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On Equality of the BLUPs under Two Linear Models with New Observations *

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Abstract

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two linear models with new observations. Through matrix rank method, we derive the necessary and sufficient conditions for the best linear unbiased predictor (BLUP) of the new observation under the model $\mathcal{M}_1$ is also BLUP under the model $\mathcal{M}_2$. As applications, the conditions of equality of the BLUPs under two mixed linear models are also given.

Keywords: General linear model, mixed linear model, best linear unbiased predictor (BLUP), matrix rank method.

AMS Subject Classification: 62J05, 15A09.

§1. Introduction

Consider the general Gauss-Markov model

$$y = X\beta + \varepsilon, \tag{1.1}$$

where y is an $n \times 1$ observable random vector, X is a known $n \times p$ model matrix, β is a $p \times 1$ vector of unknown parameters, and ε is an $n \times 1$ random error vector.

Let y_f denote an $m \times 1$ unobservable random vector containing new observations (observable in future). New observations y_f are assumed to follow linear model

$$y_f = X_f\beta + \varepsilon_f, \tag{1.2}$$

where X_f is a known $m \times p$ model matrix associated with new observations, β is the same vector of unknown parameters as in Eq. (1.1), and ε_f is an $m \times 1$ random error vector associated with new observations. The expectation vector and the covariance matrix of

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$\begin{bmatrix} y \\ y_f \end{bmatrix}$ are

$$\mathsf{E}\begin{bmatrix} y \\ y_f \end{bmatrix} = \begin{bmatrix} X \\ X_f \end{bmatrix}\beta, \qquad \mathsf{Cov}\begin{bmatrix} y \\ y_f \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} := \Sigma, \tag{1.3}$$

respectively, where Σ is a known nonnegative definite matrix with the property of $\mathscr{R}(V_{12}) \subseteq \mathscr{R}(V_{11})$.

We use the notation

$$\mathcal{M}_1 = \left\{ \begin{bmatrix} y \\ y_f \end{bmatrix}, \begin{bmatrix} X\beta \\ X_f\beta \end{bmatrix}, \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \right\} \tag{1.4}$$

to describe the general Gauss-Markov model with new observations.

Further, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. For any $A \in \mathbb{R}^{m\times n}$, the symbols A', $A^\dagger$, $r(A)$, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote the transpose, the Moore-Penrose inverse, the rank, the range (column space) and the null space of a real matrix A, respectively. $E_X = I_n - XX^\dagger$ and $F_X = I_p - X^\dagger X$ stand for the two orthogonal projectors induced by X.

We assume the model $\mathcal{M}_1$ to be consistent in the sense that

$$y \in \mathscr{R}(X, V_{11}). \tag{1.5}$$

The linear predictor Gy is unbiased for y_f if the expected prediction error is 0 : $\mathsf{E}(y_f - Gy) = 0$. This is equivalent to $GX = X_f$, i.e.,

$$X_f' = X'G'. \tag{1.6}$$

This means that $X_f\beta$ is an estimable parametric function. Now an unbiased linear predictor Gy is called the best linear unbiased predictor, BLUP, for y_f, if the Lowner ordering

$$\mathsf{Cov}\,(Gy - y_f) \leq_L \mathsf{Cov}\,(Fy - y_f) \tag{1.7}$$

holds for all F such that Fy is an unbiased linear predictor for y_f, i.e., $\mathsf{Cov}\,(Fy - y_f) - \mathsf{Cov}\,(Gy - y_f)$ is a nonnegative definite matrix.

The following lemma characterizes the BLUP; for the proof, see Isotalo and Puntanen (2006).

