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Convergence for Weighted Sums of Dependent Random Variables under Residual h-Integrability Assumption *

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Abstract

Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants. A new concept of integrability (call residually h-integrability) for an array of random variables $\{X_{ni}\}$ with respect to an array of constants $\{a_{ni}\}$ is introduced, which is weaker than other related notions of integrability, such as h-integrability, Cesàro α-integrability. Under this assumption of integrability and appropriate conditions on the array of weights, we investigate strong convergence and mean convergence for weighted sums of dependent random variables. Some related results in literature are extended and improved.

Keywords: φ-mixing sequence, pairwise LCND random sequence, r-mean convergence, uniform integrability, weighted sums.

AMS Subject Classification: 60F05, 60F15.

§1. Introduction

Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants, $\{u_n, n \geqslant 1\}$ and $\{v_n, n \geqslant 1\}$ be two sequences of integers (neither necessarily positive nor finite) such that $v_n > u_n$ for all $n \geqslant 1$ and $v_n - u_n \to \infty$ as $n \to \infty$.

Laws of large numbers for sequences of random variables play a central role in the area of limit theorems in Probability Theory, however a uniform integrability condition of some kind has played an increasingly important role as a key condition in proving laws of large numbers for a sequence of random variables.

Cabrea (1994), in order to investigate the weak convergence for weighted sums of random variables, introduces the condition of uniform integrability concerning the weights, which is weaker than uniform integrability, and leads to Cesàro uniform integrability in

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[2] as a special case. Under this condition, a weak law of large numbers for weighted sums of pairwise independent random variables is obtained.

The notion of h-integrability for an array of random variables with respect to an array of constant weights is introduced in [3] and Cabrea and Volodin proved that this concept was weaker than other previous related notions of integrability, such as Cesàro uniform integrability in [2], $\{a_{ni}\}$-uniform integrability in [1] and Cesàro α-integrability in [4]. Under appropriate conditions on the weights, they prove that h-integrability concerning the weights is sufficient for a mean convergence theorem and a weak law of large numbers to hold for weighted sums of an array of random variables with respect to some special kind of rowwise dependent array.

Definition 1.1 Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ an array of constants with $\sum\limits_{i=u_n}^{v_n} |a_{ni}| \leqslant C$ for all $n \in N$ and some constant $C > 0$. Let $\{h(n), n \geqslant 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. The array $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is said to be h-integrable concerning the array of constant $\{a_{ni}\}$ (h-integrability, in short) if

$$\sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}| \mathsf{E}|X_{ni}| < \infty \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}| \mathsf{E}|X_{ni}| I_{[|X_{ni}|>h(n)]} = 0.$$

Chandra and Goswami (2003, 2006) introduce the condition of Cesàro α-integrability ($\alpha > 0$), and also prove that Cesàro α-integrability for appropriate α is sufficient for the weak law of large numbers to hold for certain special dependent sequences of random variables.

In this paper, we introduce a new concept of integrability which deals with weighted sums of random variables, and obtain strong laws of large numbers and mean convergence theorems for the weighted sums of non-positively correlated random sequences, pairwise LCND random sequences and φ-mixing sequences.

Definition 1.2 Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ an array of constants with $\sum\limits_{i=u_n}^{v_n} |a_{ni}| \leqslant C$ for all $n \in N$ and some constant $C > 0$. Let $\{h(n), n \geqslant 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. A array $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is said to be residually h-integrable with respect to the array of constants $\{a_{ni}\}$ (Rh-integrable in short) if the following two conditions hold:

$$\text{(i)}\ \sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}| \mathsf{E}|X_{ni}| < \infty \qquad \text{and} \qquad \text{(ii)}\ \lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}| \mathsf{E}(|X_{ni}| - h(n)) I_{[|X_{ni}|>h(n)]} = 0. \tag{1.1}$$

A random array $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is said to be strong residually h-integrable with respect to the array of constants $\{a_{ni}\}$ if

$$\text{(i)}\ \sup_{n\geqslant 1}\sum_{i=u_n}^{v_n}|a_{ni}|\mathsf{E}|X_{ni}|<\infty \qquad \text{and} \qquad \text{(ii)}\ \sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n}|a_{ni}|\mathsf{E}(|X_{ni}|-h(n))I_{[|X_{ni}|>h(n)]}<\infty. \tag{1.2}$$

