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# A Generalization of Dobrushin Coefficient *

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**Abstract**

We generalize the well-known Dobrushin coefficient $\delta$ in total variation to weighted total variation $\delta_V$, which gives a criterion for the geometric ergodicity of discrete-time Markov chains.

**Keywords:** $V$-norm, $\delta_V$ coefficient, geometric ergodicity.

**AMS Subject Classification:** 37A30, 47A35.

## §1. Introduction and Main Results

In this paper, we generalize the classical Dobrushin coefficient, in order to give a criteria for geometric ergodicity of discrete-time Markov chains.

Let $X = (X_n)_{n\geq 0}$ be a discrete-time Markov chain taking values on a measurable space $(E, \mathscr{E})$. Denote by

$$P^n(x, A) := \mathsf{P}[X_n \in A | X_0 = x], \qquad x \in E,\ A \in \mathscr{E},$$

the $n$-step transition kernel. Set $P = P^1$ be the one-step kernel.

Throughout the paper, we assume that $X$ is $\psi$-irreducible and aperiodic, c.f. [1].

We are interested in the geometrical ergodicity that is, there is an invariant probability measure $\pi$ on $(E, \mathscr{E})$ and a constant $\rho \in [0, 1)$ and a function $C : E \to (0, \infty)$ such that

$$\|P^n(x, \cdot) - \pi\|_{\mathsf{Var}} \leq C(x)\rho^n, \qquad \text{for all } n > 0,\ \pi\text{-a.s. } x \in E, \tag{1.1}$$

where $\|\mu\|_{\mathsf{Var}} := 2 \sup\limits_{A\in\mathscr{E}} |\mu(A)| = \sup\left\{\displaystyle\int_E f \mathrm{d}\mu : |f| \leq 1\right\}$ is the total variation for a signed measure $\mu$.

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From [1, 2], we know that (1.1) is equivalent to that there exist constants $\rho \in [0,1)$, $C<\infty$ and a $\pi$-a.s. finite function $V: E \to [1,\infty]$ such that

$$\|P^n(x,\cdot)-\pi\|_V \le CV(x)\rho^n, \tag{1.2}$$

where $\|\mu\|_V := \sup\Big\{\displaystyle\int_E f\mathrm{d}\mu : \|f\|_V \le 1\Big\}$ denotes the $V$-norm on signed measures. Recall that for a kernel $K(x,A)$ $(x\in E,\ A\in\mathscr{E})$,

$$\|K\|_V := \sup_{x\in E}\frac{\|K(x,\cdot)\|_V}{V(x)}, \tag{1.3}$$

and for a function $f$,

$$\|f\|_V := \sup_{x\in E}\frac{|f(x)|}{V(x)}, \tag{1.4}$$

c.f. [1, Chapter 14].

Specially, if in (1.1) $C(x)\le C<\infty$ for $\pi$-a.s. $x\in E$, (1.1) is strengthened to the so-called strong ergodicity. Equivalently in (1.2), we can choose $V: E\to[1,\infty)$ to be (upper) bounded. Recall that $X$ is said to be strongly ergodic if there exist constants $\rho\in[0,1)$ and $C<\infty$ such that

$$\mathrm{ess}_\pi \sup_{x\in E}\|P^n(x,\cdot)-\pi\|_{\mathsf{Var}} \le C\rho^n, \qquad \text{for all } n>0. \tag{1.5}$$

Dobrushin (1956) gave an elegant criterion for strong ergodicity in (1.5). Let

$$\delta(P)=\frac{1}{2}\mathrm{ess}_\pi \sup_{x,y\in E}\|P(x,\cdot)-P(y,\cdot)\|_{\mathsf{Var}}.$$

Then, $X$ is strongly ergodic if and only if there exists $n$, such that $\delta(P^n)<1$. $\delta(P)$ is now in term of Dobrushin coefficient.

In the context of $E$ countable, an elegant description of $\delta(P)$ and strong ergodicity can be found in [4, § 6.1, § 6.3]. Although [4, § 6.3] only considered continuous-time Markov chains, these arguments remain valid for a general Markov process. Or this can be viewed as a special case where we will do in the following (by taking $V(x)\equiv 1$).

