Availability Analysis of a Repairable k-out-of-n:G System with a History-Dependent Critical State

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Abstract

For a repairable k-out-of-n:G system with a history-dependent critical state, this paper derives the availability, the mean up time and the mean down time in one renewal cycle when the system goes into the steady state. A comparison between this system and a repairable k-out-of-n:G system without a history-dependent critical state is conducted as well.

Keywords: Alternating renewal process, availability, exponential distribution, mean up (down) time.

AMS Subject Classification: 90B25.

§1. Introduction

Since a k-out-of-n:G system functions if and only if at least k of its components work, or equivalently, at most n-k of its components fail, the total life time of a non-repairable k-out-of-n:G system is in fact just the (n-k+1)-th order statistic of all n random life times X_1, \ldots, X_n . This fault-tolerant structure has been investigated in large scale since its earlier appearance in 1980. Quite a lot of works related to reliability analysis have been completed in the past decades, leading to a large body of research papers in literature. One may refer to Barlow and Proschan (1975), Trivedi (2001), Kuo and Zuo (2002) and Pham (2003) etc for more details.

A repairable k-out-of-n:G system fails only if the total number of failed components reaches n-k+1 at any instant of time. Assume that at least k components of the system are in working state at time 0 such that the system is in the up state at time 0. When a component fails, repair work is launched right away to restore the failed component. If the number of all components in the failure state reaches n-k+1, a threshold, the system makes a transition from the up state to the down state. When the system is down, repair work continues on the failed components and the system will return to the up state as soon as the number of failed components becomes strictly smaller than n-k+1.

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Because of a great deal of successful applications in various areas, availability analysis of the repairable k-out-of-n: G systems with or without further failures after system breaks down received much more attention from several authors. In a peer-to-peer network, a node usually connects itself to the network through several, say n, links between its neighbor nodes and itself; If it is necessary for the node to have at least k links so that it can effectively communicate the information through network, then, we have a k-out-of-n:G structure at hand when doing resilience analysis. In this model, those active links may fail even though the node fails to successfully communicate in the network. On the other hand, in industrial and electronic engineering, some major working components are usually suspended in order to protect them from being overwhelmingly damaged from a minor or accidental failure. In literature, some authors have completed interesting researching works on the k-out-of-n:G systems with or without further failure when system breaks down. For example, through using renewal theory, Angus (1988) proposed a simple method to build availability indices for k-out-of-n:G system with active components being suspended when system breaks down; Ross (1996) built the availability as well as mean time between failures a repairable parallel system without suspending any active components when system breaks down; and Barlow and Proschan (1975) also had a discussion on system with those active components suspended after system's breakdown. Afterward, Sherwin (2000) and Pham (2003) made discussions on a series system and address the difference between the availabilities of the system with and without suspending the active components when system is down; Recently, Li, Zuo and Yam (2006) further investigated the k-out-of-n:G system with some active components being suspended when system breaks down, they presented formulae for the availability, the mean time between failures and the mean working time in a cycle as well. For a comprehensive review of most research work on repairable k-out-of-n:G system models with independent and identical components, one may refer to Kuo and Zuo (2002).

This paper studies a repairable k-out-of-n:G system with a critical state which is determined by the previous history of the system in the sense that, when there are only k-1 active components, the system is active if the previous number of active components is k, and the system is in failure state if the previous number of active components is k-2. Under the assumption that all working units are suspended when system breaks down, we derive the system's availability, the mean up time and the mean down time in one renewal cycle when the system goes into the steady state. We also make a discussion on a repairable k-out-of-n:G system with a history-dependent critical state and that without such kind of state.

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§2. Model

For the sake of convenience, we first list nomenclature and notations.