Lemma 1.1 Let $\mathcal{M}_1$ be as given in Eq. (1.4). Then linear predictor Ty is the best linear unbiased predictor of y_f if and only if T satisfies the fundamental equation of the BLUP:

$$T[X, V_{11}E_X] = [X_f, V_{21}E_X], \tag{1.8}$$

in this case, the BLUP of y_f can be written as

$$\mathrm{BLUP}(y_f) = ([X_f, V_{21}E_X][X, V_{11}E_X]^{\dagger} + UE_{[X,V_{11}E_X]})y, \tag{1.9}$$

where $U \in \mathbb{R}^{m\times n}$ is arbitrary.

Consider now another linear model with new observations

$$\mathcal{M}_2 = \left\{ \begin{bmatrix} y \\ y_f \end{bmatrix}, \begin{bmatrix} \widehat{X}\beta \\ \widehat{X}_f\beta \end{bmatrix}, \begin{bmatrix} \widehat{V}_{11} & \widehat{V}_{12} \\ \widetilde{V}_{21} & \widehat{V}_{22} \end{bmatrix} \right\}. \tag{1.10}$$

This model differ from $\mathcal{M}_2$ through its covariance matrix and model matrix.

There is a rich literature on equality of ordinary least squares estimator, best linear unbiased estimator and best linear unbiased predictor in the general linear model, see Baksalary and Kala (1981), Elian (2000), Watson (1972) and Zhang and Lu (2004). Puntanen and Styan (1989) made a survey and presented various equivalent conditions. Tian and Wiens (2006) used matrix rank method to consider the equality of ordinary least squares estimator and best linear unbiased estimator, and obtained some new equivalent conditions. Recently, the present author (2009, 2011) used matrix rank method to reconsider the equality of ordinary least squares estimator, best linear unbiased estimator and best linear unbiased predictor, and gave some new equivalent conditions.

In this paper, we will use matrix rank method to investigate the equality of the BLUPs under linear model $\mathcal{M}_1$ and linear model $\mathcal{M}_2$. We will derive the necessary and sufficient conditions of the BLUP for y_f under the model $\mathcal{M}_1$ is also BLUP for y_f under the model $\mathcal{M}_2$. As applications, the conditions of equality of the BLUPs under two mixed linear models are also given. For the study of mixed linear model by matrix method, the readers can also refer to Fan and Wang (2008).

From the Eq. (1.9), we see that the BLUP for y_f is a matrix expression involving Moore-Penrose inverses. A powerful tool for simplifying matrix equality involving inverses and Moore-Penrose inverses is rank formulas for partitioned matrices. The following rank equalities for partitioned matrices due to Marsaglia and Styan (1974).

Lemma 1.2 Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$ and $C \in \mathbb{R}^{l\times n}$. Then

$$r[A, B] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.11}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.12}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C). \tag{1.13}$$

In particular,

$$r[A,B]=r(A)\Leftrightarrow E_AB=0\Leftrightarrow \mathscr{R}(B)\subseteq\mathscr{R}(A)\Leftrightarrow \mathscr{N}(A')\subseteq\mathscr{N}(B'),\tag{1.14}$$

$$r\begin{bmatrix}A\\C\end{bmatrix}=r(A)\Leftrightarrow CF_A=0\Leftrightarrow \mathscr{R}(C')\subseteq\mathscr{R}(A')\Leftrightarrow \mathscr{N}(A)\subseteq\mathscr{N}(C).\tag{1.15}$$

The following rank formula can be proved through Lemma 1.2, and has been applied in matrix analysis and statistics, for example., Tian (2007) and Liu (2009, 2011).

Lemma 1.3 Suppose A,B,C,D,P and Q are real matrices such that matrix expression $D-CP^{\dagger}AQ^{\dagger}B$ is well defined. Then

$$r(D-CP^{\dagger}AQ^{\dagger}B)=r\begin{bmatrix}P'AQ' & P'PP' & 0\\ Q'QQ' & 0 & Q'B\\ 0 & CP' & -D\end{bmatrix}-r(P)-r(Q).\tag{1.16}$$

§2. Equality of the BLUPs under Two Linear Model with New Observations

In this section, we will use the matrix rank method to derive some necessary and sufficient conditions of the equality for the BLUPs under linear model $\mathcal{M}_1$ and linear model $\mathcal{M}_2$.

