应用概率统计　第二十七卷
第五期　2011年10月

Chinese Journal of Applied Probability
and Statistics Vol.27 No.5 Oct. 2011

The Gerber-Shiu Penalty Functions for a Perturbed Risk Model with Two Classes of Risks and a Threshold Dividend Strategy *

SUN GUOHONG[1]　ZHANG CHUNSHENG[2]　JI LANPENG[2,3]

([1] *Department of Basic Science, Tianjin Agricultural College, Tianjin, 300384*)

([2] *School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071*)

([3] *Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne, UNIL-Dorigny 1015 Lausanne, Switzerland*)

Abstract

In this paper, we study the perturbed risk model with two classes of claims and a threshold dividend strategy. We assume that the two claim counting processes are, respectively, Poisson and renewal process with generalized Erlang(2) inter-claim times. Integro-differential equations and certain boundary conditions satisfied by the Gerber-Shiu penalty functions are derived in terms of matrices. Finally, we show that the closed form for the Gerber-Shiu penalty functions can be expressed by the Gerber-Shiu penalty functions without dividend payments and the matrix composed of two linearly independent solutions to the corresponding homogeneous integro-differential equations.

Keywords: Two classes of risks, Gerber-Shiu penalty functions, threshold dividend strategy, integro-differential equations.

AMS Subject Classification: 91B30.

§1. Introduction

Dividend strategy for insurance risk model is initially introduced by De Finetti (1957) for a binomial model. From then on, more general barrier strategies have been studied in a number of papers, see Lin et al. (2003), Gerber and Shiu (2006), Lin and Pavlova (2006), Li et al. (2009), Yang and Zhang (2008), Lu and Li (2009), and references therein. Lin and Pavlova (2006) study the Gerber-Shiu penalty function and related problems for the classical compound Poisson risk model with a threshold dividend strategy. For perturbed compound Poisson risk model and a threshold dividend strategy, the expected discounted dividend payments prior to ruin and the Gerber-Shiu penalty function have been studied

*The project supported by the Tianjin Natural Science Foundation (08JCYBJC02200), the Keygrant Project of Chinese Ministry of Education (309009) and the Natural Science Foundation of China (11171164).

Received October 27, 2010. Revised July 3, 2011.

by Wan (2007). Meng et al. (2007) analyze the expectation of aggregate dividends until ruin for a Sparre Andersen model perturbed by diffusion with generalized Erlang(n)-distributed inter-claim times and a threshold strategy. For the same model as in Meng et al. (2007), Gao and Yin (2008) give the integro-differential equations with boundary conditions for the moment-generating function and the Gerber-Shiu penalty functions. Li et al. (2009) give the closed form expression of the expected discounted dividend function for a jump-diffusion risk process by studying a constructed fluid flow process. The Gerber-Shiu penalty function and the moments of the total dividend payments for the Markovian regime-switching risk model with a threshold dividend strategy have been studied by Lu and Li (2009).

There has been much recent research in the actuarial literature on the analysis of ruin probabilities and Gerber-Shiu penalty functions for two classes of risks. Yuen et al. (2002) consider a risk process with two dependent classes of insurance business. By transforming this kind of risk process to a process with two independent classes of business for which one claim number process is a Poisson and the other is a renewal process with generalized Erlang(2) claim inter-arrival times, they derive explicit expressions for survival probabilities when the claim sizes are exponentially distributed. Garrido and Li (2005) also study the ruin probabilities for two independent classes of risk processes for which one claim number process is a Poisson and the other is a renewal process with generalized Erlang claim inter-arrival times. Explicit results are given when the claim amount distributions of both classes belong to the K_n family of distributions. The Gerber-Shiu penalty functions have been studied in Li and Lu (2005) for the risk process with claim counting processes are independent Poisson and renewal process with generalized Erlang(2) inter-claim times and Zhang et al. (2009) for the risk process with claim counting processes are independent Poisson and renewal process with generalized Erlang(n) inter-claim times. Explicit expressions for the Gerber-Shiu penalty functions are obtained in Zhang et al. (2009) when the claim size distributions belong to the rational family. The Gerber-Shiu penalty functions for two classes of risks under a threshold dividend strategy have been considered in Lu et al. (2009) for the risk process with claim counting processes are independent Poisson and renewal process with generalized Erlang(2) inter-claim times. Recently, Chadjiconstantinidis and Papaioannou (2009) study the Gerber-Shiu penalty functions and the moments of the discounted sum of the dividend payments until ruin for two classes of risk processes with a constant dividend barrier, for which one claim number process is a Poisson and the other is a renewal process with generalized Erlang(n) claim inter-arrival times.

In the present paper, we adopt an approach which is akin to the one used in Lu and Li (2009) to study the Gerber-Shiu penalty functions for a perturbed risk model with two classes of risks and a threshold dividend strategy.

The outline of this paper is as follows. In Section 2, the model studied is described. In

Section 3, integro-differential equations and certain boundary conditions for the Gerber-Shiu penalty functions are derived. In Section 4, we discuss the roots of a generalized Lundberg's equation. The Gerber-Shiu penalty functions without dividend payments are analyzed in Section 5, and the main results for the Gerber-Shiu penalty functions with threshold dividend strategy are given in Section 6.

§2. The Model

Consider a continuous time risk process defined by

$$U(t) = u + c_1 t - S(t) + \sigma B(t), \qquad t \ge 0, \tag{2.1}$$

where $u \ge 0$ is the initial capital of the insurance company, $c_1 > 0$ is the rate of premium which is assumed to be a constant, $\{B(t); t \ge 0\}$ is a standard Brownian motion and $\sigma > 0$ is the diffusion coefficient. In this paper, the aggregate claim amount process $\{S(t)\}$ is defined as

$$S(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \qquad t \ge 0,$$

where X_i's are claim amounts from the first class, and assumed to be i.i.d. positive random variables with common distribution function $P(x) = \mathsf{P}(X \le x)$ and density $p(x) = P'(x)$, while Y_i's are claim amounts from the second class, also assumed to be i.i.d. positive random variables with common distribution function $Q(y) = \mathsf{P}(Y \le y)$ and density $q(y) = Q'(y)$. Denote the Laplace transforms of p and q by

$$\widehat{p}(s) = \int_0^\infty e^{-sx} p(x) \mathrm{d}x \qquad \text{and} \qquad \widehat{q}(s) = \int_0^\infty e^{-sx} q(x) \mathrm{d}x,$$

respectively. The renewal processes $\{N_1(t); t \ge 0\}$ and $\{N_2(t); t \ge 0\}$ that denote the number of claims up to time t caused by the first and the second class of risk, respectively, are defined as follows.

$\{N_1(t); t \ge 0\}$ is a poisson process with parameter λ, and denote $\{W_i\}_{i\ge1}$ as the corresponding inter-claim arrival times. $\{N_2(t); t \ge 0\}$ is a renewal process with inter-claim arrival times $\{V_i; i \ge 1\}$ which are generalized Erlang(2) distributed, that is, $V_i = L_{i1} + L_{i2}$, $i = 1, 2, \ldots$, where $\{L_{i1}; i \ge 1\}$ are i.i.d. exponentially distributed random variables with parameter λ_1 and $\{L_{i2}; i \ge 1\}$ are i.i.d. exponentially distributed random variables with parameter λ_2.

Finally, we assume that $\{X_i; i \ge 1\}$, $\{Y_i; i \ge 1\}$, $\{N_1(t); t \ge 0\}$, $\{N_2(t); t \ge 0\}$ and $\{B(t); t \ge 0\}$ are mutually independent. The net profit condition is given by $c_1 > \lambda\mathsf{E}(X) + [\lambda_1\lambda_2/(\lambda_1+\lambda_2)]\mathsf{E}(Y)$.

In this paper, we suppose that the company pays dividends to its shareholders in the following way. When the surplus is below the level b, no dividends are paid. However, when the surplus exceeds b, dividends are paid continuously at rate d $(0 < d \le c_1)$, thus the net premium rate after dividend payments is $c_2 = c_1 - d$. Let $\{U_b(t); t \ge 0\}$ be the modified surplus process under the threshold dividend strategy described above, then it can be expressed as

$$\mathrm{d}U_b(t) = (c_1 I_{(U_b(t)<b)} + c_2 I_{(U_b(t)\ge b)})\mathrm{d}t - \mathrm{d}S(t) + \sigma \mathrm{d}B(t), \tag{2.2}$$

with $U_b(0) = u$.

