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The MEMMs for Markov-Modulated GBMs *

WANG BO SONG RUILI

(*School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing, 210023*)

Abstract

In this paper, we consider the option pricing problem when the risky underlying assets are driven by Markov-modulated geometric Brownian motion (GBM). That is, the market parameters, for instance, the market interest rate, the appreciation rate and the volatility of the risky asset, depend on unobservable states of the economy which are modeled by a continuous-time hidden Markov chain. The market described by the Markov-modulated GBM model is incomplete in general, and, hence, the martingale measure is not unique. We adopt the minimal relative entropy martingale measure (MEMM) for the Markov-modulated GBM model as the suitable martingale measure and we obtain the MEMM for the market in general sense.

Keywords: GBM, hidden Markov chain model, MEMM.

AMS Subject Classification: 91B28, 60H10, 60J25.

§1. Introduction

In the field of option pricing theory, the martingale measures play very important roles. For example, the price of a contingent claim in the Black and Scholes model is given as the expectation of the return function with respect to the unique risk neutral martingale measure. If the market is complete, then the equivalent martingale measure is unique, and so the prices of options are uniquely determined by this martingale measure. However, if the market is incomplete, then there are (infinitely) many equivalent martingale measures. Therefore, the pricing models for the incomplete markets are consisting of the following two parts in general. The first part is defining the price process of underlying assets, and the second one is selecting a suitable martingale measure, which determines the option prices, among the set of all equivalent martingale measure. Many kinds of processes are proposed as the examples of price processes of underlying assets. For example, diffusion processes, jump type processes and general semi-martingale processes are proposed. For the problem to select a suitable martingale measure described in

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the second part of the construction of models, there are several candidates. For example, minimal martingale measure, variance optimal martingale measure and utility martingale measure are proposed and discussed (see [5], [11]). In this paper, we adopt the minimal entropy martingale measure (MEMM) as the suitable martingale measure. The MEMM has been discussed in [18]. Many researchers have studied the MEMMs in various models and given the explicit form of the MEMMs. For example, Miyahara discussed the MEMM of the log Lévy processes in [19]; Frittelli gave sufficient conditions for the existence of a unique MEMM and provided the characterization of the density of the MEMM in [12]; Mania, Santacroce and Tevzadze considered an incomplete financial market model where the dynamics of the assets price were described by an R^n-valued continuous semimartingale in [17]. And the authors expressed the density of the MEMM in the terms of the value process of the related optimization problem and showed that this value process was determined as the unique solution of a semimartingale backward equation; Benth and Meyer-Brandis derived the density of the MEMM in the stochastic volatility model proposed by Barndorff-Nielsen and Shephard in [2]; Benth and Karlsen proved that in a stochastic volatility market the Radon-Nikodym density of the MEMM could be expressed in terms of the solution of a semilinear PDE in [1]; Rheinlander and Steiger determined the MEMM for a general class of stochastic volatility models where both price process and volatility process contained jump terms which were correlated in [21]; Fujiwara determined the MEMMs for the exponential additive processes in [13].

In recent years, there is a considerable interest in the applications of regime switching models driven by a hidden Markov chain process to various financial problems. For an overview of hidden Markov Chain processes and their financial applications, we can refer to [6] and [9]. Some work on the using of hidden Markov Chain models in finance include [3], [7], [8], [15], [24] and so on. In [25], they obtained the MEMM for Markov switching Lévy processes. There are many authors have applied Markov-modulated geometric Brownian motion model in many financial and economical problems. For example, Siu considered the fair valuation of a participating life insurance policy with surrender options when the market values of the asset were modeled by Markov-modulated geometric Brownian motion in [23]; Elliott, Siu and Chan considered a PDE approach to evaluate coherent risk measures for derivative instruments when the dynamics of the risk underlying asset were governed by a Markov-modulated geometric Brownian motion in [10]; In [8], they considered the option pricing problem when the risky underlying assets were driven by Markov-modulated geometric Brownian motion (GBM). There they adopted a regime switching random Esscher transform to determine an equivalent martingale pricing measure. In our paper, we still adopt the model in [8]. Since the market described by the

Markov-modulated GBM model is incomplete in general, hence the equivalent martingale measure is not unique. We will adopt the relative entropy as a measure of the choice and seek a martingale measure that minimizes the relative entropy with respect to the canonical measure. The definition of relative entropy in [8] is the conditional expectation of the general relative entropy with respect to the natural filtration generated by hidden Markov chain which means that the information for the hidden Markov chain process is accessible to the market's agent in advance. We will adopt the general definition of relative entropy rather than the definition of relative entropy in [8] which means that the market's agent know nothing about the information for the hidden Markov chain process in advance, and we will obtain the MEMM for the Markov-modulated GBM model in general sense which is much more general than the MEMM in [8].