Theorem 2.1 Let the linear models with new observations $\mathcal{M}_1$ and $\mathcal{M}_2$ be as given in Eq. (1.4) and Eq. (1.10). Then the following statements are equivalent:

(a) There exist an BLUP(y_f) under model $\mathcal{M}_1$ such that the BLUP(y_f) is also the BLUP of y_f under model $\mathcal{M}_2$,

(b)

$$r\begin{bmatrix}X_f & V_{21} & \widehat{X}_f & \widehat{V}_{21}\\ X & V_{11} & \widehat{X} & \widehat{V}_{11}\\ 0 & X' & 0 & 0\\ 0 & 0 & 0 & \widehat{X}'\end{bmatrix}=r[\,X\ \ V_{11}\ \ \widehat{X}\ \ \widehat{V}_{11}\,]+r(X)+r(\widehat{X}),\tag{2.1}$$

(c)

$$\mathscr{R}\begin{bmatrix}X_f'\\ V_{12}\\ \widehat{X}_f'\\ \widehat{V}_{12}\end{bmatrix}\subseteq\mathscr{R}\begin{bmatrix}X' & 0 & 0\\ V_{11} & X & 0\\ \widehat{X}' & 0 & 0\\ \widehat{V}_{11} & 0 & \widehat{X}\end{bmatrix}.$$

Proof We first prove the equivalent of (a) and (b). From Lemma 1.1, the representation of the BLUP for y_f is the Ty, where

$$T=[X_f, V_{21}E_X][X, V_{11}E_X]^{\dagger}+UE_{[X,V_{11}E_X]}.$$

Now there exist BLUP(y_f) under model $\mathcal{M}_1$ such that BLUP(y_f) is also the BLUP of y_f under model $\mathcal{M}_2$ if and only if the matrix equation

$$[X_f, V_{21}E_X][X, V_{11}E_X]^{\dagger}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}]+UE_{[X,V_{11}E_X]}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}]=[\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}}] \tag{2.2}$$

has a solution. From the solvability conditions of Eq. (2.2), we have

$$r\begin{bmatrix} E_{[X,V_{11}E_X]}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] \\ (\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}})-[X_f, V_{21}E_X][X, V_{11}E_X]^{\dagger}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] \end{bmatrix}=r\big[E_{[X,V_{11}E_X]}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}]\big]. \tag{2.3}$$

By applying Lemma 1.2 and some basic block operation, we have

$$\begin{aligned}
& r\begin{bmatrix} E_{[X,V_{11}E_X]}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] \\ (\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}})-[X_f, V_{21}E_X][X, V_{11}E_X]^{\dagger}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] \end{bmatrix} \\
= & r\begin{bmatrix} (\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}) & (X, V_{11}E_X) \\ (\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}})-(X_f, V_{21}E_X)(X, V_{11}E_X)^{\dagger}(\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}) & 0 \end{bmatrix} \\
& -r(X, V_{11}E_X) \\
= & r\begin{bmatrix} (\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}) & (X, V_{11}E_X) \\ (\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}}) & (X_f, V_{21}E_X) \end{bmatrix}-r(X, V_{11}E_X) \\
= & r\left[\begin{pmatrix} X \\ X_f \end{pmatrix} \ \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}E_X \ \begin{pmatrix} \widehat{X} \\ \widehat{X}_f \end{pmatrix} \ \begin{pmatrix} \widehat{V}_{11} \\ \widehat{V}_{21} \end{pmatrix}E_{\widehat{X}}\right]-r(X, V_{11}E_X) \\
= & r\begin{bmatrix} X & V_{11} & \widehat{X} & \widehat{V_{11}} \\ X_f & V_{21} & \widehat{X}_f & \widehat{V_{21}} \\ 0 & X' & 0 & 0 \\ 0 & 0 & 0 & \widehat{X}' \end{bmatrix}-r(X, V_{11}E_X)-r(X)-r(\widehat{X}).
\end{aligned}$$

