应用概率统计 第二十九卷
第二期 2013年4月

Chinese Journal of Applied Probability
and Statistics Vol.29 No.2 Apr. 2013

Optimal Design of Accelerated Degradation Test based on Gamma Process Models *

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Abstract

In this paper, optimal constant-stress accelerated degradation test plans are developed under the assumption that the degradation characteristic follows a Gamma processes. The test stress levels and the proportion of units allocated to each stress level are determined by D-criterion and V-criterion. The general equivalence theorem (GET) is used to verify that the optimized test plans are globally optimum. In addition, compromise test plans are also studied. Finally, an example is provided to illustrate the proposed method and a sensitivity analysis is conducted to investigate the robustness of optimal plans.

Keywords: Optimal design, accelerated degradation test, Gamma process, Fisher information matrix, reliability.

AMS Subject Classification: 62N05.

§1. Introduction

Due to the strong pressure for marketing, a manufacturer is usually asked to provide its customers with reliability information (e.g., mean-time-to-failure) of the product. However, for highly reliable products, it is difficult to assess the lifetime of the products using traditional accelerated life tests (ALTs) that record only time-to-failure. Even the technique of censoring and/or accelerating the life by testing at higher levels of stress, such as elevated temperatures or voltages, is little help, since no failures are likely to occur over a reasonable period of time. In such a case, an accelerated degradation test (ADT) can be used as an alternative. In an ADT, a reliability-related performance characteristic degraded over time is measured at several accelerated conditions, and then analyzed using the specified ADT model. It is known that an ADT generally yields a better estimate of the reliability of test units especially when no or few failures occur (Nelson (1990) and Lu et al. (1996)).

*The research is supported by Natural Science Foundation of China (11271136), Science Foundation of Fujian Educational Committee (JA12301) and Natural Science Foundation of Fujian Province (2012J01282).

Received April 19, 2012. Revised October 25, 2012.

Recently, an optimal ADT plan was proposed based on the assumption that the underlying degradation path follows a stochastic process. Liao and Tseng (2006) designed optimal ADT plans under the assumptions of step-stress loading and a Wiener process (WP) model for the degradation path. Lim and Yum (2011) developed optimal constant-stress ADT (CSADT) plans based on a Wiener process model. For the Wiener process, it is known that the degradation path is not a strictly increasing function. Generally, a Gamma process (with monotone increasing pattern) is more suitable for describing the degradation path of some specific products, and especially in the case of crack tests. Hence, designing an efficient ADT plan for a gamma degradation process is of great interest. Although Tseng et al. (2009) designed optimal ADT plans under the assumptions of step-stress loading and a Gamma process model for the degradation path, they assume that the test stress levels are given and not optimally determined, as well as their optimal plans are obtained by numerical optimization method and no theory to verify that the optimized test plans are globally optimum.

In this paper, optimal CSADT plans are developed under the assumption that the degradation characteristic follows a Gamma processes. The test stress levels and the proportion of units allocated to each stress level are determined by D-criterion and V-criterion. The general equivalence theorem (GET) is used to verify that the optimized test plans are globally optimum.

The rest of this article is organized as follows. Gamma degradation models, some assumptions and parameter standardization procedures are introduced in Section 2. Expressions for the Fisher information matrix is obtained in Section 3. Section 4 provides the ADT design criteria and describes how to use the general equivalence theorem. In Section 5, the problem of optimally designing ADT plans with two stress levels is formulated and solved by an example. In Section 6, a compromise plan in which three stress levels are involved is developed. Sensitivity analysis of the test plans are illustrated in Section 7. Finally, conclusions and future research directions are presented in Section 8.