§2. The Model

Suppose $(\Omega, \mathscr{F}, \mathsf{P})$ is a complete probability space, where P is a real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model. Let $\{B_t\}_{t\in\mathcal{T}}$ denotes a standard Brownian motion on $(\Omega, \mathscr{F}, \mathsf{P})$. We assume that the states of the economy are modeled by a continuous-time hidden Markov chain process $\{U_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathscr{F}, \mathsf{P})$ with a finite state space $\chi := (x_1, x_2, \ldots, x_N)$ which is independent of the Brownian motion $\{B_t\}_{t\in\mathcal{T}}$. Without loss of generality, we can identify the state space of $\{U_t\}_{t\in\mathcal{T}}$ to be a finite set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1_i, \ldots, 0) \in \mathbb{R}^N$.

We consider a financial model consisting of two risky underlying assets, namely a bank account and a stock, that are tradable continuously. The instantaneous market interest rate $\{r_t\}_{t\in\mathcal{T}}$ of the bank account is given by

$$r_t := \langle r, U_t \rangle,$$

where $r := (r_1, r_2, \ldots, r_N)$ with $r_i > 0$ for each $i = 1, 2, \ldots, N$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^N$.

In this case, the dynamics of the price process $\{R_t\}_{t\in\mathcal{T}}$ for the bank account are described by

$$\begin{cases} \mathrm{d}R_t = r_t R_t \mathrm{d}t, & 0 < t \le T; \\ R_0 = 1. \end{cases} \tag{2.1}$$

Thus the SDE (2.1) has a unique solution

$$R_t = e^{\int_0^t r_s \mathrm{d}s}, \qquad 0 < t \le T.$$

We suppose that the stock appreciation rate $\{\mu_t\}_{t\in\mathcal{T}}$ and the volatility $\{\sigma_t\}_{t\in\mathcal{T}}$ of the stock price process S also depend on $\{U_t\}_{t\in\mathcal{T}}$ and are described by

$$\mu_t := \langle \mu, U_t\rangle, \qquad \sigma_t := \langle \sigma, U_t\rangle,$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)$ with $\sigma_i > 0$ for each $i = 1, 2, \ldots, N$.

The dynamics of the stock price process $\{S_t\}_{t\in\mathcal{T}}$ are then given by the following Markov-modulated geometric Brownian motion:

$$\begin{cases} \mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}B_t, & 0 < t \le T; \\ S_0 = s \end{cases}$$

with $s > 0$.

Thus, the stock price dynamics can be written as

$$S_t = s\exp\Big(\int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s\Big). \tag{2.2}$$

The discounted price process is defined by

$$\widetilde{S}_t := e^{-\int_0^t r_s \mathrm{d}s} S_t.$$

By (2.2), we have

$$\widetilde{S}_t = S_0 \exp\Big(\int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2 - r_s\Big)\mathrm{d}s\Big)$$

with $0 < t \le T$.

§3. The MEMM for the Markov-Modulated GBM

The minimal entropy martingale measure (MEMM) is one of the major tools for option valuation in an incomplete market. A martingale measure here is understood to be a probability measure $\mathsf{Q} \ll \mathsf{P}$ such that $\widetilde{S}$ is a Q-martingale. We denote with $\mathcal{M}^e$ the space of all equivalent martingale measure for $\widetilde{S}$. The relative entropy $H(\mathsf{Q}, \mathsf{P})$ of a probability measure Q with respect to a probability measure P is given as

$$H(\mathsf{Q}, \mathsf{P}) := \begin{cases} \mathsf{E}_{\mathsf{P}}\Big[\dfrac{\mathrm{d}\mathsf{Q}}{\mathrm{d}\mathsf{P}}\log\dfrac{\mathrm{d}\mathsf{Q}}{\mathrm{d}\mathsf{P}}\Big], & \text{if } \mathsf{Q} \ll \mathsf{P}; \\ +\infty, & \text{otherwise.} \end{cases}$$

We are interested in optimal investment in a financial asset with the discounted price process $\widetilde{S}_t$ as well as in the valuation of contingent claims. This pricing measure is linked to the optimal investment strategy with respect to expected exponential utility by a key duality result (see [4], [16], [22]):

$$\sup_{\theta\in\Theta} \mathsf{E}_{\mathsf{P}}\Big[-\exp\Big(-\int_0^T \theta_t \mathrm{d}\widetilde{S}_t\Big)\Big] = -\exp\Big(-\inf_{\mathsf{Q}\in\mathcal{M}^e} H(\mathsf{Q},\mathsf{P})\Big).$$

