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Ruin Probabilities for the Discrete Risk Models with Markov Chain Interest *

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Abstract

In this paper, we consider a discrete time risk process with random interest force. With the assumption that the interest rate process behaves as a Markov chain, we obtain the recursive equations and integral equations for finite and ultimate ruin probabilities, and Lundberg inequalities for the ultimate ruin probabilities are also provided.

Keywords: Discrete-time risk model, ruin probability, recursive equation, Lundberg inequalities.

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§1. Introduction

In this paper, we study the ruin probabilities for a discrete time risk model, in which the surplus process is expressed by a recursive equation

$$U_n = (U_{n-1} + X_n)(1 + I_n) - Y_n, \qquad n = 1, 2, \dots,$$
(1.1)

where $U_0 = u$ is the initial surplus of an insurance company, $\{X_n; n \ge 1\}$, $\{Y_n; n \ge 1\}$, $\{I_n; n \ge 1\}$ are three sequences of independent and identically distributed nonnegative random variables and $\{X_n\}$, $\{Y_n\}$ and $\{I_n\}$ are independent. X_n denotes the amount of premiums during the *n*th period, and is received at the beginning of the *n*th period. Y_n is the amount of claims during the *n*th period, and is paid at the end of the *n*th period. X_n , Y_n have probability distribution functions G and F respectively. I_n denotes the rate of interest during the *n*th period, i.e. from time n - 1 to time n, here we assume that I_n evolves as a Markov chain with a denumerable state \mathbb{E} consisting of nonnegative integers.

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By iteration of (1.1), it follows that, for any $n \ge 1$,

$$U_n = u \prod_{k=1}^n (1+I_k) + \sum_{k=1}^n \left((X_k(1+I_k) - Y_k) \prod_{t=k+1}^n (1+I_t) \right),$$
(1.2)

where $\prod_{t=m}^{n} (1 + I_t) = 1$ if m > n.

Let (p_{ij}) be the matrix of transition probabilities of $\{I_n\}$, i.e.

$$p_{ij} = \mathsf{P}(I_{n+1} = j | I_n = i), \tag{1.3}$$

where $p_{ij} \ge 0$ and $\sum_{j} p_{ij} = 1$ for all $i, j \in \mathbb{E}$. The run probabilities when we given the initial surplus u and the initial interest rate $I_0 = i$ is defined as

$$\psi(u,i) = \mathsf{P}\Big(\bigcup_{k=1}^{\infty} \{U_k < 0\} | U_0 = u, I_0 = i\Big).$$
(1.4)

Similarly, we define the ruin probabilities in the finite-horizon as

$$\psi_n(u,i) = \mathsf{P}\Big(\bigcup_{k=1}^n \{U_k < 0\} | U_0 = u, I_0 = i\Big).$$
(1.5)

It is clear that $0 \leq \psi_1(u,i) \leq \psi_2(u,i) \leq \cdots \leq \psi_n(u,i) \leq \cdots$, and $\psi(u,i) = \lim_{n \to \infty} \psi_n(u,i)$.

The model (1.1) has been discussed in several references. Yang (1999) considered the case when $\{I_n\}$ are identical constants. Cai (2002) discussed a generalization of the model (1.1), where $\{I_n\}$ are assumed to have a dependent autoregressive structure of order 1, and derived the ruin probability. when $\{I_n\}$ are independent identical distributed random variables, Yang and Zhang (2006) studied the ruin probability are obtained. Wei and Hu (2008) considered the recursive integral equations for the finite time ruin probability and the bounds for the finite time ruin probability and the bounds for the finite time ruin probability and the bounds for the ultimate ruin probability are also derived.

In this paper, we consider the model (1.1), where the interest rate process behaves as a denumerable state Markov chain, we use a similar method to that in Diasparra and Romera (2009) to obtain the recursive equations and generalized Lundberg inequalities for the ruin probabilities.

The rest of the paper is organized as follows. In Section 2, we will present the recursive integral equations for the ruin probabilities for $\psi_n(u,i)$ and $\psi(u,i)$. The bounds for the ultimate ruin probabilities $\psi(u,i)$ will be given in Section 3.