§2. Gamma Degradation Processes and Assumptions

2.1 Gamma Degradation Processes

Let $L(t|S_0)$ denote the degradation path of the product under a use stress S_0, and its lifetime τ can be suitably defined as the first passage time when $L(t|S_0)$ crosses a critical value ω. Hence, we have

$$\tau = \inf\{t|L(t|S_0) \geq \omega\}. \tag{2.1}$$

In the following, we assume that the independent increments of the degradation path of the product follows a gamma process (Lawless and Crowder (2004)). For fixed t and

$\triangle t$,

$$\triangle L(t|S_0)=L(t+\triangle t|S_0)-L(t|S_0)\sim f(x,\alpha_0\triangle t,\lambda)=\frac{\lambda^{\alpha_0\triangle t}}{\Gamma(\alpha_0\triangle t)}(x)^{\alpha_0\triangle t-1}\exp(-\lambda x),\quad(2.2)$$

where $\alpha_0\triangle t$ and λ are the shape and scale parameters of the gamma distribution, respectively. Recently, Park and Padgett (2005) proposed a simple approximate formula for the mean time to failure (MTTF) of the product under use stress S_0 as

$$\mathrm{MTTF}_0=\mathsf{E}(\tau)\approx\frac{\omega\lambda}{\alpha_0}+\frac{1}{2\alpha_0}.\quad(2.3)$$

The following assumptions A1-A5 are considered in this paper.

A1: Constant stress loading is adopted at each stress level S_i, $i=1,\ldots,r$. The total number of test units, n, is given and n_i units are allocated to each stress level such that

$$n_i=\pi_i n,\qquad\sum_{i=1}^{r}\pi_i=1,\qquad\pi_i\geq 0,$$

where π_i is the proportion of test units allocated to the ith stress level.

A2: The maximum and use stress levels are pre-specified as S_M and S_0, respectively. Under any constant stress level S_i, $i=0,1,\ldots,r$, the degradation characteristic $y_{ij}(t)$ of the jth unit ($j=1,2,\ldots,n_i$) follows a Gamma process with shape $\alpha_i t$ and scale λ.

A3: The relationship between the parameter α_i and the stress level S_i, $i=0,1,\ldots,r$, is assumed to follow a Arrhenius model

$$\ln(\alpha_i)=a+\frac{b}{S_i+273}.\quad(2.4)$$

A4: A unit is assumed to fail when the degradation characteristic $y_{ij}(t)$ becomes greater than the critical value ω.

A5: Let m_{ij} be the number of measurements for the jth unit at the stress level S_i. It is assumed that $m_{ij}=m$ for all i and j. The measurement times ($t_{ijk},k=1,2,\ldots,m$) and maximum test duration (t_{Mij}) for the jth test unit at the stress levels S_i are predetermined. In particular, it is assumed that $t_{Mij}=t_M$, $t_{ijk}-t_{ijk-1}=z$ and $t_{ijm}=t_M$ for all i, j and k. That is $t_{ijm}=mz$.

For the jth unit at the stress level S_i, let y_{ijk} be the degradation characteristic measured at t_{ijk}, where $t_{ij(k-1)}<t_{ijk}$ and $t_{ij0}=0$ for $i=1,2,\ldots,r$, $j=1,2,\ldots,n_i$, and $k=1,2,\ldots,m$. Then, due to Assumption A2, each degradation increment $\triangle y_{ijk}=y_{ijk}-y_{ij(k-1)}$ follows a Gamma distribution with shape $\alpha_i\triangle t_{ijk}$ and scale λ, where $\triangle t_{ijk}=t_{ijk}-t_{ij(k-1)}=z$. That is, the probability density function of $\triangle y_{ijk}$ is given by

$$f(\triangle y_{ijk})=\frac{\lambda^{\alpha_i z}}{\Gamma(\alpha_i z)}(\triangle y_{ijk})^{\alpha_i z-1}\exp(-\lambda\triangle y_{ijk}).\quad(2.5)$$

2.2 Standardization

In practical ADT experiments, the experimental region is between the use level S_0 and maximum allowable level S_M. For simplicity, the accelerating variable level is often standardized as

$$x_i=\Big(\frac{1}{S_0+273}-\frac{1}{S_i+273}\Big)\Big/\Big(\frac{1}{S_0+273}-\frac{1}{S_M+273}\Big),$$

such that the experimental region of x is in the range $[0,1]$. Thus, in terms of the standardized variable level x_i, the acceleration model (2.4) can be expressed as

$$\ln(\alpha_i)=\gamma_0+\gamma_1 x_i,\qquad 0\le x_i\le 1, \tag{2.6}$$

where (γ_0,γ_1) is a re-parameterization of (a,b). That is

$$\gamma_0=a+\frac{b}{S_0+273},\qquad \gamma_1=\frac{b}{S_M+273}-\frac{b}{S_0+273}.$$

