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The Time Value of Absolute Ruin for a General Risk Model

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Abstract

In this paper, we study the absolute ruin problems in a compound Poisson risk process. The integro-differential equations for the expected discounted penalty functions are derived, and some explicit expressions are given when the claims are exponentially distributed. Finally, by a 'renewal' argument, we obtain the explicit expression for the probability of the recovery when the claims are exponentially distributed.

Keywords: Absolute ruin, expected discounted penalty function, integro-differential equation, probability of recovery.

AMS Subject Classification: 91B30, 60J25.

§1. Introduction

Consider the following compound Poisson surplus process of an insurance company

$$U(t) = u + ct - S(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus; $c > 0$ is the rate of premium; $\{N(t), t \geq 0\}$ is a Poisson claim-number process with intensity $\lambda > 0$; representing the claim amounts are i.i.d. random variables with common distribution function $F(x) = 1 - \overline{F}(x) = \mathsf{P}(X \leq x)$, density f and mean μ; $S(t) = \sum\limits_{i=1}^{N(t)} X_i$ is the aggregate claim process. Finally, we assume that $\{N(t)\}$ and $\{X_i\}$ are mutually independent.

The time to ruin for this model is defined as

$$T = \inf\{t : U(t) < 0\}, \quad \text{or} \quad \infty \ \text{otherwise}, \tag{1.2}$$

and the corresponding ruin probability is defined as $\psi(u) = \mathsf{P}(T < \infty | U(0) = u)$.

In the insurance context, various definitions of 'ruin' do exist. In most of these cases, the zero-level ruin estimate defined as above plays an important role, however, it really

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can't reflect the surplus flow more accurate. In this paper, we consider a more general model proposed by Embrechts and Schmidli (1994). We assume that, whenever the surplus is negative or the company is on deficit, it could borrow an amount of money equal to the deficit at a debit interest force $\delta' > 0$; whenever the surplus is positive, the company could earn interest with force $\delta > 0$ for capital above a certain level $b \geq 0$, where b is the amount of capital the company retains as a liquid reserve. In this model, the surplus process, denoted by $U_g(t)$ with $U_g(0) = u$, can be expressed as

$$\mathrm{d}U_g(t) = \begin{cases} c\mathrm{d}t + \delta' U_g(t)\mathrm{d}t - \mathrm{d}S(t), & U_g(t) < 0; \\ c\mathrm{d}t - \mathrm{d}S(t), & 0 \leq U_g(t) \leq b; \\ c\mathrm{d}t + \delta(U_g(t) - b)\mathrm{d}t - \mathrm{d}S(t), & U_g(t) > b. \end{cases} \tag{1.3}$$

In this paper, we call model (1.1) and (1.3) classical risk model and general risk model, respectively. The main difference between these two models is the definition of the time to ruin, since the insurer's business could go on when the surplus becomes negative in model (1.3). Note that the surplus is no longer able to become positive when the negative surplus attains the level $-c/\delta'$ or is bellow $-c/\delta'$, because the debits of the insurer at this time are greater than or equal to c/δ' that is the present value at that time for all premium income available after then. Then the time to absolute ruin for model (1.3) is defined as

$$T_g = \inf\Big\{t : U_g(t) \leq -\frac{c}{\delta'}\Big\}, \qquad \infty \text{ otherwise},$$

and the absolute ruin probability is defined as

$$\psi_g(u) = \mathsf{P}(T_g < \infty | U_g(0) = u).$$

Furthermore, let $U_g(T_g-)$ and $|U_g(T_g)|$ denote the surplus immediately before absolute ruin and the deficit at absolute ruin, respectively. Note that the deficit is at least c/δ' and the surplus immediately before absolute ruin could be in the range of $(-c/\delta', \infty)$.