Let $T_b = \inf\{t \ge 0; U_b(t) \le 0\}$ ($T_b = \infty$ if ruin does not occur) be the time of ruin for the risk process (2.2). Define, for $\delta \ge 0$,

$$\phi_k(u) = \mathsf{E}[e^{-\delta T_b}\omega_k(U_b(T_b-), |U_b(T_b)|) I_{(T_b<\infty, J=k)}|U_b(0) = u], \qquad u \ge 0,\ k = 1, 2,$$

where J is defined to be the cause-of-ruin random variable, and $J = 1$ (or 2) if the ruin is caused by a claim of class 1 (or 2), $\omega_k(x, y)$, for $x, y \ge 0$, $k = 1, 2$, are two possibly distinct non-negative valued penalty functions, and $U_b(T_b-)$, $|U_b(T_b)|$ are two important non-negative random variables in connection with the time of ruin T_b representing the surplus immediately before ruin and the deficit at ruin, respectively. Then $\phi_k(u)$ with $\phi_k(0) = 0$ is the Gerber-Shiu penalty function if ruin is caused by a claim of class k. Further define, for $\delta \ge 0$,

$$\phi_d(u) = \mathsf{E}[e^{-\delta T_b} I_{(T_b<\infty, U_b(T_b)=0)}|U_b(0) = u], \qquad u \ge 0,$$

to be the Gerber-Shiu penalty function with penalty value 1 if ruin is caused by oscillations. Then we have $\phi_d(0) = 1$. Consequently, the Gerber-Shiu penalty function for the process (2.2) can be expressed as

$$\phi(u) = \phi_1(u) + \phi_2(u) + \phi_d(u), \qquad u \ge 0,$$

with boundary condition $\phi(0) = 1$.

In this paper we will use the following functions, The description of which are given in Li and Lu (2005).

$$\xi_k(u) = \mathsf{E}[e^{-\delta(T_b-t)}\omega_k(U_b(T_b-), |U_b(T_b)|) I_{(T_b<\infty, J=k)}|L_{11} = t, U_b(t) = u], \quad u \ge 0,\ k = 1, 2,$$

and

$$\xi_d(u) = \mathsf{E}[e^{-\delta(T_b-t)} I_{(T_b<\infty, U_b(T_b)=0)}|L_{11} = t, U_b(t) = u], \qquad u \ge 0,$$

and it is obvious that the boundary conditions are given by

$$\xi_k(0) = 0, \quad k = 1, 2, \qquad \xi_d(0) = 1.$$

In the sequel, we use the following notations.

$$\phi_1(u)=\begin{cases}\phi_1^{(1)}(u), & 0\le u<b,\\ \phi_1^{(2)}(u), & b\le u<\infty,\end{cases}\qquad \phi_2(u)+\phi_d(u)=\begin{cases}\phi_3^{(1)}(u), & 0\le u<b,\\ \phi_3^{(2)}(u), & b\le u<\infty,\end{cases}$$

$$\xi_1(u)=\begin{cases}\xi_1^{(1)}(u), & 0\le u<b,\\ \xi_1^{(2)}(u), & b\le u<\infty,\end{cases}\qquad \xi_2(u)+\xi_d(u)=\begin{cases}\xi_3^{(1)}(u), & 0\le u<b,\\ \xi_3^{(2)}(u), & b\le u<\infty.\end{cases}$$

§3. Systems of Integro-Differential Equations

Considering an infinitesimal interval from 0 to t and noting that the probability with which $[W_{11}\le t]$ and $[L_{11}\le t]$ occur simultaneously is $o(t)$, we can derive that, for $0<u<b$,

$$\begin{aligned} e^{\delta t}\phi_1^{(1)}(u) \;=\;& (1-\lambda t)(1-\lambda_1 t)\mathsf{E}[\phi_1^{(1)}(u+c_1t+\sigma B(t))]\\ &+(\lambda t)(1-\lambda_1 t)\mathsf{E}[\phi_1^{(1)}(u+c_1t+\sigma B(t)-X_1)]\\ &+(1-\lambda t)(\lambda_1 t)\mathsf{E}[\xi_1^{(1)}(u+c_1t+\sigma B(t))]+o(t), \end{aligned}\tag{3.1}$$

and

$$\begin{aligned} e^{\delta t}\xi_1^{(1)}(u) \;=\;& (1-\lambda t)(1-\lambda_2 t)\mathsf{E}[\xi_1^{(1)}(u+c_1t+\sigma B(t))]\\ &+(\lambda t)(1-\lambda_2 t)\mathsf{E}[\xi_1^{(1)}(u+c_1t+\sigma B(t)-X_1)]\\ &+(1-\lambda t)(\lambda_2 t)\mathsf{E}[\phi_1^{(1)}(u+c_1t+\sigma B(t)-Y_1)]+o(t). \end{aligned}\tag{3.2}$$

With the aid of the equations

$$\mathsf{E}[\phi_1^{(1)}(u+c_1t+\sigma B(t))]=\phi_1^{(1)}(u)+c_1\phi_1^{(1)\prime}(u)t+\frac{\sigma^2}{2}\phi_1^{(1)\prime\prime}(u)t+o(t)\tag{3.3}$$

and

$$\mathsf{E}[\xi_1^{(1)}(u+c_1t+\sigma B(t))]=\xi_1^{(1)}(u)+c_1\xi_1^{(1)\prime}(u)t+\frac{\sigma^2}{2}\xi_1^{(1)\prime\prime}(u)t+o(t),\tag{3.4}$$

we get from (3.1) and (3.2) that

$$\begin{aligned} (1+\delta t)\phi_1^{(1)}(u) \;=\;& (1-\lambda t)(1-\lambda_1 t)\Big[\phi_1^{(1)}(u)+c_1\phi_1^{(1)\prime}(u)t+\frac{\sigma^2}{2}\phi_1^{(1)\prime\prime}(u)t\Big]\\ &+(\lambda t)(1-\lambda_1 t)\mathsf{E}\Big[\int_0^{u+c_1t+\sigma B(t)}\phi_1^{(1)}(u+c_1t+\sigma B(t)-x)p(x)\mathrm{d}x\\ &+\int_{u+c_1t+\sigma B(t)}^{\infty}\omega_1(u+c_1t+\sigma B(t),x-u-c_1t-\sigma B(t))p(x)\mathrm{d}x\Big]\\ &+(1-\lambda t)(\lambda_1 t)\mathsf{E}[\xi_1^{(1)}(u+c_1t+\sigma B(t))]+o(t), \end{aligned}$$

and

$$
\begin{aligned}
(1+\delta t)\xi_1^{(1)}(u) &= (1-\lambda t)(1-\lambda_2 t)\Big[\xi_1^{(1)}(u)+c_1\xi_1^{(1)'}(u)t+\frac{\sigma^2}{2}\xi_1^{(1)''}(u)t\Big] \\
&\quad +(\lambda t)(1-\lambda_2 t)\mathsf{E}\Big[\int_0^{u+c_1t+\sigma B(t)}\xi_1^{(1)}(u+c_1t+\sigma B(t)-x)p(x)\mathrm{d}x \\
&\quad +\int_{u+c_1t+\sigma B(t)}^{\infty}\omega_1(u+c_1t+\sigma B(t),x-u-c_1t-\sigma B(t))p(x)\mathrm{d}x\Big] \\
&\quad +(1-\lambda t)(\lambda_2 t)\mathsf{E}\Big[\int_0^{u+c_1t+\sigma B(t)}\phi_1^{(1)}(u+c_1t+\sigma B(t)-y)q(y)\mathrm{d}y\Big] \\
&\quad +o(t).
\end{aligned}
$$

Then we have, for $0<u<b$,

$$
\begin{aligned}
&\frac{\sigma^2}{2}\phi_1^{(1)''}(u)+c_1\phi_1^{(1)'}(u)-(\lambda_1+\lambda+\delta)\phi_1^{(1)}(u) \\
&+\lambda_1\xi_1^{(1)}(u)+\lambda\int_0^u\phi_1^{(1)}(u-x)p(x)\mathrm{d}x+\lambda w_1(u)=0
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
&\frac{\sigma^2}{2}\xi_1^{(1)''}(u)+c_1\xi_1^{(1)'}(u)-(\lambda_2+\lambda+\delta)\xi_1^{(1)}(u) \\
&+\lambda\int_0^u\xi_1^{(1)}(u-x)p(x)\mathrm{d}x+\lambda_2\int_0^u\phi_1^{(1)}(u-y)q(y)\mathrm{d}y+\lambda w_1(u)=0,
\end{aligned} \tag{3.6}
$$

where $w_1(u)=\displaystyle\int_u^\infty\omega_1(u,x-u)p(x)\mathrm{d}x$.