§3. Fisher Information

According to (2.5), the likelihood function of the ADT model for a gamma process is given by

$$L(\theta)=\prod_{i=1}^{r}\prod_{j=1}^{n_i}\prod_{k=1}^{m}\frac{\lambda^{\alpha_i\triangle t_{ijk}}}{\Gamma(\alpha_i\triangle t_{ijk})}(\triangle y_{ijk})^{\alpha_i\triangle t_{ijk}-1}\exp(-\lambda\triangle y_{ijk}), \tag{3.1}$$

where $\theta=(\gamma_0,\gamma_1,\lambda)$. Then, the maximum likelihood estimator (MLE) $\widehat{\theta}=(\widehat{\gamma}_0,\widehat{\gamma}_1,\widehat{\lambda})$ of θ can be obtained by a numerical method. The MLE of the MTTF under S_0, $\widehat{\mathrm{MTTF}}_0$, can be obtained by substituting $\widehat{\gamma}_0,\widehat{\gamma}_1,\widehat{\lambda}$ and S_0 into (2.3) and (2.6) directly.

A test plan ξ consists of r test stress levels $x_1,x_2,\ldots,x_r$, and the corresponding proportions of test units $\pi_1,\pi_2,\ldots,\pi_r$, such that $\sum\limits_{i=1}^{r}\pi_i=1$. That is

$$\xi=\begin{pmatrix} x_1 & x_2 & \ldots & x_r\\ \pi_1 & \pi_2 & \ldots & \pi_r\end{pmatrix}.$$

Theorem 3.1 The Fisher information matrix for a test plan ξ is the expected value of the second derivative of the total log-likelihood, that is

$$I(\theta,\xi)=\mathsf{E}\Big(-\frac{\partial^2\ln L(\theta)}{\partial\theta\partial\theta'}\Big)=\begin{pmatrix}\sum\limits_{i=1}^{r}P_i & \sum\limits_{i=1}^{r}x_iP_i & -\dfrac{t_M}{\lambda}\sum\limits_{i=1}^{r}\alpha_in_i\\ \sum\limits_{i=1}^{r}x_iP_i & \sum\limits_{i=1}^{r}x_i^2P_i & -\dfrac{t_M}{\lambda}\sum\limits_{i=1}^{r}x_i\alpha_in_i\\ -\dfrac{t_M}{\lambda}\sum\limits_{i=1}^{r}\alpha_in_i & -\dfrac{t_M}{\lambda}\sum\limits_{i=1}^{r}x_i\alpha_in_i & \dfrac{t_M}{\lambda^2}\sum\limits_{i=1}^{r}\alpha_in_i\end{pmatrix}, \tag{3.2}$$

where $P_i = \sum\limits_{j=1}^{n_i}\sum\limits_{k=1}^{m}(\alpha_i\triangle t_{ijk})^2\psi_1(\alpha_i\triangle t_{ijk})$, and $\psi_1(x) = \mathrm{d}^2\ln\Gamma(x)/\mathrm{d}x^2$ is a trigamma function. According to Assumption A5, P_i is simplified as $mn_i(\alpha_i z)^2\psi_1(\alpha_i z)$.

Proof According to (3.1), Expression of the elements of the Fisher information $I(\theta,\xi)$ in (3.2) are