2.1 Nomenclature

order statistics the ordered array (from the smallest one to the largest one) of a set

of random variables; For any $1 \leq k \leq n$, the k-th order statistic

represents the (n-k+1)-th smallest one among the concerned set

of random variables.

state the total number of the active components, it may be any integer in

 $\{0,1,\ldots,n\}.$

up a mode of the system which composed of those states under which

the system functions, for example, k, \ldots, n .

down a mode of the system which is composed of those states under which

the system fails to function, for example, $0, \ldots, k-2$.

history the sequence of the state changes just before the current time, for

example, $\{n, n-1, n-2, n-1, n-2, \dots, k+2, k+1, k\}$, k is the

current state of the system.

critical state a state that a change of the mode of the system may occur once it is

entered at a time.

renewal cycle the time interval spanning between two consecutive time points at

which the system enters down mode; As the time length of the interval of a renewal process, it includes both the time that system is down

and the time that system is up.

2.2 Notations

 X_i random life time of the *i*-th component.

 Y_i random repair time of the *i*-th component.

 $X_{i:n}$ the *i*-th order statistic of X_1, \ldots, X_n .

 $Y_{j:n}$ the j-th order statistic of Y_1, \ldots, Y_n .

 λ the common failure rate of all components.

 μ the common repair rate of all components.

U the time that the system with history-dependent state is up in a cycle.

D the time that the system with history-dependent state is down in a cycle.

N(t) total number of active components at time t.

S(t) total number of system's breakdowns till time t.

P(t) the probability that the system with history-dependent state is up at t.

- $p(k, \lambda, \mu)$ the limit probability that the previous adjacent state is k given N(t) = k-1.

 MTBF mean time between failures of the historical dependent system.
 - U_1 the time that the system without history-dependent state is up in a cycle.
 - D_1 the time that the system without history-dependent state is down in a cycle.
 - $P_1(t)$ probability that the system without history-dependent state is up at time t.

In some practical situations, for example, computer engineering, electronic engineering, manufacturing and defense industry etc., a special instrument is usually installed to delay the occurrence of the failure of the system so as to avoid hazardous consequence due to unexpected failure. For instance, an engineer using a computer with a UPS instrument may have enough time to safely retreat when the electricity is interrupted as a sudden because the UPS is capable of extending the supply of electricity to some extent. In reliability model, this mechanism may be described through introducing a critical state which is Markov dependent upon the previous mode, that is, the mode of this state is completely determined by the previous history of the system's mode as follows: if the system transform itself from a state at which system is up to this state, then the system is in up mode as well; however, if the system transfers itself from a state at which system is down to this state, then the system is in down mode also.

This paper deals with a k-out-of-n:G system, which has a history-dependent critical state k-1 and suspends all active components after system breaks down. To be clear, all assumptions are listed as below.

- A1 **Identical exponential life time** Random lives of all components have a common exponential distribution.
- A2 Identical exponential repair time Repair times of all components have the other common exponential distribution.
- A3 **Statistical independence** All random lives are statistically independent, all repair times are statistically independent, and the life time and the repair time of each component are statistically independent.
- A4 **History-dependent critical state** If the number of those active components falls into k-1 from k, the system keeps in up state; the system is down as soon as the number of active components goes further down to k-2 from k-1. If the number of active components is k-2, the system keeps in down state; the system keeps in down state if the number of active components goes up to k-1 from k-2.
- A5 No further failure after system breaks down When the system breaks down, all these active components are suspended, that is, there will be no further failure in the time interval that the system is down.

A6 No limitation on repair facilities There are n-k+2 repair facilities so that a component is assigned a facility as soon as it fails.

A7 **Perfect repair** Each failed component starts to function as good as a completely new as soon as it is repaired.

In this model, the state of the system is just the total number of these components in up state, and k-1 serves as the critical history-dependent state. That is, if the total number of active components goes down to k-1 from k, then the system is in up state; however, if the total number of active components goes up to k-1 from k-2, then the system is in down state, see Figure 1 for a depiction of the path of the critical state which depends upon system's previous history.

