) \times \mathbf{L}^2_{\mathcal{F},p}([0,T];\mathbf{R})$ which satisfies (2.20) (see [7]).

Applying Itô's formula to $\langle p^*(t), x^{1,\rho}(t)\rangle$, it can be checked from Lemma 2.1 that

$$\begin{aligned} &\mathsf{E}\int_0^T [H_v(t,x^*(t),v^*(t),c^*(t),p^*(t),q^*(t),k^*(t,\cdot))v(t) \\ &+ H_c(t,x^*(t),v^*(t),c^*(t),p^*(t),q^*(t),k^*(t,\cdot))c(t)]\mathrm{d}t \le 0, \end{aligned}$$

where the Hamiltonian function $H : [0,T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined as

$$\begin{aligned} H(t,x,v,c,p,q,k(\cdot)) &\doteq \langle p, b(t,x,v,c)\rangle + \langle q, \sigma(t,x,v,c)\rangle \\ &\quad + \int_{\mathbf{E}} \langle k(e), \gamma(t,x,v,c,e)\rangle \pi(\mathrm{d}e) + Le^{-\beta t}\frac{c^{1-R}}{1-R}. \end{aligned} \tag{2.21}$$

So for any $(\overline{v}(\cdot),\overline{c}(\cdot)) \in \mathbf{U}$, we have

$$\begin{cases} \langle H_v(t,x^*(t),v^*(t),c^*(t),p^*(t),q^*(t),k^*(t,\cdot)), \overline{v}(t) - v^*(t)\rangle \le 0, \quad \text{a.e., a.s.,} \\ \langle H_c(t,x^*(t),v^*(t),c^*(t),p^*(t),q^*(t),k^*(t,\cdot)), \overline{c}(t) - c^*(t)\rangle \le 0, \quad \text{a.e., a.s..} \end{cases} \tag{2.22}$$

We have proved the following result.

Theorem 2.1 Supposed that (H2.1), (H2.2) hold. Let $(v^*(t),c^*(t))$ be an optimal control for the optimal control problem (2.6)-(2.7), $x^*(\cdot)$ the corresponding optimal trajectory and $(p^*(\cdot),q^*(\cdot),k^*(\cdot,\cdot))$ be the solution of adjoint equation (2.20). Then for any $(\overline{v}(\cdot),\overline{c}(\cdot)) \in \mathbf{U}$, the maximum condition (2.22) holds.

§3. Application to Portfolio/Consumption Choice Problem

In this section, we discuss the optimal portfolio and consumption choice problem (2.3)-(2.4) using the local maximum principle obtained in Section 2. Our target is to obtain the explicit solution for the optimal portfolio and consumption choice strategy in the state feedback form. Finally we can verify that the optimal solution satisfies condition (H2.2) indeed.

Noting in this case $\mathbf{U}_1 = \mathbf{R}$, $\mathbf{U}_2 = [0, \infty)$. The Hamiltonian function (2.21) gets the form

$$\begin{aligned} H(t,x,v,c,p,q,k(\cdot)) &= \langle p, \rho_t x + (\mu_t - \rho_t)v - c\rangle + \langle q, \sigma_t v\rangle \\ &\quad + \int_{\mathbf{E}} \langle k(e), \eta_t(e)v\rangle \pi(\mathrm{d}e) + Le^{-\beta t}\frac{c^{1-R}}{1-R}, \end{aligned} \tag{3.1}$$

and the adjoint equation (2.20) becomes

$$\begin{cases} -\mathrm{d}p^*(t) = \rho_t p^*(t)\mathrm{d}t - q^*(t)\mathrm{d}B(t) - \displaystyle\int_{\mathbf{E}} k^*(t,e)\widetilde{N}(\mathrm{d}e\mathrm{d}t), \\ p^*(T) = Kx^*(T)^{-R}. \end{cases} \tag{3.2}$$