$$\begin{aligned}
\mathsf{E}\Big(-\frac{\partial^2\ln L}{\partial\gamma_0^2}\Big) &= -\sum_{i=1}^{r}(\alpha_i n_i t_M)\ln\lambda + \sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}\big[(\triangle t_{ijk}\alpha_i)^2\psi_1(\triangle t_{ijk}\alpha_i)\\
&\quad + \psi_0(\triangle t_{ijk}\alpha_i)\triangle t_{ijk}\alpha_i\big] - \sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}\triangle t_{ijk}\alpha_i\mathsf{E}(\ln(\triangle y_{ijk})),\\
\mathsf{E}\Big(-\frac{\partial^2\ln L}{\partial\gamma_0\partial\gamma_1}\Big) &= -\sum_{i=1}^{r}(x_i\alpha_i n_i t_M)\ln(\lambda) + \sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}\big[(\triangle t_{ijk}\alpha_i)^2 x_i\psi_1(\triangle t_{ijk}\alpha_i)\\
&\quad + \psi_0(\triangle t_{ijk}\alpha_i)x_i\triangle t_{ijk}\alpha_i\big] - \sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}x_i\triangle t_{ijk}\alpha_i\mathsf{E}(\ln(\triangle y_{ijk})),\\
\mathsf{E}\Big(-\frac{\partial^2\ln L}{\partial\gamma_0\partial\lambda}\Big) &= -\sum_{i=1}^{r}\alpha_i n_i t_M\frac{1}{\lambda}, \qquad \mathsf{E}\Big(-\frac{\partial^2\ln L}{\partial\gamma_1\partial\lambda}\Big) = -\sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}\triangle t_{ijk}\alpha_i x_i\frac{1}{\lambda},\\
\mathsf{E}\Big(-\frac{\partial^2\ln L}{\partial\gamma_1^2}\Big) &= -\sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}x_i^2\triangle t_{ijk}\alpha_i\ln(\lambda) + \sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}\big[(\triangle t_{ijk}\alpha_i x_i)^2\psi_1(\triangle t_{ijk}\alpha_i)\\
&\quad + \psi_0(\triangle t_{ijk}\alpha_i)x_i^2\triangle t_{ijk}\alpha_i\big] - \sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}x_i^2\triangle t_{ijk}\alpha_i\mathsf{E}(\ln(\triangle y_{ijk})),\\
\mathsf{E}\Big(-\frac{\partial^2\ln L}{\partial\lambda^2}\Big) &= \sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}\triangle t_{ijk}\alpha_i\frac{1}{\lambda^2} = \frac{1}{\lambda^2}\sum_{i=1}^{r}\alpha_i n_i t_{im},
\end{aligned}$$

where $\mathsf{E}(\ln(\triangle y_{ijk})) = \psi_0(\triangle t_{ijk}\alpha_i) - \ln(\lambda)$, and $\psi_0(x) = \mathrm{d}\ln\Gamma x/\mathrm{d}x$, $\psi_1(x) = \mathrm{d}^2\ln\Gamma x/\mathrm{d}x^2$ are the digamma and trigamma functions, respectively. According to

$$\mathsf{E}(\ln(\triangle y_{ijk})) = \psi_0(\triangle t_{ijk}\alpha_i) - \ln(\lambda),$$

we obtain the Fisher information matrix (3.2). □

§4. Optimization Criteria and Optimization Procedure

4.1 D-Criterion

The main purpose of this paper is to study the choice of $\pi_1, \pi_2, \ldots, \pi_{r-1}$, and $x_1, x_2, \ldots, x_r$, in a r-stress CSADT. D optimality criterion (Francis (2010)), often used in planning ALT, is based on the determinant of the Fisher information matrix, which is the same as the reciprocal of the determinant of the asymptotic variance covariance matrix. Note that the overall volume of the asymptotic joint confidence region of $\theta = (\gamma_0, \gamma_1, \lambda)$ is proportional to $|I(\theta,\xi)^{-1}|^{1/2}$ at a fixed confidence level. Consequently, a larger value of

$|I(\theta,\xi)|$ is equivalent to a smaller asymptotic joint confidence ellipsoid of θ and a higher joint precision of the estimators of θ. Motivated by this, our objective is to select the optimal $\pi_1,\pi_2,\dots,\pi_{r-1}$ and $x_1,x_2,\dots,x_r$ to maximum

$$|I(\theta,\xi)|. \tag{4.1}$$

4.2 V-Criterion

The mean of the failure time is an important characteristic and indispensable in reliability analysis. In CSADT, we need to estimate the MTTF at the use stress with maximum precision. We can use the asymptotic variance of MTTF at use stress as the criterion for selecting the optimal ξ. Thus, according to (2.3) and by the δ method, the approximate variance of $\widehat{\mathrm{MTTF}}_0$ is found to be

$$\mathsf{Avar}(\widehat{\mathrm{MTTF}}_0)=C'I(\theta,\xi)^{-1}C, \tag{4.2}$$

where $C'=(-(\omega\lambda/\alpha_0+1/(2\alpha_0)),0,\omega/\alpha_0)$, and C' denotes the transpose of C. A desirable test plan with a small $\mathsf{Avar}(\widehat{\mathrm{MTTF}}_0)$ value is said to be efficient for estimating MTTF_0.