Let $\omega(x_1, x_2)$ be a nonnegative measurable function defined on $(-c/\delta', \infty) \times [c/\delta', \infty)$. For a constant number $\alpha \geq 0$, we introduce the following well-known Gerber-Shiu expected discounted penalty function at absolute ruin

$$\Phi_g(u) = \mathsf{E}[e^{-\alpha T_g}\omega(U_g(T_g-), |U_g(T_g)|)I(T_g < \infty)|U_g(0) = u], \tag{1.4}$$

where $u > -c/\delta'$; $I(A)$ is an indicator function of an event A. The study of the Gerber-Shiu function has become a standard method to study the (absolute) ruin related quantities. See Gerber and Shiu (1998), Cai (2004, 2007), Cai and Dickson (2002), Yang and Zhang (2008).

We point out that $\Phi_g(u)$ behaves differently depending on the initial surplus $-c/\delta' < u < 0$, $0 \le u \le b$ and $u > b$. Hence, we distinguish these three situations by writing

$$\Phi_g(u) = \begin{cases} \Phi_{g_1}(u), & -c/\delta' < u < 0; \\ \Phi_{g_2}(u), & 0 \le u \le b; \\ \Phi_{g_3}(u), & u > b. \end{cases}$$

Similarly, we write $\psi_g(u) = \psi_{g_1}(u)$ for $-c/\delta' < u < 0$, $\psi_g(u) = \psi_{g_2}(u)$ for $0 \le u \le b$ and $\psi_g(u) = \psi_{g_3}(u)$ for $u > b$. Note that the following relation holds $\psi_{g_3}(u) \le \psi(u)$, $u > b$.

In this paper, we assume that the net profit condition $c > \lambda\mu$ holds, which implies that $\lim\limits_{u\to\infty} \psi_{g_3}(u) = 0$. Also, we assume that $\lim\limits_{u\to\infty} \Phi_{g_3}(u) = 0$. In particular, a sufficient condition for this assumption is that $\omega(x_1, x_2)$ is a bounded function.

Recently, absolute ruin has received considerable attention in the actuarial literature. For references, see Dassios and Embrechts (1989), Embrechts and Schmidli (1994), Dickson and Egidio dos Reis (1997), Cai (2007) and references therein. In this paper, we study the expected discounted penalty function (1.4) for model (1.3), which enables us to analyze many quantities related to absolute ruin, such as the absolute ruin probability, the Laplace transform of the time to absolute ruin, the deficit at absolute ruin and the surplus immediately before absolute ruin.

§2. Integro-Differential Equations

In this section, we derive the integro-differential equations for $\Phi_g(u)$.

Theorem 2.1 When $-c/\delta' < u < 0$,

$$\Phi'_{g_1}(u) = \frac{\lambda+\alpha}{\delta' u + c}\Phi_{g_1}(u) - \frac{\lambda}{\delta' u + c}\Big[\int_0^{u+c/\delta'} \Phi_g(u-x)\mathrm{d}F(x) + A(u)\Big], \tag{2.1}$$

when $0 \le u \le b$,

$$\Phi'_{g_2}(u) = \frac{\lambda+\alpha}{c}\Phi_{g_2}(u) - \frac{\lambda}{c}\Big[\int_0^{u+c/\delta'} \Phi_g(u-x)\mathrm{d}F(x) + A(u)\Big], \tag{2.2}$$

and when $u > b$,

$$\Phi'_{g_3}(u) = \frac{\lambda+\alpha}{\delta(u-b)+c}\Phi_{g_3}(u) - \frac{\lambda}{\delta(u-b)+c}\Big[\int_0^{u+c/\delta'} \Phi_g(u-x)\mathrm{d}F(x) + A(u)\Big], \tag{2.3}$$

where $A(u) = \displaystyle\int_{u+c/\delta'}^{\infty} \omega(u, x-u)\mathrm{d}F(x)$.

Proof First, let $\bar{s}_{\bar{t}|}^{(\delta)} = \int_0^t e^{\delta v}\mathrm{d}v$ and denote the solution of the following equation

$$h_{\delta'}(t,u) = ue^{\delta' t} + c\Big(\frac{e^{\delta' t}-1}{\delta'}\Big) = 0,$$

by $t_0 = t_0(u)$, then we have $t_0 = \ln[c/(c+\delta' u)]^{1/\delta'}$.