Remark 1 It is apparent that (1.2)(ii) is indeed stronger than (1.1)(ii), it also is trivially clear that h-integrable implies Rh-integrable. Let $\{h_1(n)\}$ and $\{h_2(n)\}$ be two positive sequences monotonically increasing to infinity such that $h_2(n) \geqslant h_1(n)$ for all sufficiently n, it is also clear from the definition that Rh_1-integrable implies Rh_2-integrable.

§2. Preliminary Lemmas

In this section we give some related concepts and lemmas about the following rowwise dependence structures: low case negative dependence, rowwise pairwise non-positive correction and φ-mixing.

Definition 2.1 Random variables X and Y are lower case negatively dependent (LCND, in short) if

$$\mathsf{P}(X\leqslant x, Y\leqslant y)\leqslant \mathsf{P}(X\leqslant x)\mathsf{P}(Y\leqslant y) \qquad \text{for all}\ \ x,y\in R.$$

A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be pairwise lower case negatively dependent if every pair of random variable in the sequence are LCND.

The following lemmas in [6] are well known and give the important property of pairwise LCND random variables. Lemma 2.1 states that pairwise LCND random variables are non-positive correlated.

Lemma 2.1 If $\{X_n, n \geqslant 1\}$ is a sequence of pairwise LCND random variables, then

$$\mathsf{E}(X_iX_j)\leqslant \mathsf{E}X_i\mathsf{E}X_j, \qquad i\neq j.$$

Lemma 2.2 Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise LCND random variables. If $\{f_n, n \geqslant 1\}$ is a sequence of increasing functions, then $\{f_n(X_n), n \geqslant 1\}$ is a sequence of pairwise LCND random variables.

In order to investigate the related questions about Markov process, Dobrushin introduced the notion of φ-mixing sequence in [7].

Definition 2.2 Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables in probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $\mathcal{F}^k$ be the σ-algebra generated by $\{X_n, 1 \leqslant n \leqslant k\}$, and $\mathcal{F}_k$ the

σ-algebra generated by $\{X_n, n \geqslant k\}$. $\{X_n, n \geqslant 1\}$ is said to be φ-mixing if there exists a non-negative sequence $\{\varphi(m), m \geqslant 1\}$, with $\lim\limits_{m\to\infty} \varphi(m) = 0$, such that, for each $k \geqslant 1$ and for each $m \geqslant 1$,

$$|\mathsf{P}(B|A) - \mathsf{P}(B)| \leqslant \varphi(m), \qquad \text{for } A \in \mathcal{F}^k,\ B \in \mathcal{F}_{(k+m)},\ \mathsf{P}(A) > 0.$$

The following lemma is in Billingsley (1968):

Lemma 2.3 Let X be a $\mathcal{F}^k$-measurable random variables, and Y be a $\mathcal{F}_{(k+m)}$-measurable random variable, with $|X| \leqslant C_1$ and $|Y| \leqslant C_2$. Then $|\mathsf{Cov}\,(X,Y)| \leqslant 2C_1C_2\varphi(m)$.

§3. Strong Laws of Large Numbers and Mean Convergence for Weighted Sums of Random Variables with Some Conditions of Dependence

One of the most interesting application of all these notions of integrability is connected with the strong laws of large numbers and mean convergence with weighted sums of random variables. In the following results, we suppose that all the random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Now we state and prove our main results.