4.3 Equivalence Theorems

Equivalence theorems are used to verify the global optimality of test plan ξ_{opt} over all possible test plans. Similar to the discussion in Francis (2010), general equivalence theorem (GET) (Whittle (1973)) and its extensions to nonlinear models by White (1973) and Chaloner and Larntz (1989) can be applied to the current problem. Let V denote the set of values for the transformed accelerating variable. The test plan ξ may be regarded as a probability measure over the design space V. General equivalence theorem applies to the maximization of a concave criterion function $\psi(\xi)$. The derivative of ψ at ξ in the direction of ξ' is defined by

$$d(\xi,\xi')=\lim_{\varepsilon\to 0}\frac{\psi((1-\varepsilon)\xi+\varepsilon\xi')-\psi(\xi)}{\varepsilon}.$$

Let ξ_v denote the test plan with all allocations at v. $\psi(\xi)$ is said to be differentiable at ξ, if

$$d(\xi,\xi')=\int d(\xi,\xi_v)\xi'(\mathrm{d}v),$$

and $d(\xi,\xi')$ is linear in ξ'. Furthermore, $d(\xi,v)$ represents $d(\xi,\xi_v)$ and is called the (directional) derivative function of ψ at measure ξ. The GET, in the notation of this article, is given below.

Theorem 4.1 (Whittle (1973)) 1. If ψ is concave, then an optimal design, ξ_{opt}, can be equivalently characterized by any of the three conditions:

(a) ξ_{opt} maximizes ψ. (b) ξ_{opt} minimizes $\sup\limits_{v\in V} d(\xi, v)$. (c) $\sup\limits_{v\in V} d(\xi_{\text{opt}}, v) = 0$.

2. The point $(\xi_{\text{opt}}, \xi_{\text{opt}})$ is a saddle point of d in that

$$d(\xi_{\text{opt}}, \xi_1) \le 0 = d(\xi_{\text{opt}}, \xi_{\text{opt}}) \le d(\xi_2, \xi_{\text{opt}}),$$

for designs ξ_1 and ξ_2.

3. If ψ is also differentiable, then the support of ξ_{opt} is contained in the set of v for which $d(\xi_{\text{opt}}, v) = 0$, in that $d(\xi_{\text{opt}}, v) = 0$ almost everywhere in ξ_{opt} measure.

Equivalence theorems require evaluation of a function $d(\xi, v)$ often called the directional derivative of a test plan criterion at ξ, where $v \in V$. It suffices to plot $d(\xi_{\text{opt}}, v)$ for $v \in V$, and check that $d(\xi_{\text{opt}}, v) = 0$ for all levels v of ξ_{opt}, and $d(\xi_{\text{opt}}, v) < 0$ otherwise. If these hold, then ξ_{opt} is considered to be globally optimal over V.

4.4 Optimality Procedure

4.4.1 Equivalence Theorem for D-Optimality

White (1973) extended the GET to the nonlinear design problem. She presented results for D and D_s-optimality. The results, in the notation of this article, are given below. For D-optimality, we have

$$d_D(\xi, v) = \text{tr}\{I(\theta, \xi_v)[I(\theta, \xi)]^{-1}\} - 3. \tag{4.3}$$

Theorem 4.2 The following conditions on a design measure ξ_{opt} are equivalent:

1. ξ_{opt} is D-optimal. 2. $\sup\limits_{v\in V} d_D(\xi_{\text{opt}}, v) = 0$.