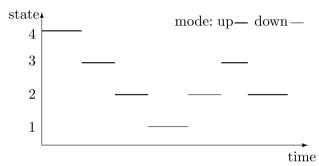


Figure 1 A path of a 3-out-of-4:G system with history-dependent state 2

It is evident that the first time to failure of the k-out-of-n:G system with history-dependent state k-1 is X_{n-k+2} , which is just the first time to failure of a k-1-out-of-n:G system without history-dependent state. However, there is an obvious difference between these two systems when repair is taken into account.

By using technique of renewal theory as in Angus (1988), we will derive, in next section, formulas for main availability indices including the mean down time in a failure-repair cycle, the steady state availability and the mean up time in a failure-repair cycle of such a k-out-of-n:G system.

§3. Main Indices

This section derives expressions of the following availability indices of the system under study: mean down time in a cycle, steady state availability, mean up time in a cycle.

It is obvious that the process of state changes of each component forms an independent alternating renewal process with the state transition time having exponential distribution. More specifically, the *i*-th component, i = 1, ..., n, is in the active state for an exponential amount of time X_i , then goes down, and stays in the down state for an exponential time

 Y_i before going back to the up state as good as new. It is evident that the state of the system N(t) varies in $\{k-2,k-1,\ldots,n\}$. As a result, the mode changes of the system constitutes another delayed alternating renewal process, which is up at time 0 until the number of components simultaneously in the failure state reaches n-k+1. The system then stays in the down mode until the number of failed components goes below n-k+1. It should be noted that repair is in progress whenever there is a failed component in the system. It is of interest to find mean time between failures of the system, mean down time in a failure-repair cycle, and mean up time in a failure-repair cycle of the system.

Random lives and repair times of all components are exponential, that is, for $i = 1, \ldots, n$,

$$P(X_i \le t) = 1 - e^{-\lambda t}, \qquad t \ge 0,$$

and

$$P(Y_i \le t) = 1 - e^{-\mu t}, \quad t \ge 0.$$

According to the limit theorems of the alternating renewal process (Ross, 1996), the limiting point availability is

$$\lim_{t \to \infty} \mathsf{P}(\text{the } i\text{-th unit is up at } t) = \frac{\mu}{\lambda + \mu}, \qquad i = 1, \dots, n, \tag{3.1}$$

and the limiting point unavailability is

$$\lim_{t \to \infty} \mathsf{P}(\text{the } i\text{-th unit is down at } t) = \frac{\lambda}{\lambda + \mu}, \qquad i = 1, \dots, n. \tag{3.2}$$

Now, we are ready to derive the main indices.

3.1 Expected Down Time

Theorem 3.1 For $2 \le k \le n$,

$$\mathsf{E}[D] = \left(\frac{1}{n-k+2} + \frac{1}{n-k+1}\right) \cdot \frac{1}{\mu}.\tag{3.3}$$

Proof Since there are k-2 active components and n-k+2 failed components when the system enters the failure state and these active ones are suspended once the system breaks down and the repair facilities are not limited, the system recovers to up mode as soon as two of those failed components are successfully repaired. According to Corollary 2.6 in Barlow and Proschan (1975), both $(n-k+2)Y_{1:n-k+2}$ and $(n-k+1)\cdot (Y_{2:n-k+2}-Y_{1:n-k+2})$ are exponential with rate μ . Thus, the mean time the system is down in a cycle is

$$\begin{split} \mathsf{E}[D] &= \mathsf{E}[Y_{2:n-k+2}] \\ &= \mathsf{E}[Y_{1:n-k+2}] + \mathsf{E}[Y_{2:n-k+2} - Y_{1:n-k+2}] \\ &= \left(\frac{1}{n-k+2} + \frac{1}{n-k+1}\right) \cdot \frac{1}{\mu}. \end{split} \ \Box$$