Let $(v^*(\cdot), c^*(\cdot))$ be an optimal control, $x^*(\cdot)$ the corresponding optimal trajectory and $(p^*(\cdot), q^*(\cdot), k^*(\cdot,\cdot))$ the solution of adjoint equation (3.2). Since the expression involving v and c are both linear, the maximum conditions (2.22) suggest that

$$(\mu_t - \rho_t)p^*(t) + \sigma_t q^*(t) + \int_{\mathbf{E}} \eta_t(e) k^*(t,e)\pi(\mathrm{d}e) = 0, \tag{3.3}$$

and

$$c^*(t) = \left(\frac{1}{L}e^{\beta t}p^*(t)\right)^{-1/R}. \tag{3.4}$$

In order to get the optimal portfolio and consumption choice strategy $(v^*(\cdot), c^*(\cdot))$ explicitly, the usual method is to give an interpretation to $p^*(t)$ via the nonlinear Feynman-Kac formula then solve the corresponding partial differential equation using the maximum condition and the relationship between the maximum principle and the dynamic programming principle (see [12]). However, it is difficult to obtain the explicit solution of such PDE. It is reasonable to believe that it is more difficult to obtain the desired result in our jump-diffusion setting.

In fact, it is convenient to guess that it is optimal to consume at a rate proportional to the current wealth $x^*(t)$. By (3.4) this suggests that

$$p^*(t) = f(t)x^*(t)^{-R} \tag{3.5}$$

for some deterministic differentiable function f to be determined. Applying Itô's formula to (3.5) we get

$$\begin{aligned} \mathrm{d}p^*(t) \;=\;& \Big\{\dot{f}(t)x^*(t)^{-R} - Rf(t)\dot{f}(t)x^*(t)^{-R-1}[\rho_t x^*(t) + (\mu_t-\rho_t)v^*(t) - c^*(t)] \\ &+\frac{1}{2}R(R+1)f(t)x^*(t)^{-R-2}\sigma_t^2 v^*(t)^2 \\ &+\int_{\mathbf{E}} f(t)[(x^*(t)+\eta_t(e)v^*(t))^{-R} - x^*(t)^{-R} + Rx^*(t)^{-R-1}\eta_t(e)v^*(t)]\pi(\mathrm{d}e)\Big\}\mathrm{d}t \\ &-Rf(t)x^*(t)^{-R-1}\sigma_t v^*(t)\mathrm{d}B(t) \\ &+\int_{\mathbf{E}} f(t)[(x^*(t-)+\eta_t(e)v^*(t-))^{-R} - x^*(t-)^{-R}]\widetilde{N}(\mathrm{d}e\mathrm{d}t). \end{aligned} \tag{3.6}$$

Comparing (3.2) with (3.6), using (3.3) we get

$$q^*(t) = -Rf(t)\sigma_t v^*(t)x^*(t)^{-R-1}, \tag{3.7}$$

$$k^*(t,e) = f(t)x^*(t)^{-R}[(1+v^*(t)\eta_t(e)x^*(t)^{-1})^{-R} - 1], \tag{3.8}$$

and

$$\begin{cases} \dot{f}(t) + \alpha_t f(t) + R(Le^{-\beta t})^{1/R} f(t)^{1-1/R} = 0, \\ f(T) = K, \end{cases} \tag{3.9}$$

where

$$\begin{aligned} \alpha_t \;=\;& (1-R)\rho_t + R(\mu_t-\rho_t)v^*(t)x^*(t)^{-1} + \frac{1}{2}R(R+1)\sigma_t^2 v^*(t)^2 x^*(t)^{-2} \\ &+\int_{\mathbf{E}}[(1+\eta_t(e)v^*(t)x^*(t)^{-1})^{-R} - 1 + R\eta_t(e)v^*(t)x^*(t)^{-1})]\pi(\mathrm{d}e). \end{aligned} \tag{3.10}$$