Proof It is easy to prove by using Theorem 4.1 and (4.3). □

4.4.2 Equivalence Theorem for V-Optimality

Whittle (1973) proved the GET in the framework of the linear design problem. However, Chaloner and Larntz (1989) remarked that the proof was valid for nonlinear design problems under some additional regularity conditions. The criterion given by Eq. (4.2) is a function of ξ that is convex. Theorem 4.1 is applied to this by considering $\psi = -\mathsf{Avar}$ which is concave and satisfies the regularity conditions stated in Chaloner and Larntz (1989). Hence, the GET applies. Observe that $\psi = -\mathsf{Avar}$ is a special case of the ϕ_2 criterion in Chaloner and Larntz (1989). The derivative function for $\psi = -\mathsf{Avar}$ is given by

$$d_V(\xi, v) = C'[I(\theta, \xi)]^{-1} I(\theta, \xi_v)[I(\theta, \xi)]^{-1} C - C'[I(\theta, \xi)]^{-1} C. \tag{4.4}$$

Theorem 4.3 The following conditions on a design measure ξ_{opt} are equivalent:

1. ξ_{opt} is V-optimal. 2. $\sup\limits_{v\in V} d_V(\xi_{\text{opt}}, v) = 0$.

Condition 2 of Theorem 4.2 and 4.3 is used to verify the D-optimality and V-optimality of test plan respectively in this article.

§5. Example

In this section, we illustrate the proposed procedure with a numerical example based on the carbon-film-resistor problem described by Meeker and Escobar (1998) on page 471. The resistance value of the carbon-film resistors over time is defined as a failure-related degradation characteristic $y(t)$. Obviously, $y(0) = 0$. The stress variable is the temperature and the degradation characteristic is assumed to follow a gamma process at a temperature level. Arrhenius model is assumed between the drift parameter and the temperature. The maximum test temperature S_M is specified as 173°C (446°k) and the use test temperature S_0 is 50°C (323°k). Changes in resistance will cause the reduction of the performance of the product, or even system failures. The lifetime of the product is typically defined as the time when the resistance value increases by a critical value from its initial value under the operating temperature $S_0 = 50$°C (323°k). The critical value is taken to be $\omega = 5$. The sample size $n = 100$ is considered for the CSADT.

For illustrative purpose, we adopt the true parameter configuration as

$$(a, b, \lambda) = (4.17, -4058.79, 16), \tag{5.1}$$

which have been adopted by Tseng et al. (2009) for optimal step-stress accelerated degradation test plan. The new parameters θ are obtained by re-parameterization of (a, b, λ). That is

$$\theta = (\gamma_0, \gamma_1, \lambda) = (-8.396, 3.465, 16). \tag{5.2}$$

The maximum test duration is assumed to be 1200h (i.e. $t_M = 1200$) and the number of measurements for each unit is assumed 40 (i.e. $m = 40$). According to Assumption A5, the interval of two measure times, $z = t_{ijk} - t_{ijk-1}$, is equal to 30. D-criterion and V-criterion can be used to find the optimum plans for the ADT experiment. With the particular problem specification, (4.1) and (4.2) are functions of the plan ξ. For a given ξ, the values of (4.1) and (4.2) can be easily calculated. Thus, we can search for the plan ξ based on D-criterion or V-criterion.

For simplicity, assume that $r = 2$ for the rest of the article. The results for $r > 2$ are similarly derived. For designs with $r = 2$, by grid search we obtained the optimal plans ξ_D and ξ_V for D-criterion and V-criterion respectively,

$$\xi_D = \begin{pmatrix} 0 & 1 \\ 0.34 & 0.66 \end{pmatrix}, \qquad \xi_V = \begin{pmatrix} 0 & 1 \\ 0.3 & 0.7 \end{pmatrix}.$$

Moreover, the optimal values of D-criterion and V-criterion are 6888878 and 357041194 respectively. The equivalence Theorem 4.2 and 4.3 are applied to verify the global optimality of the test plans. Figure 1 and Figure 2 verify the global optimality of test plans for D-criterion and V-criterion respectively. The results indicate that the optimal stress levels for the D-optimal plan ξ_D and the V-optimal plan ξ_V are the use stress ($S_0 = 50^{\circ}$C) and highest stress ($S_h = 173^{\circ}$C). The optimal assign ratios of π_1 and π_2 are almost same for the D-optimal plan and the V-optimal plan.