Now we consider the case $-c/\delta' < u < 0$. From the definition of t_0, we know that the surplus returns to zero level at time t_0 if no claim occurs prior to time t_0. Furthermore, $h_{\delta'}(t,u) < 0$ is the surplus at time $t < t_0$ if no claim occurs prior to time t_0. Thus, by conditioning on the time and amount of the first claim, we obtain when $-c/\delta' < u < 0$

$$\begin{aligned}\Phi_{g_1}(u) &= \int_0^{t_0} \lambda e^{-(\lambda+\alpha)t}\gamma(h_{\delta'}(t,u))\mathrm{d}t + \int_{t_0}^{t_0+b/c} \lambda e^{-(\lambda+\alpha)t}\gamma(c(t-t_0))\mathrm{d}t \\ &\quad + \int_{t_0+b/c}^{\infty} \lambda e^{-(\lambda+\alpha)t}\gamma\big(c\bar{s}_{\overline{t-t_0-b/c}|}^{(\delta)} + b\big)\mathrm{d}t,\end{aligned}$$

where $\gamma(u) = \int_0^{u+c/\delta'} \Phi_g(u-x)\mathrm{d}F(x) + A(u)$.

A change of variables in above equation leads to

$$\begin{aligned}\Phi_{g_1}(u) &= \lambda(\delta' u + c)^{(\lambda+\alpha)/\delta'}\int_u^0 (\delta' s + c)^{-(\lambda+\alpha)/\delta' - 1}\gamma(s)\mathrm{d}s \\ &\quad + \lambda c^{-(\lambda+\alpha)/\delta'-1}(\delta' u + c)^{(\lambda+\alpha)/\delta'}\int_0^b e^{-(\lambda+\alpha)s/c}\gamma(s)\mathrm{d}s + \lambda c^{(\lambda+\alpha)(1/\delta - 1/\delta')} \\ &\quad \times (\delta' u + c)^{(\lambda+\alpha)/\delta'} e^{-(\lambda+\alpha)b/c}\int_b^\infty (\delta(s-b)+c)^{-(\lambda+\alpha)/\delta - 1}\gamma(s)\mathrm{d}s. \end{aligned} \tag{2.4}$$

Differentiating (2.4) with respect to u yields (2.1).

Similarly, for $0 \le u \le b$, we can obtain the following integral equation,

$$\Phi_{g_2}(u) = \int_0^{(b-u)/c} \lambda e^{-(\lambda+\alpha)t}\gamma(u+ct)\mathrm{d}t + \int_{(b-u)/c}^{\infty} \lambda e^{-(\lambda+\alpha)t}\gamma\big(c\bar{s}_{\overline{t-(b-u)/c}|}^{(\delta)} + b\big)\mathrm{d}t.$$

A change of variables in above equation leads to

$$\begin{aligned}\Phi_{g_2}(u) &= \frac{\lambda}{c}\int_u^b e^{-(\lambda+\alpha)(s-u)/c}\gamma(s)\mathrm{d}s \\ &\quad + \lambda c^{(\lambda+\alpha)/\delta} e^{-(\lambda+\alpha)(b-u)/c}\int_b^\infty (\delta(s-b)+c)^{-(\lambda+\alpha)/\delta-1}\gamma(s)\mathrm{d}s.\end{aligned} \tag{2.5}$$

For $u > b$,

$$\Phi_{g_3}(u) = \int_0^\infty \lambda e^{-(\lambda+\alpha)t}\gamma\big((u-b)e^{\delta t} + c\bar{s}_{\bar{t}|}^{(\delta)} + b\big)\mathrm{d}t.$$

A change of variables in above equation leads to

$$\Phi_{g_3}(u) = \lambda(\delta(u-b)+c)^{(\lambda+\alpha)/\delta}\int_u^\infty (\delta(s-b)+c)^{-(\lambda+\alpha)/\delta-1}\gamma(s)\mathrm{d}s. \tag{2.6}$$

Finally, differentiating (2.5) and (2.6) gives (2.2) and (2.3), respectively. □

We remark that the expected discounted penalty function $\Phi_g(u)$ is continuous at $u=0$ and $u=b$, i.e.