In the dual problem we minimize the relative entropy $H(\mathsf{Q},\mathsf{P})$ over the space $\mathcal{M}^e$ of all equivalent martingale measures for $\widetilde{S}$. This duality result is robust for various choices of the space Θ of admissible strategies. For instance, we can take Θ to consist of all predictable processes θ such that $\int \theta\,\mathrm{d}\widetilde{S}$ is a Q-martingale for all $\mathsf{Q}\in\mathcal{M}^e$ with finite relative entropy. It then turns out that if there exists an equivalent martingale measure with finite relative entropy, we can find an optimal strategy $\theta\in\Theta$ for the primal problem.

A probability measure $\mathsf{Q}^E\in\mathcal{M}^e$ is the MEMM if it satisfies

$$H(\mathsf{Q}^E,\mathsf{P}) \le H(\mathsf{Q},\mathsf{P}), \qquad \text{for all } \ \mathsf{Q}\in\mathcal{M}^e.$$

In the following, we will give a very explicit representation for the MEMM of the process $\widetilde{S}$.

By Proposition 3.2 of [14], the density of the MEMM Q^E can be written in the following form

$$\frac{\mathrm{d}\mathsf{Q}^E}{\mathrm{d}\mathsf{P}} = c\cdot\exp\Big(\int_0^T \eta_t \mathrm{d}\widetilde{S}_t\Big),$$

where c is a constant.

Now we take

$$\eta_t = -\frac{\mu_t - r_t}{\sigma_t^2}(\widetilde{S}_t)^{-1}$$

and

$$c^{-1} = \mathsf{E}_{\mathsf{P}}\Big[\exp\Big(-\frac{1}{2}\int_0^T \frac{(\mu_s-r_s)^2}{\sigma_s^2}\mathrm{d}s\Big)\Big].$$

And define

$$\begin{aligned}\frac{\mathrm{d}\mathsf{P}^*}{\mathrm{d}\mathsf{P}} \quad &:= \quad c\cdot\exp\Big(\int_0^T \eta_t \mathrm{d}\widetilde{S}_t\Big)\\ &= \quad c\cdot\exp\Big(-\int_0^T \frac{\mu_s-r_s}{\sigma_s}\mathrm{d}B_s - \int_0^T \frac{(\mu_s-r_s)^2}{\sigma_s^2}\mathrm{d}s\Big).\end{aligned} \tag{3.1}$$

We denote the above density as Z_T. Now, we are in a position to prove the probability measure P^* is the MEMM for Markov-modulated GBM.

Theorem 3.1 The probability measure P^* is the MEMM for Markov-modulated GBM.

Proof Referring to the results in [20], it is enough to verify the following three statements.

(1) The expectation $\mathsf{E}_{\mathsf{P}}[Z_T]$ is equal to one.

(2) The measure induced by Z_T has finite entropy.

(3)

$$\int_0^T (\eta_t)^2 \mathrm{d}[\widetilde{S}]_t \in L_{\exp}(\mathsf{P}),$$

where $[\widetilde{S}]_t$ is the quadratic variation process of $\widetilde{S}_t$ and $L_{\exp}(\mathsf{P})$ is the Orlicz space generated by the Yong function $\exp(\cdot)$.

In the following, we will prove the above three statements one by one.

(1) Define

$$Z'_t = \exp\Big(-\int_0^t \frac{\mu_s - r_s}{\sigma_s}\mathrm{d}B_s - \frac{1}{2}\int_0^t \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big). \tag{3.2}$$

Then we know that Z'_t is a true martingale. We denote the corresponding probability measure by Q'.

Hence, we get

$$Z_T = c \cdot Z'_T \exp\Big(-\frac{1}{2}\int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big). \tag{3.3}$$

Denote $\mathscr{F}_T^U = \sigma\{U_t, 0 < t \le T\}$, then we have

$$\begin{aligned}
\mathsf{E}_{\mathsf{P}}[Z_T] &= \mathsf{E}_{\mathsf{P}}[\mathsf{E}_{\mathsf{P}}[Z_T|\mathscr{F}_T^U]] = \mathsf{E}_{\mathsf{P}}\Big[\mathsf{E}_{\mathsf{P}}\Big[c \cdot Z'_T \exp\Big(-\frac{1}{2}\int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big)\Big|\mathscr{F}_T^U\Big]\Big] \\
&= c \cdot \mathsf{E}_{\mathsf{P}}\Big[\exp\Big(-\frac{1}{2}\int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big)\mathsf{E}_{\mathsf{P}}[Z'_T|\mathscr{F}_T^U]\Big] \\
&= c \cdot \mathsf{E}_{\mathsf{P}}\Big[\exp\Big(-\frac{1}{2}\int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big)\Big] \\
&= 1.
\end{aligned}$$