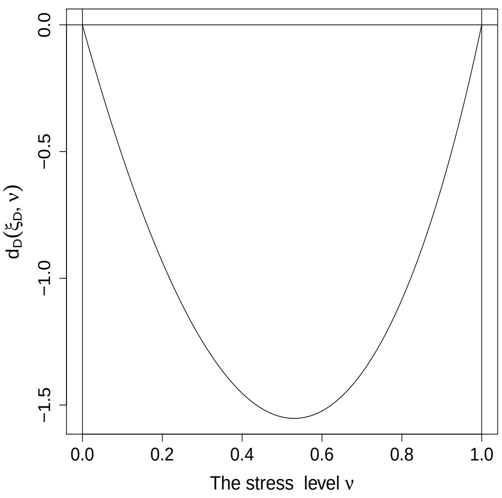

Figure 1 Plot of $d_D(\xi_{\rm opt}, v)$ versus standardized stress v to verify optimality of test plans for D-criterion

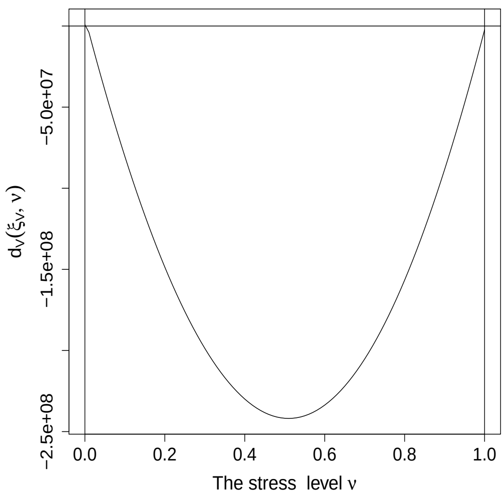

Figure 2 Plot of $d_V(\xi_{\rm opt}, v)$ versus standardized stress v to verify optimality of test plans for V-criterion

§6. Compromise Test Plan

Since the optimal ADT plan in Section 5 involves two stress levels, it dose not suitable for application. To avoid this problem, Meeker and Escobar (1998) proposed compromise test plans with three stress levels that are motivated by the optimum test plans. Thus, a compromise test plan which is similar to Meeker (1984) is proposed as follows:

1. The middle stress level x_2 is set to $(x_1 + x_3)/2$, where the high stress level x_3 is set to 1.

2. The proportion (π_2) of test units allocated to x_2 is pre-specified ($0 < \pi_2 \leq 0.3$). In this paper, we set $\pi_2 = 0.2$.

3. For a given π_2, our objective is to select the low stress level x_1 and the proportion (π_1) of test units to maximum $|I(\theta, \xi)|$ or to minimum $\mathsf{Avar}(\mathrm{MTTF}_0)$.

It is difficult to analytically determine the optimal values of x_1 and π_1 since the objective function $|I(\theta, \xi)|$ and $\mathsf{Avar}(\mathrm{MTTF}_0)$ are very complex. Therefore, a simple grid search method is employed. Grids are obtained by dividing each range of x_1 and π_1 into 100 equal parts. Then, the optimal solution, x_1^c and π_1^c, is determined as the grid point at which $|I(\theta,\xi)|$ attains the maximum or $\mathsf{Avar}(\mathrm{MTTF}_0)$ attains the minimum. Compromise ADT plans for D-criterion and V-criterion are shown in Table 1 and Table 2. The optimized D-criterion value and V-criterion value from compromise plans are 4804903 and 416938034 respectively, about 29% and 16% departure from the global optimum. Such these plans, with more than two levels of temperature, would be preferred in practice.

Table 1 ADT compromise test plan for D-criterion

Condition	Level		Allocation	
	Temp	Standardized	Proportion	Number
i	(°C)	x_i^c	π_i^c	n_i^c
Low	50	0	0.26	26
Middle	111.5	0.5	0.2	20
High	173	1	0.54	54

Table 2 ADT compromise test plan for V-criterion

Condition	Level		Allocation	
	Temp	Standardized	Proportion	Number
i	(°C)	x_i^c	π_i^c	n_i^c
Low	50	0	0.23	23
Middle	111.5	0.5	0.2	20
High	173	1	0.57	57