Similarly, we can derive, for $u<b<\infty$,

$$
\begin{aligned}
(1+\delta t)\phi_1^{(2)}(u) &= (1-\lambda t)(1-\lambda_1 t)\mathsf{E}[\phi_1^{(2)}(u+c_2t+\sigma B(t))] \\
&\quad +(\lambda t)(1-\lambda_1 t)\mathsf{E}\Big[\int_0^{u+c_2t+\sigma B(t)-b}\phi_1^{(2)}(u+c_2t+\sigma B(t)-x)p(x)\mathrm{d}x \\
&\quad +\int_{u+c_2t+\sigma B(t)-b}^{u+c_2t+\sigma B(t)}\phi_1^{(1)}(u+c_2t+\sigma B(t)-x)p(x)\mathrm{d}x \\
&\quad +\int_{u+c_2t+\sigma B(t)}^{\infty}\omega_1(u+c_2t+\sigma B(t),x-u-c_2t-\sigma B(t))p(x)\mathrm{d}x\Big] \\
&\quad +(1-\lambda t)(\lambda_1 t)\mathsf{E}[\xi_1^{(2)}(u+c_2t+\sigma B(t))]+o(t),
\end{aligned}
$$

and

$$
\begin{aligned}
(1+\delta t)\xi_1^{(2)}(u) &= (1-\lambda t)(1-\lambda_2 t)\mathsf{E}[\xi_1^{(2)}(u+c_2t+\sigma B(t))] \\
&\quad +(\lambda t)(1-\lambda_2 t)\mathsf{E}\Big[\int_0^{u+c_2t+\sigma B(t)-b}\xi_1^{(2)}(u+c_2t+\sigma B(t)-x)p(x)\mathrm{d}x \\
&\quad +\int_{u+c_2t+\sigma B(t)-b}^{u+c_2t+\sigma B(t)}\xi_1^{(1)}(u+c_2t+\sigma B(t)-x)p(x)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{u+c_2t+\sigma B(t)}^{\infty}\omega_1(u+c_2t+\sigma B(t),x-u-c_2t-\sigma B(t))p(x)\mathrm{d}x\Big]\\
&+(1-\lambda t)(\lambda_2 t)\mathsf{E}\Big[\int_0^{u+c_2t+\sigma B(t)-b}\phi_1^{(2)}(u+c_2t+\sigma B(t)-y)q(y)\mathrm{d}y\\
&+\int_{u+c_2t+\sigma B(t)-b}^{u+c_2t+\sigma B(t)}\phi_1^{(1)}(u+c_2t+\sigma B(t)-y)q(y)\mathrm{d}y\Big]+o(t).
\end{aligned}
$$

By the aid of equations similar to Eqs. (3.3) and (3.4), we can get from the last two formulas, for $b<u<\infty$,

$$
\begin{aligned}
&\frac{\sigma^2}{2}\phi_1^{(2)''}(u)+c_2\phi_1^{(2)'}(u)-(\lambda_1+\lambda+\delta)\phi_1^{(2)}(u)+\lambda_1\xi_1^{(2)}(u)\\
&+\lambda\int_0^{u-b}\phi_1^{(2)}(u-x)p(x)\mathrm{d}x+\lambda\int_{u-b}^{u}\phi_1^{(1)}(u-x)p(x)\mathrm{d}x+\lambda w_1(u)=0
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
&\frac{\sigma^2}{2}\xi_1^{(2)''}(u)+c_2\xi_1^{(2)'}(u)-(\lambda_2+\lambda+\delta)\xi_1^{(2)}(u)+\lambda\int_0^{u-b}\xi_1^{(2)}(u-x)p(x)\mathrm{d}x\\
&+\lambda\int_{u-b}^{u}\xi_1^{(1)}(u-x)p(x)\mathrm{d}x+\lambda_2\int_0^{u-b}\phi_1^{(2)}(u-y)q(y)\mathrm{d}y\\
&+\lambda_2\int_{u-b}^{u}\phi_1^{(1)}(u-y)q(y)\mathrm{d}y+\lambda w_1(u)=0.
\end{aligned}
\tag{3.8}
$$

Let

$$
\vec{\Phi}_1^{(1)}(u)=\begin{pmatrix}\phi_1^{(1)}(u)\\ \xi_1^{(1)}(u)\end{pmatrix},\qquad \vec{\Phi}_1^{(2)}(u)=\begin{pmatrix}\phi_1^{(2)}(u)\\ \xi_1^{(2)}(u)\end{pmatrix},
$$

$$
\mathbf{A}=\begin{pmatrix}-(\lambda_1+\lambda+\delta) & \lambda_1\\ 0 & -(\lambda_2+\lambda+\delta)\end{pmatrix},\qquad \mathbf{B}(x)=\begin{pmatrix}\lambda p(x) & 0\\ \lambda_2 q(x) & \lambda p(x)\end{pmatrix},
$$

and

$$
\vec{\zeta}_1(u)=\begin{pmatrix}\lambda w_1(u)\\ \lambda w_1(u)\end{pmatrix}.
$$

Rewriting Eqs. (3.5), (3.6), (3.7) and (3.8) in matrix form yields

$$
\begin{aligned}
\vec{\Phi}_1^{(1)''}(u) &= \Big(-\frac{2}{\sigma^2}\Big)\Big[c_1\vec{\Phi}_1^{(1)'}(u)+\mathbf{A}\vec{\Phi}_1^{(1)}(u)\\
&\quad+\int_0^u\mathbf{B}(x)\vec{\Phi}_1^{(1)}(u-x)\mathrm{d}x+\vec{\zeta}_1(u)\Big],\qquad 0<u<b,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\vec{\Phi}_1^{(2)''}(u) &= \Big(-\frac{2}{\sigma^2}\Big)\Big[c_2\vec{\Phi}_1^{(2)'}(u)+\mathbf{A}\vec{\Phi}_1^{(2)}(u)+\int_0^{u-b}\mathbf{B}(x)\vec{\Phi}_1^{(2)}(u-x)\mathrm{d}x\\
&\quad+\int_{u-b}^{u}\mathbf{B}(x)\vec{\Phi}_1^{(1)}(u-x)\mathrm{d}x+\vec{\zeta}_1(u)\Big],\qquad b<u<\infty,
\end{aligned}
\tag{3.10}
$$

with boundary conditions

$$\vec{\Phi}_1^{(1)}(0)=\vec{0}_2,\qquad \vec{\Phi}_1^{(1)}(b-)=\vec{\Phi}_1^{(2)}(b+)\qquad\text{and}\qquad \vec{\Phi}_1^{(1)\prime}(b-)=\vec{\Phi}_1^{(2)\prime}(b+),$$

which can be obtained by the same method as in Wan (2007), where $\vec{0}_2=(0,0)^\top$.

By similar arguments, we get

$$\begin{aligned}\vec{\Phi}_3^{(1)\prime\prime}(u) &= \Big(-\frac{2}{\sigma^2}\Big)\Big[c_1\vec{\Phi}_3^{(1)\prime}(u)+\mathbf{A}\vec{\Phi}_3^{(1)}(u)\\ &\quad+\int_0^u\mathbf{B}(x)\vec{\Phi}_3^{(1)}(u-x)\mathrm{d}x+\vec{\zeta}_3(u)\Big],\qquad 0<u<b, &(3.11)\\ \vec{\Phi}_3^{(2)\prime\prime}(u) &= \Big(-\frac{2}{\sigma^2}\Big)\Big[c_2\vec{\Phi}_3^{(2)\prime}(u)+\mathbf{A}\vec{\Phi}_3^{(2)}(u)+\int_0^{u-b}\mathbf{B}(x)\vec{\Phi}_3^{(2)}(u-x)\mathrm{d}x\\ &\quad+\int_{u-b}^u\mathbf{B}(x)\vec{\Phi}_3^{(1)}(u-x)\mathrm{d}x+\vec{\zeta}_3(u)\Big],\qquad b<u<\infty, &(3.12)\end{aligned}$$

and the boundary conditions

$$\vec{\Phi}_3^{(1)}(0)=\vec{e}_2,\qquad \vec{\Phi}_3^{(1)}(b-)=\vec{\Phi}_3^{(2)}(b+)\qquad\text{and}\qquad \vec{\Phi}_3^{(1)\prime}(b-)=\vec{\Phi}_3^{(2)\prime}(b+),$$

where

$$\vec{\Phi}_3^{(1)}(u)=\begin{pmatrix}\phi_3^{(1)}(u)\\ \xi_3^{(1)}(u)\end{pmatrix},\qquad \vec{\Phi}_3^{(2)}(u)=\begin{pmatrix}\phi_3^{(2)}(u)\\ \xi_3^{(2)}(u)\end{pmatrix},\qquad \vec{\zeta}_3(u)=\begin{pmatrix}0\\ \lambda_2w_2(u)\end{pmatrix},$$
$$w_2(u)=\int_u^\infty\omega_2(u,x-u)q(x)\mathrm{d}x\qquad\text{and}\qquad \vec{e}_2=(1,1)^\top.$$