$$\Phi_{g_1}(0-)=\Phi_{g_2}(0+), \qquad \Phi'_{g_2}(b-)=\Phi'_{g_3}(b+), \tag{2.7}$$

which can be obtained by setting $u=0$ in (2.4) and (2.5), and $u=b$ in (2.5) and (2.6), respectively. Furthermore, it is easy to see from (2.1), (2.2), (2.3) and above boundary conditions that

$$\Phi'_{g_1}(0-)=\Phi'_{g_2}(0+), \qquad \Phi'_{g_2}(b-)=\Phi'_{g_3}(b+), \tag{2.8}$$

which implies that $\Phi_g(u)$ is differentiable at $u=0$ and $u=b$.

Proposition 2.1 If

$$\lim_{u\downarrow -c/\delta'}\int_u^0 (c+\delta' s)^{-(\lambda+\alpha)/\delta'-1}A(s)\mathrm{d}s=\infty,$$

then

$$\lim_{u\downarrow -c/\delta'}\Phi_{g_1}(u)=\frac{\lambda}{\lambda+\alpha}A(-c/\delta'). \tag{2.9}$$

If

$$\lim_{u\downarrow -c/\delta'}\int_u^0 (c+\delta' s)^{-(\lambda+\alpha)/\delta'-1}A(s)\mathrm{d}s<\infty,$$

then

$$\lim_{u\downarrow -c/\delta'}\Phi_{g_1}(u)=0. \tag{2.10}$$

Proof Set

$$\begin{aligned} H(u) \;=\;& \lambda c^{-(\lambda+\alpha)/\delta'-1}(\delta' u+c)^{(\lambda+\alpha)/\delta'}\int_0^b e^{-(\lambda+\alpha)s/c}\gamma(s)\mathrm{d}s+\lambda c^{(\lambda+\alpha)(1/\delta-1/\delta')} \\ & \times(\delta' u+c)^{(\lambda+\alpha)/\delta'}e^{-(\lambda+\alpha)b/c}\int_b^\infty (\delta(s-b)+c)^{-(\lambda+\alpha)/\delta-1}\gamma(s)\mathrm{d}s. \end{aligned}$$

By noting that $\lim\limits_{u\downarrow -c/\delta'}H(u)=0$ we can complete the proof by the same arguments as that of Proposition 2.1 in Cai (2007). □

All the boundary conditions obtained above are necessary for one to solve the integro-differential equations satisfied by $\Phi_g(u)$. In Section 3, we will give some examples to illustrate the solution procedure.

§3. Explicit Results for Exponential Claims

In this section, we study two special cases in which explicit expressions for $\Phi_g(u)$ can be obtained by solving boundary-value problems.

Example 1 Let $\alpha = 0$, $F(x) = 1 - e^{-x/\mu}$ for $\mu > 0$, $\omega(x_1, x_2) = \omega(x_2)$. In this case, it is easy to check that $\mu A'(u) + A(u) = 0$. Then, integro-differential equations (2.1), (2.2) and (2.3) are equivalent to the following differential equations for $i = 1, 2, 3$,

$$\Phi''_{g_i}(u) + p_i(u)\Phi'_{g_i}(u) = 0, \tag{3.1}$$

where

$$p_1(u) = \frac{\delta' u + c + \mu(\delta' - \lambda)}{\mu(\delta' u + c)}, \qquad p_2(u) = \frac{c - \lambda\mu}{\mu c}, \qquad p_3(u) = \frac{\delta(u-b) + c + \mu(\delta - \lambda)}{\mu(\delta(u-b) + c)}.$$

The general solution of (3.1) is of the following form

$$\Phi_{g_i}(u) = C_{i1} + C_{i2} P_i(u), \tag{3.2}$$

with

$$\begin{aligned} P_1(u) &= \int_0^u e^{-x/\mu}(c + \delta' x)^{-1+\lambda/\delta'} \mathrm{d}x, \\ P_2(u) &= \frac{\mu c}{c - \lambda\mu}(1 - e^{-(c-\lambda\mu)u/(\mu c)}), \\ P_3(u) &= \int_0^{u-b} e^{-x/\mu}(c + \delta x)^{-1+\lambda/\delta} \mathrm{d}x, \end{aligned}$$

where C_{i1}, C_{i2} are arbitrary constants to be determined.