Theorem 3.1 Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of non-negative random variables and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants. Let $S_n = \sum\limits_{i=u_n}^{v_n} a_{ni}X_{ni}$. If the following conditions hold:

(i) $\mathsf{E}(X_{ni}X_{nj}) \leqslant \mathsf{E}X_{ni}\mathsf{E}X_{nj}$, for each $n \geqslant 1$, $u_n \leqslant i < j \leqslant v_n$,

(ii) $\{X_{ni}\}$ is strong residually h-integrable with respect to the array of constants $\{a_{ni}\}$,

(iii) $\sum\limits_{n=1}^{\infty} h(n) \sup\limits_{u_n \leqslant i \leqslant v_n} |a_{ni}| < \infty$,

then

$$S_n - \mathsf{E}S_n \to 0 \qquad \text{as } n \to \infty \text{ almost surely.}$$

Proof For each $n \geqslant 1$, $u_n \leqslant i \leqslant v_n$, let $Y_{ni} = X_{ni}I_{[X_{ni}\leqslant h(n)]} + h(n)I_{[X_{ni}>h(n)]}$, $T_n = \sum\limits_{i=u_n}^{v_n} a_{ni}Y_{ni}$. We prove

$$\mathsf{E}S_n - \mathsf{E}T_n \to 0, \tag{3.1}$$

$$S_n - T_n \to 0, \qquad \text{a.s.}, \tag{3.2}$$

$$T_n - \mathsf{E}T_n \to 0, \qquad \text{a.s.}. \tag{3.3}$$

These will imply that $S_n - \mathsf{E}S_n \to 0$, as $n \to \infty$ almost surely.

We will estimate each of these terms separately. For (3.1), since strong residually h-integrable implies residually h-integrable, and according to Lemma 2.1 Y_{ni} preserves the negative dependence property. it holds

$$\mathsf{E}S_n - \mathsf{E}T_n \leqslant \sum_{i=u_n}^{v_n} |a_{ni}|\mathsf{E}(X_{ni} - h(n))I_{[X_{ni}>h(n)]} \to 0, \qquad \text{as } n \to \infty.$$

To verity (3.2), $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathsf{P}(|S_n - T_n| > \varepsilon) \leqslant \sum_{n=1}^{\infty} \frac{1}{\varepsilon}\mathsf{E}\Big|\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - Y_{ni})\Big| \leqslant \sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n} \frac{1}{\varepsilon}|a_{ni}|\mathsf{E}(X_{ni} - h(n))I_{[X_{ni}>h(n)]},$$

the last sum convergence since the condition (ii), by Borel-Cantelli Lemma we get $S_n - T_n \to 0$, a.s..

Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathsf{P}\Big(\Big|\sum_{i=u_n}^{v_n} a_{ni}(Y_{ni} - \mathsf{E}Y_{ni})\Big| > \varepsilon\Big) \leqslant \sum_{n=1}^{\infty} \frac{1}{\varepsilon^2}\mathsf{E}\Big|\sum_{i=u_n}^{v_n} a_{ni}(Y_{ni} - Y_{ni})\Big|^2 \\
\leqslant\ & \sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n} a_{ni}^2 \mathsf{E}Y_{ni}^2 + \sum_{n=1}^{\infty}\sum_{u_n \leqslant i < j \leqslant v_n} a_{ni}a_{nj}(\mathsf{E}Y_{ni}Y_{nj} - \mathsf{E}Y_{ni}\mathsf{E}Y_{nj}) \\
=\ & I_1 + I_2.
\end{aligned}$$

To prove (3.3) therefore, it suffices to prove that $I_i < \infty$, $i = 1, 2$ by Borel-Cantelli Lemma.

The convergence of I_1 follows from the assumption (ii) and (iii), note that $Y_{ni} \leqslant \min\{h(n), X_{ni}\}$,

$$\begin{aligned}
I_1 &\leqslant \sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n} a_{ni}^2 h(n)\mathsf{E}X_{ni} \leqslant \sum_{n=1}^{\infty} h(n)\Big(\sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|\Big)\sum_{i=u_n}^{v_n} |a_{ni}|\mathsf{E}X_{ni} \\
&\leqslant \Big(\sup_{n \geqslant 1}\sum_{i=u_n}^{v_n} |a_{ni}|\mathsf{E}X_{ni}\Big)\sum_{n=1}^{\infty} h(n) \sup_{u_n \leqslant i \leqslant v_n} |a_{ni}| < \infty.
\end{aligned}$$