3.2 Stationary Availability and Expected Up Time

Owing to the mutual independence among all components, it follows from (3.1) and (3.2) that, as time tends to infinity, N(t), the total number of active components has the binomial distribution with the number of trials n and the probability of success $\mu/(\lambda + \mu)$. As a result, it holds that

$$\lim_{t\to\infty} \mathsf{P}(N(t)\geq k) = \left\lceil \sum_{r=k}^n \binom{n}{r} \bigg(\frac{\mu}{\lambda+\mu}\bigg)^r \bigg(\frac{\lambda}{\lambda+\mu}\bigg)^{n-r}\right\rceil \bigg/ \left\lceil \sum_{r=k-2}^n \binom{n}{r} \bigg(\frac{\mu}{\lambda}\bigg)^r\right\rceil.$$

Theorem 3.2 For $2 \le k \le n$,

$$\lim_{t \to \infty} P(t) = \left[\sum_{r=k}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda} \right)^r + p(k, \lambda, \mu) \binom{n}{k-1} \left(\frac{\mu}{\lambda} \right)^{k-1} \right] / \left[\sum_{r=k-2}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda} \right)^r \right], (3.4)$$

where

$$p(k,\lambda,\mu) = \frac{(n-k+1)(n-k+2)\mu + (k-1)(n-k+1)\lambda}{(n-k+1)(n-k+2)\mu + (k-1)(2n-2k+3)\lambda}.$$
 (3.5)

Proof The system is up if and only if the total number of active components is not smaller than k or equivalently, the total number of active components is k-1 and the previous total number of active components is k. In view of the fact that these active components are suspended instantaneously once the system enters down mode, $[N(t)|N(t) \ge k-2]$, the total number of active components in the system has the truncated binomial distribution with the number of trials n and the probability of success $\mu/(\lambda + \mu)$. Thus, by Blackwell's theorem, for small h > 0,

$$\frac{h}{\text{MTBF}} = \lim_{t \to \infty} \mathbb{E}[S(t+h) - S(t)]$$

$$= \lim_{t \to \infty} \sum_{r=0}^{\infty} r \cdot P(\text{there are } r \text{ system's breakdown in } (t, t+h))$$

$$= \lim_{t \to \infty} P(\text{there is only one system's breakdowns in } (t, t+h)) + o(h)$$

$$= \lim_{t \to \infty} P(N(t) = k-1, N(t+h) = k-2, \text{ the adjacent state before } t \text{ is } k) + o(h)$$

$$= \lim_{t \to \infty} P(N(t) = k-1) \cdot \lim_{t \to \infty} P(\text{the adjacent state before } t \text{ is } k \mid N(t) = k-1)$$

$$\cdot \lim_{t \to \infty} P(N(t+h) = k-2 \mid N(t) = k-1) + o(h)$$

$$= \frac{\binom{n}{k-1} \left(\frac{\mu}{\lambda + \mu}\right)^{k-1} \left(\frac{\lambda}{\lambda + \mu}\right)^{n-k+1}}{\sum_{r=k-2}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda + \mu}\right)^{r} \left(\frac{\lambda}{\lambda + \mu}\right)^{n-r}} \cdot (k, \lambda, \mu) \cdot (k-1)\lambda h + o(h)$$

$$= \left\{ \left[\binom{n}{k-1} \left(\frac{\mu}{\lambda}\right)^{k-1}\right] \middle/ \left[\sum_{r=k-2}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda}\right)^{r}\right] \right\} \cdot p(k, \lambda, \mu) \cdot (k-1)\lambda h + o(h).$$

Thus, it holds that

$$MTBF = \frac{\sum_{r=k-2}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda}\right)^{r}}{\binom{n}{k-2} \left(\frac{\mu}{\lambda}\right)^{k-2} \cdot (n-k+2)\mu} \cdot \frac{1}{p(k,\lambda,\mu)}.$$
 (3.6)