It is easy to see that the first condition in Proposition 2.1 holds under the conditions in Example 1, then by (2.9) we obtain

$$C_{11} + C_{12} P_1(-c/\delta') = A(-c/\delta'). \tag{3.3}$$

By the boundary conditions in (2.7), and noting $P_1(0) = P_2(0) = 0$, $P_3(b) = 0$ we have

$$C_{11} = C_{21}, \tag{3.4}$$

$$C_{21} + C_{22} P_2(b) = C_{31}. \tag{3.5}$$

While the boundary conditions in (2.8) lead to

$$c^{-1+\lambda/\delta'} C_{12} = C_{22}, \tag{3.6}$$

$$e^{-(c-\lambda\mu)/(\mu c)} C_{22} = c^{-1+\lambda/\delta} C_{32}. \tag{3.7}$$

Finally, by the assumption $\Phi_{g_3}(u) \to 0$ as $u \to \infty$, we obtain

$$C_{31} + C_{32}P_3(\infty) = 0. \tag{3.8}$$

Let $H = P_1(-c/\delta') - c^{-1+\lambda/\delta'}P_2(b) - P_3(\infty)c^{\lambda/\delta'-\lambda/\delta}e^{-(c-\lambda\mu)b/(\mu c)}$. Solving equations (3.3)-(3.8) gives

$$\begin{aligned}
C_{11} &= C_{21} = \Big(1 - \frac{P_1(-c/\delta')}{H}\Big)A(-c/\delta'),\\
C_{12} &= \frac{1}{H}A(-c/\delta'), \qquad C_{22} = \frac{c^{-1+\lambda/\delta'}}{H}A(-c/\delta'),\\
C_{31} &= -\frac{c^{\lambda/\delta'-\lambda/\delta}e^{-(c-\lambda\mu)b/(\mu c)}P_3(\infty)}{H}A(-c/\delta'),\\
C_{32} &= \frac{c^{\lambda/\delta'-\lambda/\delta}e^{-(c-\lambda\mu)b/(\mu c)}}{H}A(-c/\delta').
\end{aligned}$$

Hence, the Gerber-Shiu discounted penalty functions $\Phi_{g_i}(u)$, $i = 1, 2, 3$, are given by

$$\Phi_{g_1}(u) = \Big(1 + \frac{P_1(u) - P_1(-c/\delta')}{H}\Big)A(-c/\delta'), \qquad \frac{c}{\delta'} < u < 0, \tag{3.9}$$

$$\Phi_{g_2}(u) = \Big(1 + \frac{c^{-1+\lambda/\delta'}P_2(u) - P_1(-c/\delta')}{H}\Big)A(-c/\delta'), \qquad 0 \le u \le b, \tag{3.10}$$

$$\Phi_{g_3}(u) = \frac{c^{\lambda/\delta'-\lambda/\delta}e^{-(c-\lambda\mu)b/(\mu c)}(P_3(u) - P_3(\infty))}{H}A(-c/\delta'), \qquad u > b. \tag{3.11}$$

Proposition 3.1 Let $\alpha = 0$, $F(x) = 1 - e^{-x/\mu}$ for $\mu > 0$, $\omega(x_1, x_2) = \omega(x_2)$. Then the Gerber-Shiu discounted function can be expressed as

$$\Phi_g(u) = \psi_g(u)A(-c/\delta'). \tag{3.12}$$

Proof Set $\omega(\cdot) \equiv 1$, then $A(-c/\delta') = 1$ and $\Phi_g(u)$ is reduced to $\psi_g(u)$. Hence, we can obtain (3.12) from (3.9)-(3.11). $\square$

Proposition 3.1 shows that the Gerber-Shiu discounted penalty function is proportional to the absolute ruin probability under conditions in Example 1. This is true, due to the property of the lack of memory of exponential distribution, for some penalty functions about the deficit at absolute ruin, such as the distribution, the Laplace transform and the moments of the deficit at absolute ruin.