§4. The Roots of the Generalized Lundberg's Equation

Taking Laplace transform on both sides of Eq. (3.9) gives, for $s\in\mathbb{C}$ satisfying $\Re(s)\ge 0$,

$$\begin{aligned}&s^2\widehat{\vec{\Phi}}_1^{(1)}(s)-s\widehat{\vec{\Phi}}_1^{(1)}(s)-\widehat{\vec{\Phi}}_1^{(1)\prime}(0)\\ =&\Big(-\frac{2}{\sigma^2}\Big)\big[c_1(s\widehat{\vec{\Phi}}_1^{(1)}(s)-\widehat{\vec{\Phi}}_1^{(1)}(0))+\mathbf{A}\widehat{\vec{\Phi}}_1^{(1)}(s)+\widehat{\mathbf{B}}(s)\widehat{\vec{\Phi}}_1^{(1)}(s)+\widehat{\vec{\zeta}}_1(s)\big].\end{aligned}$$

Rewriting the last equation we get

$$\mathbf{L}_{\delta c_1}(s)\widehat{\vec{\Phi}}_1^{(1)}(s)=\frac{\sigma^2}{2}s\widehat{\vec{\Phi}}_1^{(1)}(0)+\frac{\sigma^2}{2}\widehat{\vec{\Phi}}_1^{(1)\prime}(0)+c_1\widehat{\vec{\Phi}}_1^{(1)}(0)-\widehat{\vec{\zeta}}_1(s),\qquad \Re(s)\ge 0,\tag{4.1}$$

where

$$\mathbf{L}_{\delta c_1}(s)=\frac{\sigma^2}{2}s^2+c_1s+\mathbf{A}+\widehat{\mathbf{B}}(s)\qquad\text{and}\qquad \widehat{\vec{\zeta}}_1(s)=\int_0^\infty e^{-su}\vec{\zeta}_1(u)\mathrm{d}u.$$

Similarly, we obtain from Eq. (3.11) that

$$\mathbf{L}_{\delta c_1}(s)\widehat{\vec{\Phi}}_3^{(1)}(s)=\frac{\sigma^2}{2}s\widehat{\vec{\Phi}}_3^{(1)}(0)+\frac{\sigma^2}{2}\widehat{\vec{\Phi}}_3^{(1)'}(0)+c_1\widehat{\vec{\Phi}}_3^{(1)}(0)-\widehat{\vec{\zeta}}_3(s),\qquad \Re(s)\geq 0,\tag{4.2}$$

where $\widehat{\vec{\zeta}}_3(s)=\int_0^\infty e^{-su}\vec{\zeta}_3(u)\mathrm{d}u$.

Theorem 4.1 For $\delta>0$, the generalized Lundberg's fundamental equation $\det[\mathbf{L}_{\delta c}(s)]=0$ has exactly two positive real roots denoted by $\rho_1(\delta)$, $\rho_2(\delta)$.

Proof It is easy to see that

$$\begin{aligned}\det[\mathbf{L}_{\delta c}(s)] &= \Big[\frac{\sigma^2}{2}s^2+cs-(\lambda_1+\lambda+\delta)+\lambda\widehat{p}(s)\Big]\Big[\frac{\sigma^2}{2}s^2+cs-(\lambda_2+\lambda+\delta)+\lambda\widehat{p}(s)\Big]\\ &\quad -\lambda_1\lambda_2\widehat{q}(s),\end{aligned}$$

with which we can rewrite $\det[\mathbf{L}_{\delta c}(s)]=0$ as

$$\begin{aligned}\gamma_{\delta c}(s) &:= \frac{\Big[\frac{\sigma^2}{2}s^2+cs-(\lambda_1+\lambda+\delta)+\lambda\widehat{p}(s)\Big]\Big[\frac{\sigma^2}{2}s^2+cs-(\lambda_2+\lambda+\delta)+\lambda\widehat{p}(s)\Big]}{\lambda_1\lambda_2}\\ &= \widehat{q}(s).\end{aligned}\tag{4.3}$$

Let C_r denote the right half of a circle with its center at $(0,0)$ and radius r assumed to be sufficiently large, and $\mathbb{C}_r$ denote the boundary of the contour enclosed by C_r and the imaginary axis. We first show that, for $k=1,2$, the equation

$$\frac{\sigma^2}{2}s^2+cs-(\lambda_k+\lambda+\delta)=-\lambda\widehat{p}(s)$$

has exactly one root in the right half of the complex plane. In fact, by Rouche theorem it suffices to prove

$$\Big|\frac{\sigma^2}{2}s^2+cs-(\lambda_k+\lambda+\delta)\Big|>\lambda|\widehat{p}(s)|,\qquad s\in\mathbb{C}_r.\tag{4.4}$$

It is derived that, for $s\in C_r$,

$$\Big|\frac{\sigma^2}{2}s^2+cs-(\lambda_k+\lambda+\delta)\Big|\geq|s|\Big(\frac{\sigma^2}{2}|s|-c\Big)-(\lambda_k+\lambda+\delta)>\lambda\geq\lambda|\widehat{p}(s)|,\tag{4.5}$$

and for $s=bi$, $b\in\mathbb{R}$,

$$\Big|\frac{\sigma^2}{2}s^2+cs-(\lambda_k+\lambda+\delta)\Big|\geq\frac{\sigma^2}{2}b^2+(\lambda_k+\lambda+\delta)>\lambda\geq\lambda|\widehat{p}(s)|.$$

Thus, Eq. (4.4) is proved to be valid. Consequently, we see from (4.3) that $\gamma_{\delta c}(s)=0$ has exactly two roots in the right half of the complex plane, and it is evident that these two roots are real numbers.

Similarly, by Rouche theorem, if $|\gamma_{\delta c}(s)| > |\widehat{q}(s)|$ is established for $s \in \mathbb{C}_r$, we obtain that Eq. (4.3) has exactly two roots in the right half of the complex plane. Indeed, $|\gamma_{\delta c}(s)| > 1 \geq |\widehat{q}(s)|$ for $s \in C_r$, while, for $s = bi$, $b \in \mathbb{R}$,

$$|\gamma_{\delta c}(s)| \geq \frac{(\lambda_1+\delta)(\lambda_2+\delta)}{\lambda_1\lambda_2} > 1 \geq |\widehat{q}(s)|.$$

Now it remains to establish that these two roots are real numbers. In the sequel, let $s \in [0, \infty)$. Noting that $\sigma^2 s + c + \lambda\widehat{p}'(s) \geq c - \lambda \mathsf{E}X > 0$ for $s \geq 0$, we have that

$$\gamma'_{\delta c}(s) = \frac{2[\sigma^2 s + c + \lambda\widehat{p}'(s)]\Big[\dfrac{\sigma^2}{2}s^2 + cs - \Big(\dfrac{\lambda_2+\lambda_1+2\delta}{2} + \lambda\Big) + \lambda\widehat{p}(s)\Big]}{\lambda_1\lambda_2} = 0$$

has exactly one positive real root, say s_0. And it is easy to check that $\gamma_{\delta c}(s)$ is decreasing for $s \in [0, s_0]$, increasing for $s \in (s_0, \infty)$, and has a minimum $\gamma_{\delta c}(s_0) \leq 0$ for $s \geq 0$, which with the fact that

$$\gamma_{\delta c}(0) = \frac{(\lambda_1+\delta)(\lambda_2+\delta)}{\lambda_1\lambda_2} > 1 = \widehat{q}(0)$$

imply that Eq. (4.3) has two real roots. The proof is finished. $\square$

Remark 1 If $\delta \to 0$, we have $\rho_i(\delta) \to \rho_i(0)$ for $i = 1, 2$, and $s = 0$ is one of the roots.