Notice that, if A is nonnegative definite, the rank formula

$$r\begin{bmatrix} A & B & C \\ B' & 0 & 0 \end{bmatrix}=r[\,A \ \ B \ \ C\,]+r(B)$$

holds. Thus we have

$$\begin{aligned}
r[E_{[X,V_{11}E_X]}[\widehat{X},\widehat{V}_{11}E_{\widehat{X}}]] &= r\begin{bmatrix} X & V_{11}E_X & \widehat{X} & \widehat{V}_{11}E_{\widehat{X}} \end{bmatrix} - r(X, V_{11}E_X) \\
&= r\begin{bmatrix} V_{11} & X & \widehat{V}_{11}E_{\widehat{X}} & \widehat{X} \\ X' & 0 & 0 & 0 \end{bmatrix} - r(X) - r(X, V_{11}E_X) \\
&= r\begin{bmatrix} V_{11} & X & \widehat{V}_{11}E_{\widehat{X}} & \widehat{X} \end{bmatrix} - r(X, V_{11}E_X) \\
&= r\begin{bmatrix} \widehat{V}_{11} & \widehat{X} & V_{11} & X \\ \widehat{X}' & 0 & 0 & 0 \end{bmatrix} - r(\widehat{X}) - r(X, V_{11}E_X) \\
&= r\begin{bmatrix} V_{11} & X & \widehat{V}_{11} & \widehat{X} \end{bmatrix} - r(X, V_{11}E_X).
\end{aligned} \tag{2.4}$$

Substituting them into Eq. (2.3) yields Eq. (2.1).

Notice that

$$\begin{aligned}
r\begin{bmatrix} X & V_{11} & \widehat{X} & \widehat{V_{11}} \\ 0 & X' & 0 & 0 \\ 0 & 0 & 0 & \widehat{X}' \end{bmatrix} &= r\begin{bmatrix} X & V_{11}E_X & \widehat{X} & \widehat{V}_{11}E_{\widehat{X}} \end{bmatrix} + r(X) + r(\widehat{X}) \\
&= r\begin{bmatrix} V_{11} & X & \widehat{V}_{11} & \widehat{X} \end{bmatrix} + r(X) + r(\widehat{X}).
\end{aligned} \tag{2.5}$$

Eq. (1.15) in Lemma 1.2 yields the equivalent of (b) and(c). □

Theorem 2.2 Let the linear models with new observations $\mathcal{M}_1$ and $\mathcal{M}_2$ be as given in Eq. (1.4) and Eq. (1.10). Then the following statements are equivalent:

(a) Every BLUP(y_f) under model $\mathcal{M}_1$ such that the BLUP(y_f) is also the BLUP of y_f under model $\mathcal{M}_2$,

(b)

$$\mathscr{R}\begin{bmatrix} \widehat{X} \\ \widehat{X}_f \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X & V_{11}E_X \\ X_f & V_{21}E_X \end{bmatrix} \quad \text{and} \quad \mathscr{R}\begin{bmatrix} \widehat{V}_{11}E_{\widehat{X}} \\ \widehat{V}_{21}E_{\widehat{X}} \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X & V_{11}E_X \\ X_f & V_{21}E_X \end{bmatrix}.$$

(c)

$$r\begin{bmatrix} X_f & V_{21} & \widehat{X}_f & \widehat{V}_{21} \\ X & V_{11} & \widehat{X} & \widehat{V}_{11} \\ 0 & X' & 0 & 0 \\ 0 & 0 & 0 & \widehat{X}' \end{bmatrix} = r[X \ \ V_{11} \ \ \widehat{X}\,; \widehat{V}_{11}] + r(X) + r(\widehat{X}) \tag{2.6}$$

and

$$r[X \ \ V_{11} \ \ \widehat{X} \ \ \widehat{V}_{11}] = r[X \ \ V_{11}], \tag{2.7}$$