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§2. Recursive Equations for the Ruin Probabilities

Theorem 2.1 Let $U_0 = u \ge 0$ and p_{ij} be as defined in Section 1, $\tau_j = (u+\omega)(1+j)$, then

$$\psi_1(u,i) = \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \overline{F}(\tau_j) \mathrm{d}G(\omega), \qquad (2.1)$$

and for n = 1, 2, ...,

$$\psi_{n+1}(u,i) = \sum_{j \in \mathbb{R}} p_{ij} \int_0^\infty \int_0^{\tau_j} \psi_n(\tau_j - y, j) \mathrm{d}F(y) \mathrm{d}G(\omega) + \sum_{j \in \mathbb{R}} p_{ij} \int_0^\infty \overline{F}(\tau_j) \mathrm{d}G(\omega).$$
(2.2)

Consequently, we have

$$\psi(u,i) = \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \int_0^{\tau_j} \psi(\tau_j - y, j) \mathrm{d}F(y) \mathrm{d}G(\omega) + \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \overline{F}(\tau_j) \mathrm{d}G(\omega).$$
(2.3)

Proof From (1.1), we have that $U_1 = (u + X_1)(1 + I_1) - Y_1$, then

$$\psi_1(u,i) = \mathsf{P}(Y_1 > (u+X_1)(1+I_1)|I_0 = i) = \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \overline{F}(\tau_j) \mathrm{d}G(\omega).$$

Given $Y_1 = y$, $X_1 = \omega$ and $I_i = j$, if $y > \tau_j$, then

$$\mathsf{P}(U_1 < 0 | Y_1 = y, I_1 = j, U_0 = u, I_0 = i) = 1,$$

which implies that

$$\mathsf{P}\Big(\bigcup_{k=1}^{n+1} (U_k < 0) \Big| Y_1 = y, I_1 = j, U_0 = u, I_0 = i\Big) = 1.$$

If $0 \leq y \leq \tau_j$, then $\mathsf{P}(U_1 < 0 | Y_1 = y, I_1 = j, U_0 = u, I_0 = i) = 0$, Thus, from (1.2), for $0 \leq y \leq \tau_j$,

$$\begin{split} &\mathsf{P}\Big(\bigcup_{k=1}^{n+1} (U_k < 0) \big| Y_1 = y, I_1 = j, U_0 = u, I_0 = i \Big) \\ &= \mathsf{P}\Big(\bigcup_{k=2}^{n+1} (U_k < 0) \big| Y_1 = y, I_1 = j, U_0 = u, I_0 = i \Big) \\ &= \mathsf{P}\Big(\bigcup_{k=2}^{n+1} \Big(u \prod_{k=1}^n (1+I_k) + \sum_{k=1}^n \Big((X_k(1+I_k) - Y_k) \prod_{t=k+1}^n (1+I_t) \Big) < 0 \Big) \big| U_0 = u, I_1 = j \Big) \\ &= \mathsf{P}\Big(\bigcup_{k=2}^{n+1} \Big((\tau_j - y) \prod_{t=2}^k (1+I_t) + \sum_{j=2}^n \Big((X_j(1+I_j) - Y_j) \prod_{t=j+1}^k (1+I_t) \Big) < 0 \Big) \big| U_0 = u, I_1 = j \Big) \\ &= \psi_n(\tau_j - y, j). \end{split}$$

Let the event $A = \{Y_1 = y, X_1 = \omega, I_1 = j, U_0 = u, I_0 = i\}$, then

$$\begin{split} \psi_{n+1}(u,i) &= \mathsf{P}\Big(\bigcup_{k=1}^{n+1} (U_k < 0) \big| U_0 = u, I_0 = i\Big) \\ &= \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \int_0^\infty \mathsf{P}\Big(\bigcup_{k=1}^{n+1} (U_k < 0) \big| A\Big) \mathrm{d}F(y) \mathrm{d}G(\omega) \\ &= \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \Big(\int_0^{\tau_j} \mathsf{P}\Big(\bigcup_{k=1}^{n+1} (U_k < 0) \big| A\Big) \mathrm{d}F(y) + \int_{\tau_j}^\infty \mathrm{d}F(y)\Big) \mathrm{d}G(\omega). \end{split}$$

This gives (2.1). In particular,

$$\psi_1(u,i) = \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \overline{F}(\tau_j) \mathrm{d}G(\omega)$$

Finally, by letting $n \to \infty$ and using dominated convergence theorem, we obtain (2.3).