In the following, we use ρ_1, ρ_2 to denote the positive real roots of $\det[\mathbf{L}_{\delta c_1}(s)] = 0$, and γ_1, γ_2 to denote the positive real roots of $\det[\mathbf{L}_{\delta c_2}(s)] = 0$. We also assume that $\rho_1 \neq \rho_2$ and $\gamma_1 \neq \gamma_2$.

§5. Gerber-Shiu Penalty Functions without Dividend Payments

In this section, we get some results for surplus process (2.1) where no dividends are involved, that is $b = \infty$. We denote $\vec{\phi}_1(u)$ and $\vec{\phi}_3(u)$ to be the corresponding Gerber-Shiu penalty functions when $b = \infty$. Then we get, from Eqs (3.9) and (3.11),

$$\begin{aligned}\vec{\phi}''_1(u) &= \Big(-\frac{2}{\sigma^2}\Big)\Big[c_1\vec{\phi}'_1(u) + \mathbf{A}\vec{\phi}_1(u) \\ &\quad + \int_0^u \mathbf{B}(x)\vec{\phi}_1(u-x)\mathrm{d}x + \vec{\zeta}_1(u)\Big], \qquad 0 < u < \infty, \qquad (5.1)\\ \vec{\phi}''_3(u) &= \Big(-\frac{2}{\sigma^2}\Big)\Big[c_1\vec{\phi}'_3(u) + \mathbf{A}\vec{\phi}_3(u) \\ &\quad + \int_0^u \mathbf{B}(x)\vec{\phi}_3(u-x)\mathrm{d}x + \vec{\zeta}_3(u)\Big], \qquad 0 < u < \infty, \qquad (5.2)\end{aligned}$$

and the boundary conditions $\vec{\phi}_1(0) = \vec{0}_2$ and $\vec{\phi}_3(0) = \vec{e}_2$.

5.1 Expressions for $\vec{\phi}_1'(0)$ and $\vec{\phi}_3'(0)$

Now we recall the divided differences of a matrix $\mathbf{B}(s)$ with respect to distinct numbers $r_1, r_2, \ldots$ which are defined recursively as follows (see Lu and Li (2009)).

$$\begin{aligned}
\mathbf{B}[r_1, s] &= \frac{\mathbf{B}(s) - \mathbf{B}(r_1)}{s - r_1}, \\
\mathbf{B}[r_1, r_2, s] &= \frac{\mathbf{B}[r_1, s] - \mathbf{B}[r_1, r_2]}{s - r_2}, \\
\mathbf{B}[r_1, r_2, r_3, s] &= \frac{\mathbf{B}[r_1, r_2, s] - \mathbf{B}[r_1, r_2, r_3]}{s - r_3},
\end{aligned} \tag{5.3}$$

and so on.

Theorem 5.1 The closed form for $\vec{\phi}_1'(0)$ and $\vec{\phi}_3'(0)$ are given by

$$\vec{\phi}_1'(0) = \frac{2}{\sigma^2}\big[\vec{\zeta}_1(\rho_2) + \mathbf{L}^*_{\delta c_1}[\rho_1, \rho_2]^{-1}\mathbf{L}^*_{\delta c_1}(\rho_1)\vec{\zeta}_1[\rho_1, \rho_2]\big], \tag{5.4}$$

and

$$\begin{aligned}
\vec{\phi}_3'(0) \;\; = \;\; & -\Big[\Big(\rho_2 + \frac{2}{\sigma^2}c_1\Big) + \mathbf{L}^*_{\delta c_1}[\rho_1, \rho_2]^{-1}\mathbf{L}^*_{\delta c_1}(\rho_1)\Big]\vec{e}_2^{\top} \\
& + \frac{2}{\sigma^2}\big[\vec{\zeta}_3(\rho_2) + \mathbf{L}^*_{\delta c_1}[\rho_1, \rho_2]^{-1}\mathbf{L}^*_{\delta c_1}(\rho_1)\vec{\zeta}_3[\rho_1, \rho_2]\big].
\end{aligned} \tag{5.5}$$

where $\mathbf{A}^*$ denotes the adjoint matrix of matrix $\mathbf{A}$.

Proof By the fact that each element of $\widehat{\vec{\phi}}_1(s)$ is finite for any $s \in \mathbb{C}$ satisfying $\Re(s) \geq 0$, we get from Eq. 4.1 that

$$\Big(-\frac{\sigma^2}{2}\Big)\mathbf{L}^*_{\delta c_1}(\rho_i)\vec{\phi}_1'(0) = \mathbf{L}^*_{\delta c_1}(\rho_i)\Big(\frac{\sigma^2}{2}\rho_i + c_1\Big)\vec{\phi}_1(0) - \mathbf{L}^*_{\delta c_1}(\rho_i)\widehat{\vec{\zeta}}_1(\rho_i), \qquad i = 1, 2.$$

Using (5.3), we derive that

$$\begin{aligned}
\Big(-\frac{\sigma^2}{2}\Big)\mathbf{L}^*_{\delta c_1}[\rho_1, \rho_2]\vec{\phi}_1'(0) \;\; &= \;\; \Big(-\frac{\sigma^2}{2}\Big)\frac{\mathbf{L}^*_{\delta c_1}(\rho_2) - \mathbf{L}^*_{\delta c_1}(\rho_1)}{\rho_2 - \rho_1}\vec{\phi}_1'(0) \\
&= \frac{\mathbf{L}^*_{\delta c_1}(\rho_2)\Big(\frac{\sigma^2}{2}\rho_2 + c_1\Big) - \mathbf{L}^*_{\delta c_1}(\rho_1)\Big(\frac{\sigma^2}{2}\rho_1 + c_1\Big)}{\rho_2 - \rho_1}\vec{\phi}_1(0) \\
&\quad - \frac{\mathbf{L}^*_{\delta c_1}(\rho_2)\widehat{\vec{\zeta}}_1(\rho_2) - \mathbf{L}^*_{\delta c_1}(\rho_1)\widehat{\vec{\zeta}}_1(\rho_1)}{\rho_2 - \rho_1} \\
&= -\big[\mathbf{L}^*_{\delta c_1}[\rho_1, \rho_2]\widehat{\vec{\zeta}}_1(\rho_2) + \mathbf{L}^*_{\delta c_1}(\rho_1)\hat{\vec{\zeta}}_1[\rho_1, \rho_2]\big],
\end{aligned}$$

where in the last equation we have used the boundary condition $\vec{\phi}_1(0) = \vec{0}_2$. Consequently, we have

$$\vec{\phi}_1'(0) = \frac{2}{\sigma^2}\big[\vec{\zeta}_1(\rho_2) + \mathbf{L}^*_{\delta c_1}[\rho_1, \rho_2]^{-1}\mathbf{L}^*_{\delta c_1}(\rho_1)\vec{\zeta}_1[\rho_1, \rho_2]\big].$$

By similar arguments, we get Eq. (5.5) from Eq. (4.2) and $\vec{\phi}_3(0) = \vec{e}_2$. □

5.2 Expressions for $\vec{\phi}_1(u)$ and $\vec{\phi}_3(u)$

The common homogeneous integro-differential equations of Eqs. (5.1) and (5.2) are of the form

$$\vec{\Phi}_1''(u) = \Big(-\frac{2}{\sigma^2}\Big)\Big[c_1\vec{\Phi}_1'(u) + \mathbf{A}\vec{\Phi}_1(u) + \int_0^u \mathbf{B}(x)\vec{\Phi}_1(u-x)\mathrm{d}x\Big], \qquad 0 \le u < \infty. \tag{5.6}$$

Using the same method as in Burton (2005), we get the following theorem.