While the convergence of I_2, by the assumption (i) and (ii) we note that

$$\begin{aligned}
I_2 &\leqslant \sum_{n=1}^{\infty}\sum_{u_n \leqslant i < j \leqslant v_n} |a_{ni}a_{nj}|(\mathsf{E}X_{ni}X_{nj} - \mathsf{E}Y_{ni}\mathsf{E}Y_{nj}) \\
&\leqslant \sum_{n=1}^{\infty}\sum_{u_n \leqslant i < j \leqslant v_n} |a_{ni}a_{nj}|(\mathsf{E}X_{ni}\mathsf{E}X_{nj} - \mathsf{E}Y_{ni}\mathsf{E}Y_{nj}) \\
&= \sum_{n=1}^{\infty}\sum_{u_n \leqslant i < j \leqslant v_n} |a_{ni}a_{nj}|[(\mathsf{E}X_{ni} - \mathsf{E}Y_{ni})\mathsf{E}X_{nj} + \mathsf{E}Y_{ni}(\mathsf{E}X_{nj} - \mathsf{E}Y_{nj})] \\
&\leqslant \sum_{n=1}^{\infty}\sum_{i,j} |a_{ni}a_{nj}|\mathsf{E}X_{nj}\mathsf{E}[X_{ni} - h(n)]I_{[X_{ni}>h(n)]} \\
&\quad + \sum_{n=1}^{\infty}\sum_{i,j} |a_{ni}a_{nj}|\mathsf{E}X_{ni}\mathsf{E}[X_{nj} - h(n)]I_{[X_{nj}>h(n)]} \\
&\leqslant 2\Big(\sup_{n \geqslant 1}\sum_{i=u_n}^{v_n} |a_{ni}|\mathsf{E}X_{ni}\Big)\sum_{n=1}^{\infty}\sum_{j=u_n}^{v_n} |a_{nj}|\mathsf{E}[X_{nj} - h(n)]I_{[X_{nj}>h(n)]} < \infty,
\end{aligned}$$

Because $\{X_{ni}\}$ is strong residually h-integrable with respect to the array of constants $\{a_{ni}\}$, this completes the proof of the theorem. $\square$

Lemma 2.2 states that pairwise LCND random variables are non-positive correlated. For a general sequence $\{X_n, n \geqslant 1\}$ of random variables, noting that if $\{X_n, n \geqslant 1\}$ is residually h-integrable, respectively strong residually h-integrable, then $\{X_n^+, n \geqslant 1\}$ and $\{X_n^-, n \geqslant 1\}$ are residually h-integrable, respectively strong residually h-integrable. By Lemma 2.3 $\{X_{ni}^+, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ and $\{X_{ni}^-, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ are arrays of rowwise pairwise LCND where $a^+ = a \vee 0$, $a^- = -a \vee 0$, and applying Theorem 3.1 to the arrays $\{X_{ni}^+\}$ and $\{X_{ni}^-\}$ separately, we get the following result.

Corollary 3.1 Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of rowwise pairwise LCND and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants. Suppose that

(i) $\{X_{ni}\}$ is strong residually h-integrable concerning the array of constants $\{a_{ni}\}$,

(ii) $\sum\limits_{n=1}^{\infty} h(n) \sup\limits_{u_n \leqslant i \leqslant v_n} |a_{ni}| < \infty$,

then $S_n - \mathsf{E}S_n \to 0$ as $n \to \infty$ almost surely.

A special case of pairwise LCND random variables is the case of pairwise independent random variables. We also have the following corollary.

Corollary 3.2 Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of rowwise pairwise independent random variables and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants. Suppose that

(i) $\{X_{ni}\}$ is strong residually h-integrable with respect to the array of constants $\{a_{ni}\}$,

(ii) $\sum\limits_{n=1}^{\infty} h(n) \sup\limits_{u_n \leqslant i \leqslant v_n} |a_{ni}| < \infty$,

then $S_n - \mathsf{E}S_n \to 0$ as $n \to \infty$ almost surely.