On the other hand, the limit probability for the system to be up is

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \mathsf{P}(N(t) \ge k) + \lim_{t \to \infty} \mathsf{P} \binom{\text{the adjacent state before}}{\text{time } t \text{ is } k \text{ and } N(t) = k-1}$$

$$= \frac{\sum\limits_{r=k}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda}\right)^r + \binom{n}{k-1} \left(\frac{\mu}{\lambda}\right)^{k-1} \cdot p(k,\lambda,\mu)}{\sum\limits_{r=k-2}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda}\right)^r}.$$

According to the limit theorem of the alternate renewal process (see Ross (1996) and Karlin and Taylor (2002)),

$$\lim_{t\to\infty}P(t)=\frac{\mathsf{E}[U]}{\mathsf{E}[U]+\mathsf{E}[D]}.$$

By (3.3) and (3.4), we have

$$\mathsf{E}[U] = \frac{\lim_{t \to \infty} P(t)}{1 - \lim_{t \to \infty} P(t)} \mathsf{E}[D]$$

$$= \frac{\sum_{r=k}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda}\right)^{r} + \binom{n}{k-1} \left(\frac{\mu}{\lambda}\right)^{k-1} \cdot p(k,\lambda,\mu)}{\binom{n}{k-2} \left(\frac{\mu}{\lambda}\right)^{k-2} + \binom{n}{k-1} \left(\frac{\mu}{\lambda}\right)^{k-1} [1 - p(k,\lambda,\mu)]} \cdot \left(\frac{1}{n-k+2} + \frac{1}{n-k+1}\right) \cdot \frac{1}{\mu}$$

$$(3.7)$$

and hence

MTBF =
$$\mathsf{E}[U] + \mathsf{E}[D]$$

= $\frac{\sum\limits_{r=k-2}^{n} \binom{n}{r} \left(\frac{\mu}{\lambda}\right)^r \cdot \left(\frac{1}{n-k+2} + \frac{1}{n-k+1}\right) \cdot \frac{1}{\mu}}{\binom{n}{k-2} \left(\frac{\mu}{\lambda}\right)^{k-2} + \binom{n}{k-1} \left(\frac{\mu}{\lambda}\right)^{k-1} [1 - p(k, \lambda, \mu)]}$ (3.8)

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Setting the right sides of (3.6) and (3.8) equal, (3.5) follows immediately from the resulted equation. Now, the desired result in (3.4) is validated.

The expected up time in one cycle is an immediate corollary of Theorem 3.1 and Theorem 3.2.

Corollary 3.1 For $2 \le k \le n$, $\mathsf{E}[U]$ is determined by (3.5) and (3.7).

§4. Discussion

In order to get a better understanding of the main results, we list as below those corresponding results on the system without any history-dependent critical state, reader may see Angus (1988) and Li, Zuo and Yam (2006) for more details.

The stationary availability is

$$\lim_{t \to \infty} P_1(t) = \left[\sum_{r=k}^n \binom{n}{r} \left(\frac{\mu}{\lambda} \right)^r \right] / \left[\sum_{r=k-1}^n \binom{n}{r} \left(\frac{\mu}{\lambda} \right)^r \right], \tag{4.1}$$

the mean down time in a cycle is

$$\mathsf{E}[D_1] = \frac{1}{(n-k+1)\mu},\tag{4.2}$$

and the mean up time in a cycle is

$$\mathsf{E}[U_1] = \left\{ \left[\sum_{r=k}^n \binom{n}{r} \left(\frac{\mu}{\lambda} \right)^r \right] \middle/ \left[\binom{n}{k-1} \left(\frac{\mu}{\lambda} \right)^{k-1} \right] \right\} \cdot \frac{1}{(n-k+1)\mu}. \tag{4.3}$$

By (3.3) and (4.2), we have the following evident corollary. The system with history-dependent critical state has a larger down time in a cycle since, in this case two of n-k+2 failed components need to be repaired, however, it is enough to repair only one of n-k+1 failed components. In view of the fact that in the case with history-dependent critical state, system is also up when it transforms from state k to k-1, the up time of the system with history-dependent critical state in a cycle is stochastically larger than that of the system without any history-dependent critical state.