Theorem 5.2 Let $\mathbf{V}(u) = (\vec{V}_1(u), \vec{V}_2(u))$, $0 \le u < \infty$, be a 2×2 matrix, where $\vec{V}_1(u)$ and $\vec{V}_2(u)$ are two linearly independent solutions to Eq. (5.6) such that $\mathbf{V}(0) = \mathbf{0}$ and $\mathbf{V}'(0) = \mathbf{I}$ which is the 2×2 identity matrix. We have

$$\vec{\phi}_1(u) = \mathbf{V}(u)\vec{\phi}_1'(0) - \frac{2}{\sigma^2}\int_0^u \mathbf{V}(u-x)\vec{\zeta}_1(x)\mathrm{d}x, \qquad 0 \le u < \infty, \tag{5.7}$$

$$\vec{\phi}_3(u) = \Big[\mathbf{V}'(u) + \frac{2c_1}{\sigma^2}\mathbf{V}(u)\Big]\vec{e}_2 + \mathbf{V}(u)\vec{\phi}_3'(0) - \frac{2}{\sigma^2}\int_0^u \mathbf{V}(u-x)\vec{\zeta}_3(x)\mathrm{d}x, \qquad 0 \le u < \infty. \tag{5.8}$$

Proof Taking Laplace transform on both sides of the equation

$$\mathbf{V}''(u) = \Big(-\frac{2}{\sigma^2}\Big)\Big[c_1\mathbf{V}'(u) - \mathbf{A}\mathbf{V}(u) + \int_0^u \mathbf{B}(x)\mathbf{V}(u-x)\mathrm{d}x\Big]$$

yields, for $s \in \mathbb{C}$ satisfying $\Re(s) \ge 0$,

$$[s^2\widehat{\mathbf{V}}(s) - s\mathbf{V}(0) - \mathbf{V}'(0)] = \Big(-\frac{2}{\sigma^2}\Big)[c_1(s\widehat{\mathbf{V}}(s) - \mathbf{V}(0)) + \mathbf{A}\widehat{\mathbf{V}}(s) + \widehat{\mathbf{B}}(s)\widehat{\mathbf{V}}(s)]$$

which can be rewritten as

$$\widehat{\mathbf{V}}(s) = \frac{\sigma^2}{2}[\mathbf{L}_{\delta c_1}(s)]^{-1}, \tag{5.9}$$

where we have used the boundary conditions $\mathbf{V}(0) = \mathbf{0}$ and $\mathbf{V}'(0) = \mathbf{I}$. It is easy to see from Eqs. (4.1), (4.2) and (5.9) that

$$\widehat{\vec{\phi}}_1(s) = s\widehat{\mathbf{V}}(s)\vec{\phi}_1(0) + \widehat{\mathbf{V}}(s)\vec{\phi}_1'(0) + \frac{2c_1}{\sigma^2}\widehat{\mathbf{V}}(s)\vec{\phi}_1(0) - \frac{2}{\sigma^2}\widehat{\mathbf{V}}(s)\widehat{\vec{\zeta}}_1(s),$$

$$\widehat{\vec{\phi}}_3(s) = s\widehat{\mathbf{V}}(s)\vec{\phi}_3(0) + \widehat{\mathbf{V}}(s)\vec{\phi}_3'(0) + \frac{2c_1}{\sigma^2}\widehat{\mathbf{V}}(s)\vec{\phi}_3(0) - \frac{2}{\sigma^2}\widehat{\mathbf{V}}(s)\widehat{\vec{\zeta}}_3(s),$$

from which and the boundary conditions $\vec{\phi}_1(0) = \vec{0}_2$, $\vec{\phi}_3(0) = \vec{e}_2$, $\mathbf{V}(0) = \mathbf{0}$ and $\mathbf{V}'(0) = \mathbf{I}$ we conclude that Eqs. (5.7) and (5.8) are proved to be valid. □

Remark 2 Eq. (5.9) implies

$$\mathbf{V}(u) = \frac{\sigma^2}{2}\mathcal{L}^{-1}\{[\mathbf{L}_{\delta c_1}(s)]^{-1}\},$$

where $\mathcal{L}$ denotes the Laplace operator. Hence $\mathbf{V}(u)$ can be obtained by inverting each element of $\widehat{\mathbf{V}}(u)$ through partial fractions when both of the two claim amounts are from the rational family.

§6. Gerber-Shiu Penalty Functions under Threshold Dividend Strategy

It is obvious that Eqs. (3.9) and (3.11) have the common homogeneous integro-differential equation (5.6). By the same derivations of Theorem 5.2, we get

$$\vec{\Phi}_1^{(1)}(u) = \mathbf{V}(u)\vec{\Phi}_1^{(1)'}(0) - \frac{2}{\sigma^2}\int_0^u \mathbf{V}(u-x)\vec{\zeta}_1(x)\mathrm{d}x, \qquad 0 \le u < b, \tag{6.1}$$

$$\vec{\Phi}_3^{(1)}(u) = \Big[\mathbf{V}'(u) + \frac{2c_1}{\sigma^2}\mathbf{V}(u)\Big]\vec{e}_2 + \mathbf{V}(u)\vec{\Phi}_3^{(1)'}(0) - \frac{2}{\sigma^2}\int_0^u \mathbf{V}(u-x)\vec{\zeta}_3(x)\mathrm{d}x, \qquad 0 \le u < b. \tag{6.2}$$

It is calculated from Eqs. (6.1) and (5.7) that

$$\begin{aligned}\vec{\Phi}_1^{(1)}(u) &= \vec{\phi}_1(u) + \mathbf{V}(u)[\vec{\Phi}_1^{(1)'}(0) - \vec{\phi}_1'(0)] \\ &= \vec{\phi}_1(u) + \mathbf{V}(u)\vec{K}_1(b), \qquad 0 \le u < b.\end{aligned} \tag{6.3}$$

Similarly, we have from Eqs. (6.2) and (5.8) that

$$\vec{\Phi}_3^{(1)}(u) = \vec{\phi}_3(u) + \mathbf{V}(u)\vec{K}_3(b), \qquad 0 \le u < b. \tag{6.4}$$

Here $\vec{K}_1(b)$ and $\vec{K}_3(b)$ are two unknown vectors whose expressions will be given latter. Using similar arguments as in Lu and Li (2009), we can get the expressions for $\vec{\Phi}_1^{(2)}(u)$ and $\vec{\Phi}_3^{(2)}(u)$ as follows.

For $b \le u < \infty$, let $y = u - b$, and $\vec{\varphi}_1(y) = \vec{\Phi}_1^{(2)}(u) = \vec{\Phi}_1^{(2)}(y+b)$ for $y > 0$. Eq. (3.10) can be rewritten as

$$\vec{\varphi}_1''(y) = \Big(-\frac{2}{\sigma^2}\Big)\Big[c_2\vec{\varphi}_1'(y) + \mathbf{A}\vec{\varphi}_1(y) + \int_0^y \mathbf{B}(x)\vec{\varphi}_1(y-x)\mathrm{d}x + \vec{\eta}_1(y)\Big], \qquad y > 0, \tag{6.5}$$

with boundary conditions $\vec{\varphi}_1(0) = \vec{\Phi}_1^{(2)}(b+) = \vec{\Phi}_1^{(1)}(b-)$ and $\vec{\varphi}_1'(0) = \vec{\Phi}_1^{(2)'}(b+) = \vec{\Phi}_1^{(1)'}(b-)$, where

$$\vec{\eta}_1(y) = \int_y^{y+b} \mathbf{B}(x)\vec{\Phi}_1^{(1)}(y+b-x)\mathrm{d}x + \vec{\zeta}_1(y+b).$$

By the same arguments as in Theorem 5.2, the solution to Eq. (6.5) can be expressed as

$$\vec{\varphi}_1(y) = \Big[\mathbf{W}'(y) + \frac{2c_2}{\sigma^2}\mathbf{W}(y)\Big]\vec{\varphi}_1(0) + \mathbf{W}(y)\vec{\varphi}_1'(0) - \frac{2}{\sigma^2}\int_0^y \mathbf{W}(y-x)\vec{\eta}_1(x)\mathrm{d}x, \qquad y \ge 0,$$

that is,

$$\begin{aligned}\vec{\Phi}_1^{(2)}(u) &= \Big[\mathbf{W}'(u-b) + \frac{2c_2}{\sigma^2}\mathbf{W}(u-b)\Big]\vec{\Phi}_1^{(1)}(b) + \mathbf{W}(u-b)\vec{\Phi}_1^{(1)'}(b) \\ &\quad - \frac{2}{\sigma^2}\int_0^{u-b} \mathbf{W}(u-b-x)\vec{\eta}_1(x)\mathrm{d}x, \qquad b \le u < \infty,\end{aligned} \tag{6.6}$$

where

$$\mathbf{W}(y) = \frac{\sigma^2}{2}\mathcal{L}^{-1}\{[\mathbf{L}_{\delta c_2}(s)]^{-1}\}, \qquad y \geq 0. \tag{6.7}$$