Set $\omega(x_2) = I(x_2 \le y)$ for $y \ge c/\delta'$, we obtain

$$G_g(y, u) = \mathsf{P}(|U_g(T_g)| \le y, T_g < \infty | U_g(0) = u),$$

which is the (defective) distribution function of the deficit at absolute ruin when absolute ruin occurs. From (3.12), we obtain

$$G_g(y, u) = \psi_g(u)(1 - e^{c/(\delta'\mu)}e^{-y/\mu}).$$

Then the conditional distribution function of the deficit at absolute ruin, given that absolute ruin occurs, satisfies for $y \geq c/\delta'$,

$$\mathsf{P}(|U_g(T_g)| \leq y | T_g < \infty) = \frac{G_g(y,u)}{\psi_g(u)} = (1 - e^{c/(\delta'\mu)} e^{-y/\mu}).$$

While the expected deficit at absolute ruin, given that absolute ruin occurs, is given by

$$\mathsf{E}(|U_g(T_g)||T_g < \infty) = \frac{1}{\mu}\int_{c/\delta'}^{\infty} y e^{c/(\delta'\mu)} e^{-y/\mu} \mathrm{d}y = \mu + \frac{c}{\delta'},$$

which is the formula (5.23) in Cai (2007).

Now we consider several limit cases.

Case 1 $b \to \infty$. We denote the Gerber-Shiu discounted penalty function by $\Phi_{\infty,-}(u)$ and $\Phi_{\infty,+}(u)$ for $-c/\delta' < u < 0$ and $u \geq 0$, respectively. Note that as $b \to \infty$

$$P_2(b) \to \frac{\mu c}{c - \lambda\mu}, \qquad H \to P_1(-c/\delta') - \frac{\mu c^{\lambda/\delta'}}{c - \lambda\mu},$$

from which, and (3.9), (3.10), one obtains

$$\Phi_{\infty,-}(u) = \frac{1 + \dfrac{c-\lambda\mu}{\mu c}\displaystyle\int_u^0 e^{-x/\mu}(1+\delta' x/c)^{-1+\lambda/\delta'}\mathrm{d}x}{1 + \dfrac{c-\lambda\mu}{\mu c}\displaystyle\int_{-c/\delta'}^0 e^{-x/\mu}(1+\delta' x/c)^{-1+\lambda/\delta'}\mathrm{d}x} A(-c/\delta'), \quad -c/\delta' < u < 0,$$

$$\Phi_{\infty,+}(u) = \frac{e^{-(c-\lambda\mu)u/(\mu c)}}{1 + \dfrac{c-\lambda\mu}{\mu c}\displaystyle\int_{-c/\delta'}^0 e^{-x/\mu}(1+\delta' x/c)^{-1+\lambda/\delta'}\mathrm{d}x} A(-c/\delta'), \quad u \geq 0.$$

We remark that above two formulas recover the corresponding results obtained in Cai (2007).

Case 2 $b \to 0$. In this case, we denote the Gerber-Shiu discounted penalty function by $\Phi_{0-}(u)$ and $\Phi_{0+}(u)$ for $-c/\delta' < u < 0$ and $u \geq 0$ respectively. As $b \to 0$, we have

$$P_2(b) \to 0, \qquad H \to P_1(-c/\delta') - c^{\lambda/\delta' - \lambda/\delta} P_3(\infty).$$