Now, we will generalize $\delta(P)$ to $\delta_V(P)$, which gives a criteria for geometric ergodicity in (1.1) or (1.2).

**Definition 1.1** For a transition kernel $P$, and a $\pi$-a.s. finite function $V: E\to[1,\infty]$, we define the generalized Dobrushin coefficient $\delta_V(P)$:

$$\delta_V(P) := \mathrm{ess}_\pi \sup_{x,y\in E}\frac{1}{V(x)+V(y)}\|P(x,\cdot)-P(y,\cdot)\|_V. \tag{1.6}$$

Here is the main result.

**Theorem 1.1** A $\psi$-irreducible and aperiodic Markov chain $X = (X_n)_{n\geq 0}$ is geometrically ergodic if and only if $X$ is ergodic with stationary distribution $\pi$, and there is a $\pi$-a.s. finite function $V : E \to [1, \infty]$ such that $\pi(V) < \infty$ and $\delta_V(P^n) < 1$ for $n$ large enough.

Note that when $V(x) \equiv 1$, $\delta_V(P) = \delta(P)$ and hence Theorem 1.1 generalizes the classical Dobrushin coefficient.

The key ingredient for the proof of Theorem 1.1 comes from an excellent observation. By defining a metric on $E$ as in [5], we can transfer $\delta_V(P)$ to be a Wasserstein distance on space of probability measures.

## §2. Proof of Theorem

For simplicity, in the following, sup is referred to $\text{ess}_\pi \sup$.

Following [5], we introduce a metric on $E$. Let $V : E \to [1, \infty)$. For $x, y \in E$,

$$d_V(x,y) = \begin{cases} 0, & x = y; \\ V(x) + V(y), & x \neq y. \end{cases}$$

Then $(E, d_V)$ is a complete metric space. Let $\varphi$ be a function on $(E, d_V)$, the Lipschitz norm of $\varphi$ is

$$\|\varphi\|_{\text{Lip}(d_V)} := \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{d_V(x,y)} = \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{V(x) + V(y)}.$$

For two (probability) measures $\mu_1, \mu_2$ on $(E, \mathcal{E})$, Wasserstein metric $W_{d_V}$ is defined by

$$W_{d_V}(\mu_1, \mu_2) = \sup_{\varphi:\|\varphi\|_{\text{Lip}(d_V)}\leq 1} \int \varphi \mathrm{d}(\mu_1 - \mu_2).$$

The following was proved in [5, Lemma 2.1].

**Lemma 2.1** For two probability measures $\mu_1, \mu_2$ on $(E, \mathcal{E})$,

$$\|\mu_1 - \mu_2\|_V = W_{d_V}(\mu_1, \mu_2). \tag{2.1}$$

**Remark 1** (1) For $V(x) \equiv 1$, both sides of (2.1) are reduced to the total variation of $\mu_1 - \mu_2$. For the left side, when $V(x) \equiv 1$,

$$\|\mu_1 - \mu_2\|_V = \sup\left\{\int_E f \mathrm{d}(\mu_1 - \mu_2) : |f| \leq 1\right\} = \|\mu_1 - \mu_2\|_{\mathsf{Var}}.$$

For the right side, $d_V$ is nothing but discrete metric. Hence by [6, Theorem 5.7], $W_{d_V}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{\mathsf{Var}}$.

(2) As pointed out in [5], (2.1) is also true for measures $\mu_1$, $\mu_2$ with equal mass.

The following properties are essential to the proof of Theorem 1.1.

**Proposition 2.1** Let $P$ and $\widetilde{P}$ be probability transition kernels on $(E, \mathcal{E})$. Then

(i) $\delta_V(P\widetilde{P}) \le \delta_V(P)\delta_V(\widetilde{P})$,

(ii) $|\delta_V(P) - \delta_V(\widetilde{P})| \le \|P - \widetilde{P}\|_V$,

(iii) Let $R$ be a transition kernel on $(E, \mathcal{E})$ such that $R(x, E) = 0$, for all $x \in E$ and $\|R\|_V < \infty$. Then $\|RP\|_V \le \|R\|_V \delta_V(P)$.