§3. Bounds for the Ruin Probabilities

Theorem 3.1 Suppose that there is a constant R > 0 satisfying

$$\mathsf{E}(\exp(R(Y_1 - X_1))) = 1, \tag{3.1}$$

then for any $u \ge 0$

$$\psi(u,i) \leq \beta \mathsf{E}(\exp(RY_1))\mathsf{E}(\exp(-R(u+X_1)(1+I_1))|I_0=i) \leq \beta e^{-Ru},$$
 (3.2)

where

$$\beta^{-1} = \inf_{t \ge 0} \frac{\int_t^\infty \exp(Ry) \mathrm{d}F(y)}{\exp(Rt)\overline{F}(t)}$$

Proof It is sufficient to prove that rightmost term in (3.2) is an upper bound for $\psi_n(u,i)$ for all $n \ge 1$. We will show that by induction. First, for any $u \ge 0$, we have

$$\overline{F}(x) = \left(\frac{\int_{x}^{\infty} \exp(Ry) dF(y)}{\exp(Rx)\overline{F}(x)}\right)^{-1} \exp(-Rx) \int_{x}^{\infty} \exp(Ry) dF(y)$$

$$\leqslant \quad \beta \exp(-Rx) \int_{x}^{\infty} \exp(Ry) dF(y)$$

$$\leqslant \quad \beta \exp(-Rx) \mathsf{E}(\exp(RY_{1})). \tag{3.3}$$

Thus, by (2.1),

$$\begin{split} \psi_1(u,i) &= \sum_{j \in \mathbb{R}} p_{ij} \int_0^\infty \overline{F}(\tau_j) \mathrm{d}G(\omega) \\ &\leqslant \quad \beta \sum_{j \in \mathbb{R}} p_{ij} \int_0^\infty \exp(-R\tau_j) \mathsf{E}(\exp(RY_1)) \mathrm{d}G(\omega) \\ &= \quad \beta \mathsf{E}(\exp(RY_1)) \mathsf{E}(\exp(-R(u+X_1)(1+I_1)) | I_0 = i) \end{split}$$

This shows that the results holds for n = 1. To prove the result for general $n \ge 1$, the induction hypothesis is that for some $n \ge 1$ and every $x \ge 0$ and $i \in \mathbb{E}$,

$$\psi_n(u,i) \leq \beta \mathsf{E}(\exp(RY_1))\mathsf{E}(\exp(-R(u+X_1)(1+I_1))|I_0=i).$$
 (3.4)

By (2.2),

$$\begin{split} \psi_{n+1}(u,i) &= \beta \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \int_0^{\tau_j} \mathsf{E}(\exp(RY_1)) \exp(-R(u+\omega)(1+j)) \mathrm{d}F(y) \mathrm{d}G(\omega) \\ &+ \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \overline{F}(\tau_j) \mathrm{d}G(\omega) \\ &= \beta \sum_{j \in \mathbb{E}} p_{ij} \int_0^\infty \sum_{j \in \mathbb{E}} p_{ij} \mathsf{E}(\exp(RY_1)) \exp(-R(u+\omega)(1+j)) \mathrm{d}G(\omega) \\ &= \beta \mathsf{E}(\exp(RY_1)) \mathsf{E}(\exp(-R(u+X_1)(1+I_1)) | I_0 = i) \\ &\leqslant \ \mathsf{E}(\exp(RY_1)) \mathsf{E}(\exp(-R(u+X_1))) \\ &= \beta \exp(-Ru). \end{split}$$

Hence (3.4) holds for all n = 1, 2, ..., then by letting $n \to \infty$, we obtain the result in the theorem. \Box

We can also present the upper bounds by a martingale approach.