(d)

$$\mathscr{R}\begin{bmatrix} X_f' \\ V_{12} \\ \widehat{X}_f' \\ \widehat{V}_{12} \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X' & 0 & 0 \\ V_{11} & X & 0 \\ \widehat{X}' & 0 & 0 \\ \widehat{V}_{11} & 0 & \widehat{X} \end{bmatrix} \qquad \text{and} \qquad \mathscr{R}[\,\widehat{X}\ \ \widehat{V}_{11}\,] \subseteq \mathscr{R}[\,X\ \ V_{11}\,],$$

Proof We first prove the equivalent of (a) and (b). If every BLUP(y_f) under model $\mathcal{M}_1$ such that the BLUP(y_f) is also the BLUP of y_f under model $\mathcal{M}_2$, then for any U, Eq. (2.2) holds. This means that

$$E_{[X,V_{11}E_X]}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] = 0 \tag{2.8}$$

and

$$[X_f, V_{21}E_X][X, V_{11}E_X]^{\dagger}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] = [\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}}] \tag{2.9}$$

hold simultaneously. From Eq. (2.8), we see that

$$[X, V_{11}E_X][X, V_{11}E_X]^{\dagger}[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] = [\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}]. \tag{2.10}$$

From Eq. (2.9) and Eq. (2.10), we easily get

$$\mathscr{R}\begin{bmatrix} \widehat{X} & \widehat{V}_{11}E_{\widehat{X}} \\ \widehat{X}_f & \widehat{V}_{21}E_{\widehat{X}} \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X & V_{11}E_X \\ X_f & V_{21}E_X \end{bmatrix}. \tag{2.11}$$

The Eq. (2.11) yields (b).

Conversely, if (b) holds, then there are matrices K_1, K_2 and L_1, L_2 such that

$$\begin{bmatrix} \widehat{X} \\ \widehat{X}_f \end{bmatrix} = \begin{bmatrix} X & V_{11}E_X \\ X_f & V_{21}E_X \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \widehat{V}_{11}E_{\widehat{X}} \\ \widehat{V}_{21}E_{\widehat{X}} \end{bmatrix} = \begin{bmatrix} X & V_{11}E_X \\ X_f & V_{21}E_X \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

If Ty is an BLUP of y_f under the model $\mathcal{M}_1$, then T satisfies the Eq. (1.8). Hence we have

$$\begin{aligned} T[\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}] &= T[X, V_{11}E_X]\begin{bmatrix} K_1 & L_1 \\ K_2 & L_2 \end{bmatrix} \\ &= [X_f, V_{21}E_X]\begin{bmatrix} K_1 & L_1 \\ K_2 & L_2 \end{bmatrix} \\ &= [\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}}]. \end{aligned}$$

From Lemma 1.1, we have proved (a).

Next we derive the equivalent of (a) and (c). By applying Lemma 1.3 to the Eq. (2.9) and notice that the Eq. (2.8) and the Eq. (1.8), we have

$$\begin{aligned}
& r[(\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}}) - (X_f, V_{21}E_X)(X, V_{11}E_X)^{\dagger}(\widehat{X}, \widehat{V}_{11}E_{\widehat{X}})] \\
= \; & r\begin{bmatrix} (X, V_{11}E_X)'(X, V_{11}E_X)(X, V_{11}E_X)' & (X, V_{11}E_X)'(\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}) \\ (X_f, V_{21}E_X)(X, V_{11}E_X)' & (\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}}) \end{bmatrix} \\
& - r[X, V_{11}E_X] \\
= \; & r\begin{bmatrix} (X, V_{11}E_X) & (\widehat{X}, \widehat{V}_{11}E_{\widehat{X}}) \\ (X_f, V_{21}E_X) & (\widehat{X}_f, \widehat{V}_{21}E_{\widehat{X}}) \end{bmatrix} - r[X, V_{11}E_X] \\
= \; & r\left[\begin{pmatrix} X \\ X_f \end{pmatrix} \quad \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} E_X \quad \begin{pmatrix} \widehat{X} \\ \widehat{X}_f \end{pmatrix} \quad \begin{pmatrix} \widehat{V}_{11} \\ \widehat{V}_{21} \end{pmatrix} E_{\widehat{X}}\right] - r(X, V_{11}E_X) \\
= \; & r\begin{bmatrix} X & V_{11} & \widehat{X} & \widehat{V_{11}} \\ X_f & V_{21} & \widehat{X}_f & \widehat{V_{21}} \\ 0 & X' & 0 & 0 \\ 0 & 0 & 0 & \widehat{X}' \end{bmatrix} - r(X, V_{11}E_X) - r(X) - r(\widehat{X}). \qquad (2.12)
\end{aligned}$$