Thus, one can obtain from (3.9) and (3.11) that

$$\Phi_{0-}(u) = \frac{c^{\lambda/\delta'-\lambda/\delta}\displaystyle\int_0^{\infty} e^{-x/\mu}(c+\delta x)^{-1+\lambda/\delta}\mathrm{d}x - P_1(u)}{c^{\lambda/\delta'-\lambda/\delta}\displaystyle\int_0^{\infty} e^{-x/\mu}(c+\delta x)^{-1+\lambda/\delta}\mathrm{d}x - P_1(-c/\delta')} A(-c/\delta'), \quad -c/\delta' < u < 0,$$

$$\Phi_{0+}(u) = \frac{c^{\lambda/\delta'-\lambda/\delta}\displaystyle\int_u^{\infty} e^{-x/\mu}(c+\delta x)^{-1+\lambda/\delta}\mathrm{d}x}{c^{\lambda/\delta'-\lambda/\delta}\displaystyle\int_0^{\infty} e^{-x/\mu}(c+\delta x)^{-1+\lambda/\delta}\mathrm{d}x - P_1(-c/\delta')} A(-c/\delta'), \quad u \geq 0.$$

Case 3 $\delta' \to \infty$. Note that in this case ruin occurs immediately when the surplus process first drops below zero. We denote the Gerber-Shiu discounted penalty function by $\Phi_{b-}(u)$ and $\Phi_{b+}(u)$ for $0 \le u \le b$ and $u > b$, respectively. As $\delta' \to \infty$ one obtains

$$A(u) \to w(u) := \int_u^{\infty} \omega(u, x-u)\mathrm{d}F(x),$$

and

$$P_1(u) \to 0, \qquad H \to -\frac{1}{c}P_2(b) - c^{-\lambda/\delta}e^{-(c-\lambda\mu)b/(\mu c)}P_3(\infty).$$

Thus, one can obtain from (3.10) and (3.11) that

$$\Phi_{b-}(u) = \frac{P_3(\infty) - \dfrac{\mu c^{\lambda/\delta}}{c-\lambda\mu}(1-e^{-(c-\lambda\mu)(u-b)/(\mu c)})}{P_3(\infty) - c^{-1+\lambda/\delta}P_2(b)}w(0), \qquad 0 \le u \le b,$$

$$\Phi_{b+}(u) = \frac{P_3(\infty) - P_3(u)}{P_3(\infty) - c^{-1+\lambda/\delta}P_2(b)}w(0), \qquad u > b.$$

Case 4 $\delta' \to \infty$, $b \to 0$. In this case, we obtain the classical compound Poisson model with constant interest force. We denote the Gerber-Shiu discounted penalty function by $\Phi_\delta(u)$ for $u \ge 0$. As $\delta' \to \infty$ and $b \to 0$, we have

$$P_1(u) \to 0, \qquad P_2(b) \to 0, \qquad H \to -c^{-\lambda/\delta}P_3(\infty).$$

One can obtain from (3.11) that

$$\Phi_\delta(u) = \frac{\displaystyle\int_u^{\infty} e^{-x/\mu}(c+\delta x)^{-1+\lambda/\delta}\mathrm{d}x}{\displaystyle\int_0^{\infty} e^{-x/\mu}(c+\delta x)^{-1+\lambda/\delta}\mathrm{d}x}w(0), \qquad u \ge 0.$$

Case 5 $\delta' \to \infty$, $b \to \infty$. In this case, we obtain the classical compound Poisson model. We denote the Gerber-Shiu discounted penalty function by $\Phi(u)$ for $u \ge 0$. As $\delta' \to \infty$, $b \to 0$, we have

$$P_1(u) \to 0, \qquad P_2(b) \to \frac{\mu c}{c-\lambda\mu}, \qquad H \to -\frac{\mu}{c-\lambda\mu}.$$

Finally, one can obtain from (3.10) that

$$\Phi(u) = e^{-(c-\lambda\mu)u/(\mu c)}w(0), \qquad u \ge 0.$$

§4. Probability of Recovery

In this section, we will discuss the probability of recovery for risk model (1.3). Let T^* be the time when the surplus process $U_g(t)$ first drops below zero, i.e.

$$T^* = \inf\{t > 0, U_g(t) < 0\}.$$

Denote by $\widehat{T}$ the time of recovery for risk model (1.3)

$$\widehat{T} := \inf\{t > 0, U_g(T^* + t) = 0\}, \tag{4.1}$$

and by $\phi(u)$ the probability of recovery

$$\phi(u) := \mathsf{P}(\widehat{T} < \infty | U_g(0) = u), \qquad u \geq 0. \tag{4.2}$$