**Proof** (i) By the definition of $\delta_V(P)$ and Lemma 2.1, we have

$$W_{d_V}(P(x,\cdot), P(y,\cdot)) = \|P(x,\cdot) - P(y,\cdot)\|_V \le \delta_V(P)(V(x) + V(y)), \qquad \pi\text{-a.s. } x, y \in E.$$

Hence by the definition of $W_{d_V}$, we get

$$|P\varphi(x) - P\varphi(y)| \le \delta_V(P)\|\varphi\|_{\mathrm{Lip}(d_V)}(V(x) + V(y))$$

or

$$\frac{1}{V(x) + V(y)}|P\varphi(x) - P\varphi(y)| \le \delta_V(P)\|\varphi\|_{\mathrm{Lip}(d_V)},$$

and

$$\|P\varphi\|_{\mathrm{Lip}(d_V)} \le \delta_V(P)\|\varphi\|_{\mathrm{Lip}(d_V)}. \tag{2.2}$$

Since $\displaystyle\int (\widetilde{P}P)(x, \mathrm{d}z)\varphi(z) = \int (P\varphi)(z)\widetilde{P}(x, \mathrm{d}z)$, we get from (2.2) that

$$\begin{aligned} W_{d_V}((\widetilde{P}P)(x,\cdot), (\widetilde{P}P)(y,\cdot)) &= \sup_{\|\varphi\|_{\mathrm{Lip}(d_V)} \le 1} \int (P\varphi)(z)(\widetilde{P}(x, \mathrm{d}z) - \widetilde{P}(y, \mathrm{d}z)) \\ &\le \delta_V(P) \sup_{\|\psi\|_{\mathrm{Lip}(d_V)} \le 1} \int \psi(z)(\widetilde{P}(x, \mathrm{d}z) - \widetilde{P}(y, \mathrm{d}z)) \\ &= \delta_V(P) W_{d_V}(\widetilde{P}(x,\cdot), \widetilde{P}(y,\cdot)). \end{aligned}$$

Therefore by Lemma 2.1 again, we have

$$\begin{aligned} \delta_V(\widetilde{P}P) &= \sup_{x,y \in E} \frac{1}{V(x) + V(y)} \|(\widetilde{P}P)(x,\cdot) - (\widetilde{P}P)(y,\cdot)\|_V \\ &= \sup_{x,y \in E} \frac{1}{V(x) + V(y)} W_{d_V}((\widetilde{P}P)(x,\cdot), (\widetilde{P}P)(y,\cdot)) \\ &\le \delta_V(P) \sup_{x,y \in E} \frac{1}{V(x) + V(y)} W_{d_V}(\widetilde{P}(x,\cdot), \widetilde{P}(y,\cdot)) \\ &= \delta_V(P) \sup_{x,y \in E} \frac{1}{V(x) + V(y)} \|\widetilde{P}(x,\cdot) - \widetilde{P}(y,\cdot)\|_V \\ &= \delta_V(P)\delta_V(\widetilde{P}). \end{aligned}$$

(ii) Using the fact that $\big|\sup_x |u(x)| - \sup_x |v(x)|\big| \le \sup_x |u(x) - v(x)|$, we get

$$\begin{aligned}
&|\delta_V(P) - \delta_V(\widetilde{P})| \\
= &\Big|\sup_{x,y\in E} \frac{1}{V(x)+V(y)}\|P(x,\cdot) - P(y,\cdot)\|_V - \sup_{x,y\in E} \frac{1}{V(x)+V(y)}\|\widetilde{P}(x,\cdot) - \widetilde{P}(y,\cdot)\|_V\Big| \\
\le &\sup_{x,y\in E} \frac{1}{V(x)+V(y)} \sup_{|f|\le V} |(Pf(x) - Pf(y)) - (\widetilde{P}f(x) - \widetilde{P}f(y))| \\
\le &\sup_{x,y\in E} \frac{1}{V(x)+V(y)} \sup_{|f|\le V} [|Pf(x) - \widetilde{P}f(x)| + |Pf(y) - \widetilde{P}f(y)|] \\
\le &\sup_{x,y\in E} \frac{\|P(x,\cdot) - \widetilde{P}(x,\cdot)\|_V + \|P(y,\cdot) - \widetilde{P}(y,\cdot)\|_V}{V(x)+V(y)} \\
\le &\sup_{x\in E} \frac{\|P(x,\cdot) - \widetilde{P}(x,\cdot)\|_V}{V(x)} \\
= &\|P - \widetilde{P}\|_V,
\end{aligned}$$