Substituting (3.5), (3.7), 3.8 into (3.3), we get $F(v^*(t)x^*(t)^{-1}) = 0$, where

$$F(\chi) \doteq \mu_t - \rho_t - R\sigma_t^2\chi + \int_{\mathbf{E}} \eta_t(e)[(1+\eta_t(e)\chi)^{-R} - 1]\pi(\mathrm{d}e),$$

which is easily seen to have a zero $\chi^*(t) > 0$, i.e.:

$$F(\chi^*(t)) = 0. \tag{3.11}$$

With the choice of $v^*(t)x^*(t)^{-1} = \chi^*(t)$ and α_t given by (3.10), we can obtain that

$$f(t) = Le^{-\beta t}e^{\int_t^T(\beta-\alpha_s)\mathrm{d}s}\Big(\int_t^T -\frac{1}{L}e^{\beta s}e^{\int_s^T(\alpha_r-\beta)/R\mathrm{d}r}\mathrm{d}s + (Le^{-\beta T})^{-1/R}K^{1/R}\Big)^R. \tag{3.12}$$

Using (3.4) and (3.5) we get that

$$c^*(t) = (Le^{-\beta t})^{1/R} f(t)^{-1/R} x^*(t). \tag{3.13}$$

The corresponding wealth equation (2.3) for $x^*(t)$ becomes

$$\begin{cases} \mathrm{d}x^*(t) = x^*(t-)\Big\{(\rho_t + (\mu_t - \rho_t)\chi^*(t) - (Le^{-\beta t})^{1/R} f(t)^{-1/R})\mathrm{d}t \\ \qquad + \sigma_t \chi^*(t)\mathrm{d}B(t) + \displaystyle\int_{\mathbf{E}} \eta_t(e)\chi^*(t)\widetilde{N}(\mathrm{d}e\mathrm{d}t)\Big\}, \\ x^*(0) = x_0^* > 0. \end{cases} \tag{3.14}$$

The solution of this equation is

$$\begin{aligned} x^*(t) \;&=\; x_0^* \exp\Big\{\int_0^t \Big[\rho_s + (\mu_s - \rho_s)\chi^*(s) - (Le^{-\beta s})^{1/R} f(s)^{-1/R} - \frac{1}{2}\sigma_s^2 \chi^*(s)^2\Big]\mathrm{d}s \\ &\quad + \int_0^t \sigma_s \chi^*(s)\mathrm{d}B(s) + \int_0^t \Big[\int_{\mathbf{E}} \ln(1 + \chi^*(s-)\eta_s(e))N(\mathrm{d}e\mathrm{d}s) \\ &\quad - \ln(\chi^*(s)\eta_s(e))\pi(\mathrm{d}e)\mathrm{d}s\Big]\Big\}. \end{aligned} \tag{3.15}$$

Finally, we can check that $\forall\, t \in [0,T]$, $x^*(t) > 0$ and $\mathsf{E}[x^*(T)^{-2R}] < \infty$. So $\forall\, t \in [0,T]$, $c^*(t) > 0$ and $\mathsf{E}\big[\displaystyle\int_0^T c^*(t)^{-2R}\mathrm{d}t\big] < \infty$. Therefore, assumption (H2.2) holds indeed. Therefore we have the following theorem.

Theorem 3.1 The optimal solution $(v^*(\cdot), c^*(\cdot))$ to the portfolio and consumption choice problem (2.3)-(2.4) is given in state feedback form by

$$\begin{cases} v^*(t,x) = \chi^*(t)x, \\ c^*(t,x) = (Le^{-\beta t})^{1/R} f(t)^{-1/R} x, \end{cases} \tag{3.16}$$

where $\chi^*(t)$ given by (3.11) and $f(t)$ given by (3.12).

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跳扩散最优控制的随机最大值原理及在金融中的应用

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讨论了由金融市场中投资组合和消费选择问题引出的一类最优控制问题, 投资者的期望效用是常数相对风险厌恶(CRRA)情形. 在跳扩散框架下, 利用古典变分法得到了一个局部随机最大值原理. 结果应用到最优投资组合和消费选择策略问题, 得到了状态反馈形式的显式最优解.

关键词: 随机最大值原理, 最优控制, 跳扩散, 投资组合和消费选择, CRRA效用.

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