Corollary 4.1 For
$$2 \le k \le n$$
, it holds that $E[D_1] < E[D]$ and $E[U_1] < E[U]$.

As far as the mean up time in a cycle is concerned, we have the next corollary, which asserts that system with history-dependent critical state has a larger stationary availability under some circumstance.

Corollary 4.2 For $2 \le k \le n$, if

$$\frac{\mu}{\lambda} \ge \frac{k-1}{n-k+2} \sqrt{\frac{2n-2k+3}{n-k+1}},\tag{4.4}$$

then $\lim_{t\to\infty} P_1(t) < \lim_{t\to\infty} P(t)$.

Proof Since

$$\frac{\mu}{\lambda} \ge \frac{k-1}{n-k+2} \sqrt{\frac{2n-2k+3}{n-k+1}},$$

it holds that

$$p(k,\lambda,\mu) \binom{n}{k-1} \left(\frac{\mu}{\lambda}\right)^{k-1} \ge \binom{n}{k-2} \left(\frac{\mu}{\lambda}\right)^{k-2}.$$

Note that $a/b < (a + c_a)/(b + c_b)$ for b > a > 0 and $c_a \ge c_b > 0$, by (3.4) and (4.1), it can be concluded that $\lim_{t\to\infty} P_1(t) < \lim_{t\to\infty} P(t)$.

Figure 2 plots availability of a 6-out-of-10 system with respect to the Example 1 ratio μ/λ . As can be seen, (i) the availability in (3.4) is not necessarily large than that in (4.1), (ii) condition (4.4) holds for $1.2 < \mu/\lambda \le 2.0$ and $\lim_{t \to \infty} P_1(t) < \lim_{t \to \infty} P(t)$, (iii) for $0.6 \le \mu/\lambda \le 1.2$, condition (4.4) does not hold. By numerical method, it holds that $\lim_{t \to \infty} P_1(t) > \lim_{t \to \infty} P(t) \text{ for } \mu/\lambda \le 0.5732 \text{ and } \lim_{t \to \infty} P_1(t) < \lim_{t \to \infty} P(t) \text{ for } 0.5732 \le \mu/\lambda \le 1.2.$

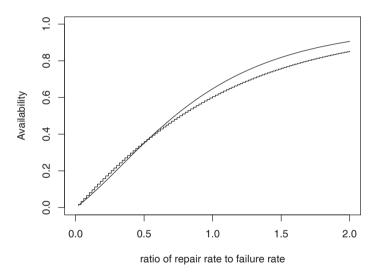


Figure 2 Solid curve and stepwise curve correspond to limits of P(t) and $P_1(t)$, respectively.

To end this paper, we make the following remarks.

1. State changes of the repairable k-out-of-n:G system under investigation can also be characterized as a Markov process with state k-1 being defined as two completely different states according to the history of the system, one can easily write out the transition matrix of process and thus gets the desired result through traditional analysis of the transition matrix (see Karlin and Taylor (2002)); Nonetheless, this is usually both laborious and unwieldy when the system size n is relatively large.

2. It is of interest to study the behavior of the two-dimensional k-out-of-n:G system which has history-dependent critical states though we don't know any concrete applications at this time yet.

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具有历史相依临界状态的可维修n中取k:G系统的可用度分析

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对于一个具有历史相依临界状态的可维修n中取k:G系统,论文给出了当系统平稳时它的可用度,一个循环中的平均工作时间和平均失效时间.并且和不具有年龄相依临界状态的可维修n中取k:G系统进行了比较.

关键词: 交替更新过程,可用度,指数分布,平均工作(失效)时间.

学科分类号: O213.2.