where in the last inequality, we use the element that

$$\frac{u(x)+u(y)}{v(x)+v(y)} \le \max\Big\{\frac{u(x)}{v(x)}, \frac{u(y)}{v(y)}\Big\}.$$

(iii) Since $R(x,E) = 0$, for all $x \in E$, we have Jordan-Hahn decomposition $R = R^+ - R^-$, hence

$$\begin{aligned}
\|RP\|_V &= \sup_{x\in E} \frac{1}{V(x)}\|(R^+P)(x,\cdot) - (R^-P)(x,\cdot)\|_V \\
&= \sup_{x\in E} \frac{1}{V(x)} W_{d_V}((R^+P)(x,\cdot), (R^-P)(x,\cdot)).
\end{aligned}$$

It follows from (2.2) and Lemma 2.1 that,

$$\begin{aligned}
W_{d_V}((R^+P)(x,\cdot), (R^-P)(x,\cdot)) &= \sup_{\varphi:\|\varphi\|_{\mathrm{Lip}(d_V)}\le 1} \int \varphi(y)((R^+P)(x,\mathrm{d}y) - (R^-P)(x,\mathrm{d}y)) \\
&= \sup_{\varphi:\|\varphi\|_{\mathrm{Lip}(d_V)}\le 1} \int (P\varphi)(y)[R^+(x,\mathrm{d}y) - R^-(x,\mathrm{d}y)] \\
&\le \delta_V(P) \sup_{\phi:\|\phi\|_{\mathrm{Lip}(d_V)}\le 1} \int \phi(y)[R^+(x,\mathrm{d}y) - R^-(x,\mathrm{d}y)] \\
&= \delta_V(P) W_{d_V}(R^+(x,\cdot), R^-(x,\cdot)) \\
&= \delta_V(P)\|R^+(x,\cdot) - R^-(x,\cdot)\|_V.
\end{aligned}$$

Finally we obtain

$$\|RP\|_V \le \delta_V(P) \sup_{x\in E} \frac{1}{V(x)}\|R^+(x,\cdot) - R^-(x,\cdot)\|_V = \delta_V(P)\|R\|_V. \qquad \square$$

**Proof of Theorem 1.1** Set $\Pi(x,\cdot)=\pi$, for all $x\in E$. As noted in [1], the function $V$ in (1.6) can be chosen to be such that $\pi(V)<\infty$. Since $\delta_V(\Pi)=0$, we obtain

$$\delta_V(P^n)=|\delta_V(P^n)-\delta_V(\Pi)|\le\|P^n-\Pi\|_V=\sup_{x\in E}\frac{1}{V(x)}\|P^n(x,\cdot)-\pi\|_V\to 0.$$

Conversely, we have $\|I-\Pi\|_V\le\sup\limits_{x\in E}[V(x)+\pi(V)]/V(x)=1+\pi(V)<\infty$ and $(I-\Pi)(x,E)=0$ for all $x\in E$. Then

$$\|P^n-\Pi\|_V=\|P^n-\Pi P^n\|_V=\|(I-\Pi)P^n\|_V\le\|I-\Pi\|_V\delta_V(P^n),$$

and

$$\|P^n(x,\cdot)-\pi\|_V\le\|P^n-\Pi\|_V V(x)\le\|I-\Pi\|_V V(x)\delta_V(P^n). \qquad \square$$

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# Dobrushin系数的推广


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本文将一般的全变差距离下的Dobrushin系数$\delta$推广到加权的全变差下的$\delta_V$, 并利用$\delta_V$系数得到了离散时间马氏链的几何遍历的判定准则.



**关键词:** $V$范数, $\delta_V$系数, 几何遍历.

**学科分类号:** O211.62.