From the Eq. (2.4) and the Eq. (2.8), we get

$$r\begin{bmatrix} V_{11} & X & \widehat{V}_{11} & \widehat{X} \end{bmatrix} = r(X, V_{11}E_X). \qquad (2.13)$$

The Eq. (2.12) and the Eq. (2.13) yield (c). The equivalent of (c) and (d) follows from Lemma 1.2 and Eq. (2.5). $\square$

§3. Equality of the BLUPs under Two Mixed Linear Models

A mixed linear model can be presented as

$$y = X\beta + Z\gamma + \varepsilon, \qquad (3.1)$$

where $X \in \mathbb{R}^{n\times p}$ and $Z \in \mathbb{R}^{n\times q}$ are known matrices, $\beta \in \mathbb{R}^p$ is a vector of unknown fixed effects, γ is an unobservable vector of random effects with $\mathsf{E}(\gamma) = 0 \in \mathbb{R}^q$, $\mathsf{Cov}\,(\gamma) = D \in \mathbb{R}^{q\times q}$, and

$$\mathsf{E}(\varepsilon) = 0 \in \mathbb{R}^n, \qquad \mathsf{Cov}\,(\varepsilon) = R \in \mathbb{R}^{n\times n}, \qquad \mathsf{Cov}\,(\gamma, \varepsilon) = 0 \in \mathbb{R}^{q\times n}. \qquad (3.2)$$

We may denote this mixed model briefly as

$$\mathcal{N} = \{y, X\beta + Z\gamma, D, R\}. \tag{3.3}$$

Various properties of the BLUP in the mixed effects model are discussed, for example, by Searle (1997).

The mixed model can be presented as a version of the model with "new observations". The new observations being now in γ:

$$\mathcal{M} = \left\{ \begin{bmatrix} y \\ \gamma \end{bmatrix}, \begin{bmatrix} X\beta \\ 0 \end{bmatrix}, \begin{bmatrix} ZDZ' + R & ZD \\ DZ' & D \end{bmatrix} \right\}. \tag{3.4}$$

Consider two mixed linear models:

$$\mathcal{N}_1 = \{y, X_1\beta + Z_1\gamma, D_1, R_1\}, \qquad \mathcal{N}_2 = \{y, X_2\beta + Z_2\gamma, D_2, R_2\}. \tag{3.5}$$

The only difference above concerns the covariance matrices and model matrices. We may denote $\Sigma_i = Z_iD_iZ_i' + R_i$, $(i = 1, 2)$. We can get two models with new observations as follows:

$$\mathcal{M}_{11} = \left\{ \begin{bmatrix} y \\ \gamma \end{bmatrix}, \begin{bmatrix} X_1\beta \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & Z_1D_1 \\ D_1Z_1' & D_1 \end{bmatrix} \right\}. \tag{3.6}$$

$$\mathcal{M}_{12} = \left\{ \begin{bmatrix} y \\ \gamma \end{bmatrix}, \begin{bmatrix} X_2\beta \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_2 & Z_2D_2 \\ D_2Z_2' & D_2 \end{bmatrix} \right\}. \tag{3.7}$$