Theorem 3.2 Let$\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of mixing random variables such that for each $n \geq 1$ the row $\{X_{ni}, u_n \leqslant i \leqslant v_n\}$ is a φ_n-mixing sequences of random variables. Let $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants with $a_{ni} \leqslant a_{nj}$ for all $n \geqslant 1$ and $u_n \leqslant i < j \leqslant v_n$. Suppose that

(i) $\limsup\limits_{n\to\infty} \sum\limits_{i=1}^{v_n-u_n} \varphi_n(i) < \infty$ for each $n \geqslant 1$,

(ii) $\{X_{ni}\}$ is strong residually h-integrable with respect to $\{a_{ni}\}$,

(iii) $\sum\limits_{n=1}^{\infty} h^2(n) \sum\limits_{i=u_n}^{v_n} a_{ni}^2 < \infty$,

then $S_n - \mathsf{E}S_n \to 0$ as $n \to \infty$ almost surely.

Proof Let Y_{ni}, S_n, T_n be the same as in the proof of Theorem 3.1, we only need to prove that

$$\sum_{n=1}^{\infty}\sum_{u_n\leqslant k<j\leqslant v_n} a_{nk}a_{nj}\mathsf{Cov}\,(Y_{nk}Y_{nj})<\infty.$$

By applying Lemma 2.3 and note that $Y_{ni}\leqslant h(n)$, $u_n\leqslant i\leqslant v_n$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{u_n\leqslant k<j\leqslant v_n} a_{nk}a_{nj}\mathsf{Cov}\,(Y_{nk}Y_{nj})\\
=\;&\sum_{n=1}^{\infty}\sum_{i=1}^{v_n-u_n}\sum_{k=u_n}^{v_n-i} a_{nk}a_{n(k+i)}\mathsf{Cov}\,(Y_{nk}Y_{n(k+i)})\\
\leqslant\;&2\sum_{n=1}^{\infty}h^2(n)\sum_{i=1}^{v_n-u_n}\sum_{k=u_n}^{v_n-i}a_{nk}^2\varphi_n(i)\leqslant 2\sum_{n=1}^{\infty}h^2(n)\sum_{k=u_n}^{v_n}a_{nk}^2\sum_{i=1}^{v_n-u_n}\varphi_n(i)\\
\leqslant\;&\Big(\limsup_{n\to\infty}\sum_{i=1}^{v_n-u_n}\varphi_n(i)\Big)\sum_{n=1}^{\infty}h^2(n)\sum_{k=u_n}^{v_n}a_{nk}^2<\infty.
\end{aligned}$$

The last inequality is according to the assumption (i) and (iii). The rest of the proof is similar to the proof of Theorem 3.1. □

Now we discuss r-mean convergence for the randomly weighted sums $\sum\limits_{i=u_n}^{v_n}(a_{ni}X_{ni}-\mathsf{E}a_{ni}X_{ni})$ under the condition of residually h-integrable.

Theorem 3.3 Let $\{X_{ni}, u_n\leqslant i\leqslant v_n, n\geqslant 1\}$ be an array of non-negative random variables and $\{a_{ni}, u_n\leqslant i\leqslant v_n, n\geqslant 1\}$ be an array of constants. Suppose that

(i) $\mathsf{E}(X_{ni}X_{nj})\leqslant \mathsf{E}X_{ni}\mathsf{E}X_{nj}$, for each $n\geqslant 1$, $u_n\leqslant i<j\leqslant v_n$,

(ii) $\{X_{ni}\}$ is residually h-integrable with respect to the array of constants $\{a_{ni}\}$,

(iii) $\lim\limits_{n\to\infty}h(n)\sup\limits_{u_n\leqslant i\leqslant v_n}|a_{ni}|=0$,

then $S_n-\mathsf{E}S_n\to 0$ in L^1 and, hence, in probability as $n\to\infty$.