Proposition 3.1 Suppose that $\mathsf{E}(X_1 - Y_1) > 0$, and for each $i \in \mathbb{E}$, there exists $\rho_i > 0$ satisfying that

$$\mathsf{E}[\exp(-\rho_i(X_1 - Y_1)(1 + I_1)^{-1})|I_0 = i] = 1.$$

Then

$$R_1 := \min \rho_i \geqslant R,\tag{3.5}$$

and for all $i \in \mathbb{E}$,

$$\mathsf{E}[\exp(-R_1(X_1 - Y_1)(1 + I_1)^{-1})|I_0 = i] \leq 1.$$
(3.6)

Proof For each $i \in \mathbb{E}$ and r > 0, let

$$l_i(r) := \mathsf{E}[\exp(-r(X_1 - Y_1)(1 + I_1)^{-1})|I_0 = i] - 1$$

It is easy to check that $l_i(r)$ is a convex function, and by the assumption of the theorem, we have

$$l_i(0) = \mathsf{E}[-(X_1 - Y_1)]\mathsf{E}[(1 + I_1)^{-1}|I_0 = i] < 0.$$

Let ρ_i be the unique positive root of the equation $l_i(r) = 0$ on $(0, \infty)$, then for $0 < \rho \leq \rho_i$, $l_i(\rho) < 0$. On the other hand, by Jensen's inequality,

$$\mathsf{E}[\exp(-R(X_1 - Y_1)(1 + I_1)^{-1})|I_0 = i] = \sum_{j \in \mathbb{E}} p_{ij} \mathsf{E}[\exp(-R(X_1 - Y_1)(1 + j)^{-1})]$$

$$\leqslant \sum_{j \in \mathbb{E}} p_{ij} \mathsf{E}[\exp(-R(X_1 - Y_1))]^{(1+j)^{-1}}.$$

By (3.1) and $\sum_{j} p_{ij} = 1$,

$$\mathsf{E}[\exp(-R_1(X_1 - Y_1)(1 + I_1)^{-1})|I_0 = i] \le 1$$

This implies that $l_i(R) \leq 0$. Moreover, for all $i, R \leq \rho_i$, so $R_1 := \min_{i \in \mathbb{E}} \rho_i \geq R$, thus (3.5) holds. In addition, $R_1 \leq \rho_i$ for all $i \in \mathbb{E}$, which implies (3.6).

Theorem 3.2 Under the assumption of Proposition 3.1, for all $i \in \mathbb{E}$, and u > 0,

$$\psi(u,i) \leqslant \exp(-R_1 u). \tag{3.7}$$

Proof Let $V_k := U_k \prod_{t=1}^k (1+I_t)^{-1}$, then

$$V_k = u + \sum_{l=1}^k \left((X_1 - Y_1) \prod_{t=1}^l (1 + I_t)^{-1} \right).$$

Let $S_n = \exp(-R_1 V_n)$, then

$$S_{n+1} = S_n \exp\left(-R_1(X_1 - Y_{n+1})\prod_{t=1}^{n+1}(1+I_t)^{-1}\right).$$

Thus for any $n \ge 1$,

$$\mathsf{E}[S_{n+1}|Y_1, \dots, Y_n, I_1, \dots, I_n]$$

$$= S_n \mathsf{E}\Big[\exp\Big(-R_1(X_1 - Y_{n+1})\prod_{t=1}^{n+1}(1 + I_t)^{-1}\Big)|Y_1, \dots, Y_n, I_1, \dots, I_n\Big]$$

$$= S_n \mathsf{E}\Big[\exp\Big(-R_1(X_1 - Y_{n+1})(1 + I_{n+1})^{-1}\prod_{t=1}^{n}(1 + I_t)^{-1}\Big)|I_1, \dots, I_n\Big]$$

$$\leq S_n \mathsf{E}[\exp(-R_1(X_1 - Y_{n+1})(1 + I_{n+1})^{-1})|I_1, \dots, I_n]^{\prod_{t=1}^{n}(1 + I_t)^{-1}}$$

$$\leq S_n.$$

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Let $T_i = \min\{n : V_n < 0 | I_0 = i\}$, then T_i is a stopping time and $n \wedge T_i := \min\{n, T_i\}$ is a finite stopping time. Thus by the optional stopping time theorem,

$$\mathsf{E}S_{n\wedge T_i} \leqslant \mathsf{E}[S_0] = \exp(-R_1 x)$$

Hence we have $\psi(u,i) \leq \exp(-R_1 u)$ by the corresponding lines in Diasparra and Romera (2009).

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利率为马氏链的离散时间风险模型的破产概率

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本文考虑了带随机利率的离散时间风险模型. 在假设利率为马氏链条件下, 得到了有限时间和最终破产 概率所满足的递推积分方程,以及最终破产概率的Lundberg不等式.

关键词: 离散时间风险模型, 破产概率, 递推方程, Lundberg不等式. 学科分类号: O211.67.