Now we can apply Theorem 2.1 and Theorem 2.2 to models (3.6) and (3.7) yields the following results:

Theorem 3.1 Let the mixed linear models $\mathcal{N}_1$ and $\mathcal{N}_2$ be given in Eq. (3.5). Then the following statements are equivalent:

(a) There exist an BLUP(γ) under mixed model $\mathcal{N}_1$ such that the BLUP(γ) is also the BLUP of γ under mixed model $\mathcal{N}_2$,

(b)

$$r\begin{bmatrix} 0 & D_1Z_1' & 0 & D_2Z_2' \\ X_1 & \Sigma_1 & X_2 & \Sigma_2 \\ 0 & X_1' & 0 & 0 \\ 0 & 0 & 0 & X_2' \end{bmatrix} = r[\,X_1 \;\; \Sigma_1 \;\; X_2 \;\; \Sigma_2\,] + r(X_1) + r(X_2),$$

(c)

$$\mathscr{R}\begin{bmatrix} 0 \\ Z_1D_1 \\ 0 \\ Z_1D_2 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X_1' & 0 & 0 \\ \Sigma_1 & X_1 & 0 \\ X_2' & 0 & 0 \\ \Sigma_2 & 0 & X_2 \end{bmatrix}.$$

Theorem 3.2 Let the mixed linear models $\mathcal{N}_1$ and $\mathcal{N}_2$ be as given in Eq. (3.5). Then the following statements are equivalent:

(a) Every BLUP(γ) under mixed model $\mathcal{N}_1$ is also the BLUP of γ under mixed model $\mathcal{N}_2$,

(b)

$$\mathscr{R}\begin{bmatrix} X_2 \\ 0 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X_1 & \Sigma_1E_{X_1} \\ 0 & D_1Z_1'E_{X_1} \end{bmatrix} \quad \text{and} \quad \mathscr{R}\begin{bmatrix} \Sigma_2E_{X_2} \\ D_2Z_2'E_{X_2} \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X_1 & \Sigma_1E_{X_1} \\ 0 & D_1Z_1'E_{X_1} \end{bmatrix}.$$

(c)

$$r\begin{bmatrix} 0 & D_1Z_1' & 0 & D_2Z_2' \\ X_1 & \Sigma_1 & X_2 & \Sigma_2 \\ 0 & X_1' & 0 & 0 \\ 0 & 0 & 0 & X_2' \end{bmatrix} = r[\,X_1\ \ \Sigma_1\ \ X_2\ \ \Sigma_2\,] + r(X_1) + r(X_2),$$

and

$$r[\,X_1\ \ \Sigma_1\ \ X_2\ \ \Sigma_2\,] = r[\,X_1\ \ \Sigma_1\,],$$

(d)

$$\mathscr{R}\begin{bmatrix} 0 \\ Z_1D_1 \\ 0 \\ Z_1D_2 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} X_1' & 0 & 0 \\ \Sigma_1 & X_1 & 0 \\ X_2' & 0 & 0 \\ \Sigma_2 & 0 & X_2 \end{bmatrix} \quad \text{and} \quad \mathscr{R}[\,X_2\ \ \Sigma_2\,] \subseteq \mathscr{R}[\,X_1\ \ \Sigma_1\,].$$

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两个线性模型间最优线性无偏预测的等价性

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设$\mathcal{M}_1$和$\mathcal{M}_2$是两个带有预测量的线性模型, 通过使用矩阵秩方法, 本文给出了模型$\mathcal{M}_1$下预测量的最优线性无偏预测同时也是模型$\mathcal{M}_2$下的最优线性无偏预测的充分必要条件. 作为这个结果的应用, 我们给出了两个线性混合模型间最优线性无偏预测等价性的充分必要条件.

关 键 词: 一般线性模型, 混合线性模型, 最优线性无偏预测, 矩阵秩方法.

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