Proof Let Y_{ni}, S_n, T_n be the same as in the proof of Theorem 3.1. It suffices to prove that

$$\mathsf{E}S_n-\mathsf{E}T_n\to 0,\qquad S_n-T_n\to 0,\ \text{ in }\ L^1,\ \text{ as }\ n\to\infty. \tag{3.4}$$

$$T_n-\mathsf{E}T_n\to 0,\qquad \text{in }\ L^1,\ \text{ as }\ n\to\infty. \tag{3.5}$$

We first prove (3.4), note that the $\{X_{ni}\}$ is an array of non-negative random variables, it holds

$$|\mathsf{E}S_n-\mathsf{E}T_n|\leqslant \mathsf{E}|S_n-T_n|\leqslant\sum_{i=u_n}^{v_n}|a_{ni}|\mathsf{E}[X_{ni}-h(n)]I_{[X_{ni}>h(n)]},$$

and the last expression above tends 0 as $n\to\infty$ by the assumption (ii).

Turning now to (3.5), we actually show that $T_n - \mathsf{E}T_n \to 0$ in L^2 and hence in L^1.

$$\begin{aligned}\mathsf{E}|T_n - \mathsf{E}T_n|^2 &\leqslant \sum_{i=u_n}^{v_n} a_{ni}^2 \mathsf{E}Y_{ni}^2 + \sum_{n=1}^{\infty} \sum_{u_n\leqslant i<j\leqslant v_n} |a_{ni}a_{nj}|(\mathsf{E}Y_{ni}Y_{nj} - \mathsf{E}Y_{ni}\mathsf{E}Y_{nj})\\ &= A_{n1} + A_{n2}.\end{aligned}$$

We only need to prove that $A_{ni} \to 0$, $i = 1, 2$ when $n \to \infty$. For A_{n1}, we have

$$A_{n1} \leqslant \sum_{i=u_n}^{v_n} a_{ni}^2 h(n)\mathsf{E}X_{ni} \leqslant \Big(\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n} |a_{ni}|\mathsf{E}X_{ni}\Big)\Big[h(n) \sup_{u_n\leqslant i\leqslant v_n} |a_{ni}|\Big] \to 0.$$

Next we estimate A_{n2}, since every random variable X_{ni} is non-negative:

$$\begin{aligned}A_{n2} &\leqslant \sum_{u_n\leqslant i<j\leqslant v_n} |a_{ni}a_{nj}|(\mathsf{E}X_{ni}X_{nj} - \mathsf{E}Y_{ni}\mathsf{E}Y_{nj})\\ &\leqslant \sum_{u_n\leqslant i<j\leqslant v_n} |a_{ni}a_{nj}|(\mathsf{E}X_{ni}\mathsf{E}X_{nj} - \mathsf{E}Y_{ni}\mathsf{E}Y_{nj})\\ &= \sum_{u_n\leqslant i<j\leqslant v_n} |a_{ni}a_{nj}|[(\mathsf{E}X_{ni} - \mathsf{E}Y_{ni})\mathsf{E}X_{nj} + \mathsf{E}Y_{ni}(\mathsf{E}X_{nj} - \mathsf{E}Y_{nj})]\\ &\leqslant \sum_{i,j} |a_{ni}a_{nj}|\mathsf{E}X_{nj}\mathsf{E}[X_{ni} - h(n)]I_{[X_{ni}>h(n)]}\\ &\quad + \sum_{n=1}^{\infty}\sum_{i,j} |a_{ni}a_{nj}|\mathsf{E}X_{ni}\mathsf{E}[X_{nj} - h(n)]I_{[X_{nj}>h(n)]}\\ &\leqslant 2\Big(\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n} |a_{ni}|\mathsf{E}X_{ni}\Big)\sum_{j=u_n}^{v_n} |a_{nj}|\mathsf{E}[X_{nj} - h(n)]I_{[X_{nj}>h(n)]} \to 0,\end{aligned}$$

the second inequality follows from the assumption (i), the last series go to 0 because $\{X_{ni}\}$ is residually h-integrable with respect to the array of constants $\{a_{ni}\}$. □

Remark 2 Cabrera and Volodin (2005) prove Theorem 3.3 when $\{X_{ni}\}$ is h-integrable with respect to the array of constants $\{a_{ni}\}$ and $h^2(n)\sum_{i=u_n}^{v_n} a_{ni}^2 \to 0$ are satisfied as $n \to \infty$. The condition $h^2(n)\sum_{i=u_n}^{v_n} a_{ni}^2 \to 0$ is stronger than $h(n) \sup_{u_n\leqslant i\leqslant v_n} a_{ni} \to 0$ as $n \to \infty$, however the condition residually h-integrable concerning the array of constants $\{a_{ni}\}$ is weaker than the condition h-integrable concerning the array of constants $\{a_{ni}\}$ in Remark 1. Hence Theorem 3.3 contains as a particular case [3, Theorem 2]. According to the Remark 4 in [3], we also show that Theorem 3.3 contains as a particular case [4, Theorem 2.1(a)].