(2)

$$\begin{aligned}
&\mathsf{E}_{\mathsf{P}}[Z_T|\ln Z_T|] \\
= \;& c \cdot \mathsf{E}_{\mathsf{Q}'}\Big[\exp\Big(-\frac{1}{2}\int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big)\Big|\ln c - \int_0^t \frac{\mu_s - r_s}{\sigma_s}\mathrm{d}B_s - \frac{1}{2}\int_0^t \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big|\Big] \\
= \;& c \cdot \mathsf{E}_{\mathsf{Q}'}\Big[\exp\Big(-\frac{1}{2}\int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2}\mathrm{d}s\Big)\Big|\ln c - \int_0^t \frac{\mu_s - r_s}{\sigma_s}\mathrm{d}B_s^{\mathsf{Q}'}\Big|\Big] \\
< \;& \infty,
\end{aligned}$$

where $B_t^{\mathsf{Q}'}$ is a Q' Brownian motion.

(3) Since we have

$$\mathrm{d}[\widetilde{S}]_t = \widetilde{S}_t^2\sigma_t^2\mathrm{d}t.$$

Hence,

$$\int_0^T (\eta_t)^2\mathrm{d}[\widetilde{S}]_t = \int_0^T \Big(\frac{\mu_t - r_t}{\sigma_t^2}\Big)^2(\widetilde{S}_t^{-1})^2\mathrm{d}[\widetilde{S}_t] = \int_0^T \frac{(\mu_t - r_t)^2}{\sigma_t^2}\mathrm{d}t.$$

Thus

$$\mathsf{E}_{\mathsf{P}}\Big[\exp\Big(\int_0^T (\eta_t)^2\mathrm{d}[\widetilde{S}]_t\Big)\Big] = \mathsf{E}_{\mathsf{P}}\Big[\exp\Big(\int_0^T \frac{(\mu_t - r_t)^2}{\sigma_t^2}\mathrm{d}t\Big)\Big] < \infty. \qquad \square$$

Remark 1 Z_t' in (3.2) is the density process of the MEMM in [8]. (3.3) expresses the relation between the density of the general MEMM and the density of the MEMM in [8].

Remark 2 If the process U has only one state, that is

$$r_t \equiv r > 0, \qquad \mu_t \equiv \mu, \qquad \sigma_t \equiv \sigma > 0, \qquad t \in \mathcal{T}.$$

Then the dynamics of the stock price process $\{S_t\}_{t\in\mathcal{T}}$ are given by the standard geometric Brownian motion:

$$\begin{cases} \mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}B_t, & 0 < t \le T; \\ S_0 = s. \end{cases}$$

That is,

$$S_t = s\exp\Big(\Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma B_t\Big), \qquad 0 < t \le T.$$

Thus the market is complete, hence there is a unique equivalent martingale measure Q such that

$$\widetilde{S}_t := e^{-rt}S_t$$

is a martingale. As we all known, the probability measure Q is defined by

$$\frac{\mathrm{d}\mathsf{Q}}{\mathrm{d}\mathsf{P}} := \exp\Big(-\frac{\mu - r}{\sigma}B_T - \frac{(\mu - r)^2}{2\sigma^2}T\Big).$$

That is just the case P^* in (3.1) when the process U has only one state.

It expresses that the standard geometric Brownian motion is a special case of our model, and the MEMMs that we obtain are much more general than the risky neutral equivalent martingale measures in the Black-Scholes models.

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马尔可夫调制的几何布朗运动的最小熵鞅测度

王 波 宋瑞丽

(南京财经大学应用数学学院, 南京, 210023)

本文中, 我们考虑风险资产由马尔可夫调制的几何布朗运动驱动的期权定价问题. 在此模型中, 市场参数如市场利率、升值幅度和风险资产的波动率都依赖于不可观的经济状态, 而这些经济状态是由连续时间隐马尔可夫链来描述. 由马尔可夫调制的几何布朗运动描述的市场一般不是完备的, 因此鞅测度不唯一. 我们采用最小熵鞅测度作为马尔可夫调制的几何布朗运动模型的适宜的鞅测度, 并且得到了一般意义上的最小熵鞅测度.

关 键 词: 几何布朗运动, 隐马尔可夫链模型, 最小熵鞅测度.

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