Similarly, we have

$$\begin{aligned}\vec{\Phi}_3^{(2)}(u) &= \Big[\mathbf{W}'(u-b) + \frac{2c_2}{\sigma^2}\mathbf{W}(u-b)\Big]\vec{\Phi}_3^{(1)}(b) + \mathbf{W}(u-b)\vec{\Phi}_3^{(1)\prime}(b) \\ &\quad - \frac{2}{\sigma^2}\int_0^{u-b}\mathbf{W}(u-b-x)\vec{\eta}_3(x)\mathrm{d}x, \qquad b \leq u < \infty, \end{aligned} \tag{6.8}$$

where

$$\vec{\eta}_3(y) = \int_y^{y+b}\mathbf{B}(x)\vec{\Phi}_3^{(1)}(y+b-x)\mathrm{d}x + \vec{\zeta}_3(y+b) \qquad \text{for} \qquad y > 0.$$

Now we discuss the expressions for $\vec{K}_1(b)$ and $\vec{K}_3(b)$. We recall the operator T_r defined in Lu and Li (2009) for a matrix function $\mathbf{B}(y)$ whose elements are real-valued integrable functions of y.

$$T_r\mathbf{B}(y) = \int_y^{\infty} e^{-r(x-y)}\mathbf{B}(x)\mathrm{d}x, \qquad \text{for} \qquad r \in \mathbb{C},\ y \geq 0.$$

It is easy to see that, for distinct $r_1, r_2 \in \mathbb{C}$ and $y \geq 0$,

$$T_{r_1}T_{r_2}\mathbf{B}(y) = T_{r_2}T_{r_1}\mathbf{B}(y) = \frac{T_{r_1}\mathbf{B}(y) - T_{r_2}\mathbf{B}(y)}{r_2 - r_1}. \tag{6.9}$$

Multiplying both sides of Eq. (3.10) by $e^{-s(u-b)}$ and integrating with respect to u from b to ∞, we can obtain, for $s \in \mathbb{C}$ satisfying $\Re(s) \geq 0$,

$$\begin{aligned}\mathbf{L}_{\delta c_2}(s)T_s\vec{\Phi}_1^{(2)}(b) &= \Big[\Big(\frac{\sigma^2 s}{2} + c_2\Big)\vec{\Phi}_1^{(2)}(b) + \frac{\sigma^2}{2}\vec{\Phi}_1^{(2)\prime}(b)\Big] \\ &\quad - \int_0^b T_s\mathbf{B}(b-y)\vec{\Phi}_1^{(1)}(y)\mathrm{d}y - T_s\vec{\zeta}_1(b)\end{aligned}$$

which implies

$$T_s\vec{\Phi}_1^{(2)}(b) = \frac{\mathbf{L}^*_{\delta c_2}(s)\Big\{\Big[\Big(\frac{\sigma^2 s}{2} + c_2\Big)\vec{\Phi}_1^{(2)}(b) + \frac{\sigma^2}{2}\vec{\Phi}_1^{(2)\prime}(b)\Big] - \vec{\beta}_1(s)\Big\}}{\det[\mathbf{L}_{\delta c_2}(s)]}, \tag{6.10}$$

where

$$\vec{\beta}_1(s) = \int_0^b T_s\mathbf{B}(b-y)\vec{\Phi}_1^{(1)}(y)\mathrm{d}y + T_s\vec{\zeta}_1(b).$$

By the fact that the two elements of $T_s\vec{\Phi}_1^{(2)}(b)$ are finite for $s \in \mathbb{C}$ such that $\Re(s) \geq 0$, we have, from (6.10),

$$\mathbf{L}^*_{\delta c_2}(\gamma_i)\vec{\Phi}_1^{(2)\prime}(b) = \Big(-\frac{2}{\sigma^2}\Big)\Big[\Big(\frac{\sigma^2\gamma_i}{2} + c_2\Big)\mathbf{L}^*_{\delta c_2}(\gamma_i)\vec{\Phi}_1^{(2)}(b) - \mathbf{L}^*_{\delta c_2}(\gamma_i)\vec{\beta}_1(\gamma_i)\Big],$$

for $i=1,2$, where γ_i, $i=1,2$, have been given in Section 4.

Using Eq. (5.3) we derive that

$$\begin{aligned}
&\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\vec{\Phi}_1^{(2)\prime}(b)\\
=\ &\Big(-\frac{2}{\sigma^2}\Big)\Bigg[\frac{\Big(\dfrac{\sigma^2\gamma_2}{2}+c_2\Big)\mathbf{L}^*_{\delta c_2}(\gamma_2)-\Big(\dfrac{\sigma^2\gamma_1}{2}+c_2\Big)\mathbf{L}^*_{\delta c_2}(\gamma_1)}{\gamma_2-\gamma_1}\vec{\Phi}_1^{(2)}(b)\\
&-\frac{\mathbf{L}^*_{\delta c_2}(\gamma_2)\vec{\beta}_1(\gamma_2)-\mathbf{L}^*_{\delta c_2}(\gamma_1)\vec{\beta}_1(\gamma_1)}{\gamma_2-\gamma_1}\Bigg]\\
=\ &\Big(-\frac{2}{\sigma^2}\Big)\Big\{\Big[\Big(\frac{\sigma^2\gamma_2}{2}+c_2\Big)\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]+\frac{\sigma^2}{2}\mathbf{L}^*_{\delta c_2}(\gamma_1)\Big]\vec{\Phi}_1^{(2)}(b)\\
&-\big[\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\vec{\beta}_1(\gamma_2)+\mathbf{L}^*_{\delta c_2}(\gamma_1)\vec{\beta}_1[\gamma_1,\gamma_2]\big]\Big\}. \qquad (6.11)
\end{aligned}$$

With boundary conditions at b, and Eqs. (6.3) and (6.9) in hand, we can rewrite (6.11) as

$$\begin{aligned}
&\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\vec{\phi}_1'(b)+\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\mathbf{V}'(b)\vec{K}_1(b)\\
=\ &\Big[\Big(-\gamma_2-\frac{2c_2}{\sigma^2}\Big)\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]-\mathbf{L}^*_{\delta c_2}(\gamma_1)\Big]\vec{\phi}_1(b)\\
&+\Big[\Big(-\gamma_2-\frac{2c_2}{\sigma^2}\Big)\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]-\mathbf{L}^*_{\delta c_2}(\gamma_1)\Big]\mathbf{V}(b)\vec{K}_1(b)\\
&+\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\int_0^b T_{\gamma_2}\mathbf{B}(b-y)\vec{\phi}_1(y)\mathrm{d}y\\
&+\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\int_0^b T_{\gamma_2}\mathbf{B}(b-y)\mathbf{V}(y)\mathrm{d}y\vec{K}_1(b)+\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]T_{\gamma_2}\vec{\zeta}_1(b)\\
&-\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)\int_0^b T_{\gamma_1}T_{\gamma_2}\mathbf{B}(b-y)\vec{\phi}_1(y)\mathrm{d}y\\
&-\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)\int_0^b T_{\gamma_1}T_{\gamma_2}\mathbf{B}(b-y)\mathbf{V}(y)\mathrm{d}y\vec{K}_1(b)-\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)T_{\gamma_1}T_{\gamma_2}\vec{\zeta}_1(b),
\end{aligned}$$

from which we get

$$\vec{K}_1(b)=\mathbf{H}^{-1}\vec{R}_1, \qquad (6.12)$$

where

$$\begin{aligned}
\mathbf{H}\ =\ &\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\mathbf{V}'(b)+\Big[\Big(\gamma_2+\frac{2c_2}{\sigma^2}\Big)\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]+\mathbf{L}^*_{\delta c_2}(\gamma_1)\Big]\mathbf{V}(b)\\
&-\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\int_0^b T_{\gamma_2}\mathbf{B}(b-y)\mathbf{V}(y)\mathrm{d}y\\
&+\frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)\int_0^b T_{\gamma_1}T_{\gamma_2}\mathbf{B}(b-y)\mathbf{V}(y)\mathrm{d}y
\end{aligned}$$

and

$$
\begin{aligned}
\vec{R}_1 \;=\; & -\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\vec{\phi}_1'(b) - \Big[\Big(\gamma_2 + \frac{2c_2}{\sigma^2}\Big)\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2] + \mathbf{L}^*_{\delta c_2}(\gamma_1)\Big]\vec{\phi}_1(b) \\
& + \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\int_0^b T_{\gamma_2}\mathbf{B}(b-y)\vec{\phi}_1(y)\mathrm{d}y \\
& - \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)\int_0^b T_{\gamma_1}T_{\gamma_2}\mathbf{B}(b-y)\vec{\phi}_1(y)\mathrm{d}y \\
& + \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]T_{\gamma_2}\vec{\zeta}_1(b) - \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)T_{\gamma_1}T_{\gamma_2}\vec{\zeta}_1(b).
\end{aligned}
$$