§7. Sensitivity Analysis

In this section, we follow the framework of sensitivity analysis in Tseng et al. (2009). In practice, the estimated parameters $\widehat{\theta}=(\widehat{\gamma}_0,\widehat{\gamma}_1,\widehat{\lambda})$ would depart from the true parameters $\theta=(\gamma_0,\gamma_1,\lambda)$. Hence, it is important to investigate the effects of these unknown parameters on the optimal test plan. Without loss of generality, we assume that $\varepsilon_1,\varepsilon_2$ and ε_3 denote the predicted errors for γ_0,γ_1, and λ, respectively. Under the same configuration $(n,m,z,\omega)=(100,40,30,5)$, Table 3 and Table 4 present the optimal plan under various combinations of $((1+\varepsilon_1)\gamma_0,(1+\varepsilon_2)\gamma_1,(1+\varepsilon_3)\lambda)$ according to a $L_9(3^{3-1})$ orthogonal array with $\theta=(\gamma_0,\gamma_1,\lambda)$ in (5.2). From these results, it shows that the D-test plan is quite robust for a moderate departure from the assumed values of these parameters. Note that the optimal plans for V-criterion are slightly affected by the parameters of $\theta=(\gamma_0,\gamma_1,\lambda)$. Hence, to design a better ADT plan, we need more precise estimation of θ.

Table 3 Optimal ADT plan for D-criterion under various combinations of parameters $((1+\varepsilon_1)\gamma_0,(1+\varepsilon_2)\gamma_1,(1+\varepsilon_3)\lambda)$

ε_1	ε_2	ε_3	x_1	x_2	π_1	π_2	D-criterion value
−5%	−5%	−5%	0	1	0.34	0.66	9287719
−5%	0	0	0	1	0.34	0.66	9666715
−5%	+5%	+5%	0	1	0.34	0.66	10097190
0	−5%	0	0	1	0.34	0.66	5824802
0	0	+5%	0	1	0.34	0.66	6125322
0	+5%	−5%	0	1	0.34	0.66	8659532
+5%	−5%	+5%	0	1	0.34	0.66	3626175
+5%	0	−5%	0	1	0.34	0.66	5163548
+5%	+5%	0	0	1	0.34	0.66	5422761
0	0	0	0	1	0.34	0.66	6753168

Table 4 Optimal ADT plan for V-criterion under various combinations of parameters $((1+\varepsilon_1)\gamma_0,(1+\varepsilon_2)\gamma_1,(1+\varepsilon_3)\lambda)$

ε_1	ε_2	ε_3	x_1	x_2	π_1	π_2	V-criterion value
−5%	−5%	−5%	0	1	0.33	0.67	122634983
−5%	0	0	0	1	0.34	0.66	125656311
−5%	+5%	+5%	0	1	0.35	0.65	129058176
0	−5%	0	0	1	0.29	0.71	396043518
0	0	+5%	0	1	0.30	0.70	397002563
0	+5%	−5%	0	1	0.32	0.68	297373093
+5%	−5%	+5%	0	1	0.25	0.65	1317078785
+5%	0	−5%	0	1	0.27	0.63	966020478
+5%	+5%	0	0	1	0.28	0.72	964409566
0	0	0	0	1	0.30	0.70	357041194

§8. Conclusion and Areas for Further Research

In this paper, we obtain the optimal CSADT plan based on D-criterion and V-criterion. The sensitivity analysis reveals that the optimal test plans are quite robust to moderate departures from the assumed values of the model parameters. In practice, the optimal designs often depend on the preestimates of its parameters. In order to overcome this problem, attractive future research is to follow a Bayesian approach to planning by assigning prior distributions to experimental conditions and deriving the corresponding optimal designs.

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基于Gamma过程加速退化试验的优化设计

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本文研究了基于Gamma过程恒定应力加速退化试验的优化设计问题. 在D–最优和V–最优为准则下, 确定了试验最优应力和各个应力下所分配的最优比例数. 广义等价性定理被用来确保最优点的全局最优性. 另外我们还研究了其平衡试验. 最后, 通过一个例子说明本文所提的方法, 同时通过敏感性分析研究了优化点的稳键性.

关 键 词: 最优设计, 加速退化试验, Gamma过程, Fisher信息量, 可靠性.

学 科 分 类 号: O213.2.