We point out that $\phi(u)$ behaves differently depending on the initial surplus $0 \leq u \leq b$, $u > b$. Hence, we distinguish these two situations by writing

$$\phi(u) = \begin{cases} \phi_1(u), & 0 \leq u \leq b, \\ \phi_2(u), & u > b. \end{cases}$$

Denote by T^0_{-y} the first passage time for the surplus process to reach zero from a given surplus level $-y$, $0 < y < c/\delta'$. Define

$$\zeta(-y) := \mathsf{P}(T^0_{-y} < \infty | U_g(0) = -y), \qquad 0 < y < \frac{c}{\delta'}. \tag{4.3}$$

By the total expectation formula, one can easily obtain an expression for the probability of recovery $\phi(u)$ as follows:

$$\begin{aligned} \phi(u) &= \mathsf{P}(\widehat{T} < \infty | U_g(0) = u) \\ &= \int_0^{c/\delta'} \mathsf{P}(T^0_{-y} < \infty | U_g(0) = -y) \mathsf{P}(T^* < \infty | U_g(T^*) | \in \mathrm{d}y | U_g(0) = u). \end{aligned} \tag{4.4}$$

If $F(x) = 1 - e^{-x/\mu}$ for $\mu > 0$, from (3.9) in Yuan and Hu (2008), we conclude that

$$\zeta(-y) = \frac{\displaystyle\int_0^{-y+c/\delta'} e^{-t/\mu} t^{\lambda/\delta'-1} \mathrm{d}t}{\displaystyle\int_0^{c/\delta'} e^{-t/\mu} t^{\lambda/\delta'-1} \mathrm{d}t}, \qquad 0 \leq y \leq \frac{c}{\delta'}. \tag{4.5}$$

It is easy to obtain from Case 3 that

$$\mathsf{P}(T^* < \infty | U_g(T^*) | \in \mathrm{d}y | U_g(0) = u) = \begin{cases} \dfrac{M_1(u)}{\mu} e^{-y/\mu}, & 0 \leq u \leq b, \\ \dfrac{M_2(u)}{\mu} e^{-y/\mu}, & u > b. \end{cases} \tag{4.6}$$

Where

$$M_1(u) = \frac{P_3(\infty) - \dfrac{\mu c^{\lambda/\delta}}{c - \lambda\mu}(1 - e^{-(c-\lambda\mu)(u-b)/(\mu c)})}{P_3(\infty) - c^{-1+\lambda/\delta} P_2(b)}, \qquad M_2(u) = \frac{P_3(\infty) - P_3(u)}{P_3(\infty) - c^{-1+\lambda/\delta} P_2(b)}.$$

Consequently, substituting (4.5) and (4.6) into (4.4) yields

$$\phi_1(u) = M_1(u) \cdot \left(1 - \frac{\dfrac{\delta'}{\lambda}\left(\dfrac{c}{\delta'}\right)^{\lambda/\delta'}}{\displaystyle\int_0^{c/\delta'} e^{-(1/\mu)(t-c/\delta')} t^{\lambda/\delta'-1} \mathrm{d}t} \right), \qquad 0 \le u \le b,$$

and

$$\phi_2(u) = M_2(u) \cdot \left(1 - \frac{\dfrac{\delta'}{\lambda}\left(\dfrac{c}{\delta'}\right)^{\lambda/\delta'}}{\displaystyle\int_0^{c/\delta'} e^{-(1/\mu)(t-c/\delta')} t^{\lambda/\delta'-1} \mathrm{d}t} \right), \qquad u > b.$$

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一般风险模型的绝对破产时间

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本论文研究了关于复合Possion风险模型中绝对破产的问题. 得到了关于罚金折现期望函数的积分微分方程, 并在索赔函数为指数分布时, 得到了关于罚金折现期望函数的确切解. 最后, 作为一个新的讨论, 当索赔函数为指数分布时, 得到了关于恢复概率的确切值.

关键词: 绝对破产, 罚金折现期望函数, 积分微分方程, 恢复概率.

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