By the same arguments, we obtain

$$\vec{K}_3(b) = \mathbf{H}^{-1}\vec{R}_3, \tag{6.13}$$

where

$$
\begin{aligned}
\vec{R}_3 \;=\; & -\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\vec{\phi}_3'(b) - \Big[\Big(\gamma_2 + \frac{2c_2}{\sigma^2}\Big)\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2] + \mathbf{L}^*_{\delta c_2}(\gamma_1)\Big]\vec{\phi}_3(b) \\
& + \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]\int_0^b T_{\gamma_2}\mathbf{B}(b-y)\vec{\phi}_3(y)\mathrm{d}y \\
& - \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)\int_0^b T_{\gamma_1}T_{\gamma_2}\mathbf{B}(b-y)\vec{\phi}_3(y)\mathrm{d}y \\
& + \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}[\gamma_1,\gamma_2]T_{\gamma_2}\vec{\zeta}_3(b) - \frac{2}{\sigma^2}\mathbf{L}^*_{\delta c_2}(\gamma_1)T_{\gamma_1}T_{\gamma_2}\vec{\zeta}_3(b).
\end{aligned}
$$

Finally, we summarize our main results in the following theorem.

Theorem 6.1 The analytical expressions for $\vec{\Phi}_1^{(1)}(u)$, $\vec{\Phi}_1^{(2)}(u)$,$\vec{\Phi}_3^{(1)}(u)$ and $\vec{\Phi}_3^{(2)}(u)$ are given by

$$
\begin{cases}
\vec{\Phi}_1^{(1)}(u) = \vec{\phi}_1(u) + \mathbf{V}(u)\vec{K}_1(b), \qquad 0 \le u < b, \\
\vec{\Phi}_1^{(2)}(u) = \Big[\mathbf{W}'(u-b) + \dfrac{2c_2}{\sigma^2}\mathbf{W}(u-b)\Big]\vec{\Phi}_1^{(1)}(b) + \mathbf{W}(u-b)\vec{\Phi}_1^{(1)\prime}(b) \\
\qquad\qquad -\dfrac{2}{\sigma^2}\displaystyle\int_0^{u-b}\mathbf{W}(u-b-x)\vec{\eta}_1(x)\mathrm{d}x, \qquad b \le u < \infty,
\end{cases}
$$

and

$$
\begin{cases}
\vec{\Phi}_3^{(1)}(u) = \vec{\phi}_3(u) + \mathbf{V}(u)\vec{K}_3(b), \qquad 0 \le u < b, \\
\vec{\Phi}_3^{(2)}(u) = \Big[\mathbf{W}'(u-b) + \dfrac{2c_2}{\sigma^2}\mathbf{W}(u-b)\Big]\vec{\Phi}_3^{(1)}(b) + \mathbf{W}(u-b)\vec{\Phi}_3^{(1)\prime}(b) \\
\qquad\qquad -\dfrac{2}{\sigma^2}\displaystyle\int_0^{u-b}\mathbf{W}(u-b-x)\vec{\eta}_3(x)\mathrm{d}x, \qquad b \le u < \infty,
\end{cases}
$$

where $\vec{\phi}_1(u)$ and $\vec{\phi}_3(u)$ are given by Theorem 5.2, and $\mathbf{V}(u)$, $\mathbf{W}(u)$, $\vec{K}_1(b)$, $\vec{K}_3(b)$ are given by (5.9), (6.7), (6.12), (6.13), respectively.

Acknowledgements The authors would like to thank the editor and the anonymous referees for their helpful suggestions on the earlier version of this paper.

References

[1] De Finetti, B., Su un'impostazione alternativa dell teoria colletiva del rischio, *Transactions of the XV International Congress of Actuaries*, **2**(1957), 433–443.

[2] Lin, X.S., Willmot, G.E. and Drekic, S., The classical risk model with a constant dividend barrier: Analysis of the Gerber-Shiu discounted penalty function, *Insurance: Mathematics and Economics*, **33(3)**(2003), 551–566.

[3] Gerber, H.U. and Shiu, E.S.W., On optiomal dividend strategies in the compound Poisson model, *North American Actuarial Journal*, **10(2)**(2006), 76–93.

[4] Lin, X.S. and Pavlova, K.P., The compound Poisson risk model with a threshold dividend strategy, *Insurance: Mathematics and Economics*, **38(1)**(2006), 57–80.

[5] Li, B., Wu, R. and Song, M., A renewal jump-diffusion process with threshold dividend strategy, *Journal of Computational and Applied Mathematics*, **228(1)**(2009), 41–55.

[6] Yang, H. and Zhang, Z., Gerber-Shiu discounted penalty function in a Sparre Andersen model with multi-layer dividend strategy, *Insurance: Mathematics and Economics*, **42(3)**(2008), 984–991.

[7] Lu, Y. and Li, S., The Markovian regime-switching risk model with a threshold dividend strategy, *Insurance: Mathematics and Economics*, **44(2)**(2009), 296–303.

[8] Wan, N., Dividend payments with a threshold strategy in the compound Poisson risk model perturbed by diffusion, *Insurance: Mathematics and Economics*, **40(3)**(2007), 509–532.

[9] Meng, H., Zhang, C. and Wu, R., The expectation of aggregate discounted dividends for a Sparre Andersen risk process perturbed by diffusion, *Applied Stochastic Models in Business and Industry*, **23(4)**(2007), 273–291.

[10] Gao, H. and Yin, C., The perturbed Sparre Anderssen model with a threshold dividend strategy, *Journal of Computational and Applied Mathematics*, **220(1-2)**(2008), 394–408.

[11] Yuen, K.C., Guo, J. and Wu, X., On a correlated aggragate claims model with Poisson and Erlang risk processes, *Insurance: Mathematics and Economics*, **31(2)**(2002), 205–214.

[12] Garrido, J. and Li, S., Ruin probabilities for two classes of risk processes, *ASTIN Bulletin*, **35(1)**(2005), 61–77.

[13] Li, S. and Lu, Y., On the expected discounted penalty functions for two classes of risk processes, *Insurance: Mathematics and Economics*, **36(2)**(2005), 179–193.

[14] Zhang, Z., Li, S. and Yang, H., The Gerber-Shiu discounted penalty functions for a risk model with two classes of claims, *Journal of Computational and Applied Mathematics*, **230(2)**(2009), 643–655.

[15] Lu, Z., Xu, W., Sun, D. and Han, W., On the expected discounted penalty functions for two classes of risk processes under a threshold dividend strategy, *Journal of Computational and Applied Mathematics*, **232(2)**(2009), 582–593.

[16] Chadjiconstantinidis, S. and Papaioannou, A.D., Analysis of the Gerber-Shiu function and dividend barrier problems for a risk process with two classes of claims, *Insurance: Mathematics and Economics*, **45(3)**(2009), 470–484.

[17] Burton, T.A., *Volterra Integral and Differential Equations*, World Scientific, Singapore, 2005.

Threshold分红策略下带干扰的两类索赔风险模型的Geber-Shiu函数

孙国红[1] 张春生[2] 季兰朋[2,3]

([1]天津农学院基础科学学系, 天津, 300384; [2]南开大学数学学院, 天津, 300071;

[3]瑞士洛桑大学精算系, 洛桑, 1015)

本文研究了在threshold分红策略下带干扰的两类索赔风险模型的Geber-Shiu函数. 这里假设两个索赔计数过程为独立的更新过程, 其中一个为Poisson过程另一个为时间间隔服从广义Erlang(2)分布的更新过程. 本文得到了threshold分红策略下Gerber-Shiu函数所满足的积分–微分方程及其边界条件. 最后, 本文指出threshold分红策略下Gerber-Shiu函数可以由不分红(即: $b=\infty$)时所对应的Geber-Shiu函数和一个齐次积分–微分方程的线性独立解表示出来.

关键词: 两类索赔, Geber-Shiu函数, threshold分红策略, 积分–微分方程.

学科分类号: O211.6.