Corollary 3.3 Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of rowwise pairwise LCND and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants. Suppose that

(i) $\{X_{ni}\}$ is h-integrable with respect to the array of constants $\{a_{ni}\}$,

(ii) $\lim_{n\to\infty} h(n) \sup_{u_n\leqslant i\leqslant v_n} |a_{ni}| = 0$,

then $S_n - \mathsf{E}S_n \to 0$ in L^1 and, hence, in probability as $n \to \infty$.

Remark 3 According to Remark 1 and Remark 2, Corollary 3.3 extends Theorem 1 in [3]. The proof is similar to the proof of Theorem 3.3.

A special case of pairwise LCND random variables is the case of pairwise independent random variables. We also have the following corollary.

Corollary 3.4 Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of rowwise pairwise independent random variables and $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants. If

(i) $\{X_{ni}\}$ is h-integrable with respect to the array of constants $\{a_{ni}\}$,

(ii) $\lim\limits_{n\to\infty} h(n) \sup\limits_{u_n\leqslant i\leqslant v_n} |a_{ni}| = 0$,

then $S_n - \mathsf{E}S_n \to 0$ in L^1 and, hence, in probability as $n \to \infty$.

Theorem 3.4 Let$\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of mixing random variables such that for each $n \geq 1$ the row $\{X_{ni}, u_n \leqslant i \leqslant v_n\}$ is a φ_n-mixing sequences of random variables. Let $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of constants with $a_{ni} \leqslant a_{nj}$ for all $n \geqslant 1$ and $u_n \leqslant i < j \leqslant v_n$. Suppose that

(i) $\limsup\limits_{n\to\infty} \sum\limits_{i=1}^{v_n-u_n} \varphi_n(i) < \infty$ for each $n \geqslant 1$,

(ii) $\{X_{ni}\}$ is residually h-integrable with respect to $\{a_{ni}\}$,

(iii) $\lim\limits_{n\to\infty} h^2(n) \sum\limits_{i=u_n}^{v_n} a_{ni}^2 = 0$,

then $S_n - \mathsf{E}S_n \to 0$ in L^1 and, hence, in probability as $n \to \infty$.

Proof Let Y_{ni}, S_n, T_n be the same as in the proof of Theorem 3.1. We proceed as in the proof of Theorem 3.3 in order to prove that $S_n - T_n \to 0$, $T_n - \mathsf{E}T_n \to 0$ in L^1, $\mathsf{E}S_n - \mathsf{E}T_n \to 0$ as $n \to \infty$. It suffices to for us show that

$$\limsup_{n\to\infty} \sum_{u_n\leqslant k<j\leqslant v_n} a_{nk}a_{nj}\mathsf{Cov}\,(Y_{nk}Y_{nj}) \leqslant 0. \tag{3.6}$$

By applying Lemma 2.3 and note that $Y_{ni} \leqslant h(n)$, $u_n \leqslant i \leqslant v_n$, we have

$$\begin{aligned}\sum_{u_n\leqslant k<j\leqslant v_n} a_{nk}a_{nj}\mathsf{Cov}\,(Y_{nk}Y_{nj}) &= \sum_{i=1}^{v_n-u_n}\sum_{k=u_n}^{v_n-i} a_{nk}a_{n(k+i)}\mathsf{Cov}\,(Y_{nk}Y_{n(k+i)}) \\ &\leqslant 2h^2(n)\sum_{i=1}^{v_n-u_n}\sum_{k=u_n}^{v_n-i} a_{nk}^2\varphi_n(i) \\ &\leqslant 2h^2(n)\sum_{k=u_n}^{v_n} a_{nk}^2 \sum_{i=1}^{v_n-u_n}\varphi_n(i) \\ &\leqslant 2\Big(\sup_{n\geqslant 1}\sum_{i=1}^{v_n-u_n}\varphi_n(i)\Big)\Big[h^2(n)\sum_{k=u_n}^{v_n} a_{nk}^2\Big].\end{aligned}$$

By the condition (i) and (iii), we know (3.6) is satisfied. □

Corollary 3.5 Let$\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables such that for each $n \geq 1$ the row $\{X_{ni}, u_n \leqslant i \leqslant v_n\}$ is a $m(n)$-dependent sequences of random variables. Let $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of non-negative constants with $a_{ni} \leqslant a_{nj}$ for all $n \geqslant 1$ and $u_n \leqslant i < j \leqslant v_n$. Suppose that

(i) $\limsup\limits_{n\to\infty} m(n) < \infty$ for each $n \geqslant 1$,

(ii) $\{X_{ni}\}$ is residually h-integrable with respect to $\{a_{ni}\}$,

(iii) $\lim\limits_{n\to\infty} h^2(n) \sum\limits_{i=u_n}^{v_n} a_{ni}^2 = 0$,

then $S_n - \mathsf{E}S_n \to 0$ in L^1 and, hence, in probability as $n \to \infty$.

Proof We only need to note that we can consider $\varphi_n(i) = 0$ for $i > m(n)$ and $\varphi_n(i) = 1$ for $i \leqslant m(n)$, and so $\sum\limits_{i=1}^{v_n-u_n} \varphi_n(i) \leqslant m(n)$ for all $n \geqslant 1$. □

Remark 4 According to Remark 1, it is clear that Theorem 3.4 (respectively Corollary 3.5) extends Theorem 3 in [3] (respectively Corollary 2 in [3]).

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References

[1] Cabrera, M.O., Convergence of weighted sums of random variables and uniform integrability concerning the weights, *Collectanea Mathematica*, **45(2)**(1994), 121–132.

[2] Chandra, T.K., Uniform integrability in the Cesàro sence and the weak law of large numbers, *Sankhya: the Indian Journal of Statistics* (*Series A*), **51**(1989), 309–317.

[3] Cabrera, M.O. and Volodin, A., Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability, *Journal of Mathematical Analysis and Applications*, **305**(2005), 644–658.

[4] Chandra, T.K. and Goswami, A., Cesàro α-integrability and laws of large numbers I, *Journal of Theoretical Probability*, **16(3)**(2003), 655–669.

[5] Chandra, T.K. and Goswami, A., Cesàro α-integrability and laws of large numbers II, *Journal of Theoretical Probability*, **19(4)**(2006), 789–816.

[6] Lehmamn, E.L., Some concepts of dependence, *The Annals of Mathematical Statistics*, **37**(1966), 1137–1153.

[7] Dobrushin, R.L., Central limit theorem for nonstationary Markov chains (I), *Theory of Probability & Its Applications*, **1(1)**1956, 65–80.

[8] Billingsley, P., *Convergence of Probability Measures*, Wiley, New York, 1968.

[9] Matula, P., A note on the almost sure convergence of sums of negatively dependent random variables, *Statistics & Probability Letters*, **15(3)**(1992), 209–213.

[10] Yuan, D.M., An, J. and Tao, B., Laws of large numbers for residual Cesàro alpha-integrable sequences under dependence assumptions, *Statistics & Probability Letters*, **79(5)**(2009), 611–618.

剩余h-可积条件下相依随机变量加权和的收敛性质

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设$\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$与$\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$分别为一个随机阵列和一个常数阵列. 本文首先引入了随机阵列$\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$关于常数阵列$\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$剩余$h$-可积的概念, 它是弱于$h$-可积, Cesàro α-可积等其它相关可积的定义. 然后在这一可积的定义和适当的条件下, 我们研究了相依随机序列加权和的强收敛性和平均收敛性, 推广并改进了相关文献已有结果.

关键词: φ-混合序列, 两两LCND序列, r-阶平均收敛, 一致可积, 加权和.

学科分类号: O211.4.