NA样本下半参数EV模型小波估计的渐近性 *

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摘　要

本文用小波光滑的方法研究了误差为NA序列情形下的半参数EV模型, 得到了参数、非参数和误差方差的估计量分别为$\widehat{\beta}_n$, $\widehat{g}_n(t)$, $\widehat{\sigma}_n^2$, 并证明了它们的强相合性和渐近正态性.

关键词: 半参数EV模型, 小波估计, NA序列, 强相合性, 渐近正态性.

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§1. 引　言

考虑如下半参数EV (Errors-in-Variables)模型

$$\begin{cases} y_i = x_i\beta + g(t_i) + \varepsilon_i, \\ X_i = x_i + u_i, \qquad i = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

其中(t_i, X_i, y_i)为样本观测值, x_i为潜在的不能直接观测的变量, $\beta \in R$为未知参数, 随机误差$\{\varepsilon_i\}$为NA序列(定义见下文), 误差变量$\{u_i\}$独立同分布, 且$\mathsf{E}\varepsilon_i = \mathsf{E}u_i = 0$, $\mathsf{E}\varepsilon_i^2 = \sigma^2 < \infty$, $\mathsf{E}u_i^2 = \sigma_u^2 < \infty$ (其中σ^2和σ_u^2未知, 常常假定比例$\sigma^2/\sigma_u^2 = \lambda$已知, 本文不妨像文献[1]一样假定$\lambda = 1$), $g(\cdot)$和$h(\cdot)$ (其中$h(\cdot)$在后面的估计中与i无关)为$[0, 1]$上的未知函数, $\{t_i\}$和$\{w_i\}$为$[0, 1]$上的两个非随机序列.

最近十几年来, 许多学者都在研究EV模型: 如文献[2]考虑了线性EV模型最小二乘估计的强相合性和渐近正态性; 文献[3]研究了线性EV模型估计量的相合性, 同时得到其估计量的强弱相合性有等价关系; 文献[4]研究了线性EV模型的T-型回归估计和EM算法; 文献[5]系统地讨论了非线性EV模型. 近些年来, 半参数EV模型也受到了广泛的关注(如文献[6–10]等), 此模型在经济、生物及林业方面都有着广泛的应用, 如文献[11]就是研究此模型在经济及心理学方面的应用, 其主要探讨了功能和结构关系; 文献[12]则用此模型研究了温度和电流销量之间的关系. 在研究过程中, 人们常常假定误差是独立(同分布)的, 然而在很多实际应用中误差常常表现出某种相依性, 因此研究相依误差的半参数EV模型具有很

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重要的理论与实际意义. 大家知道, NA序列是一种重要的相依序列, 它在可靠性理论、概率过程、随机过程、多元统计等领域中有着广泛的应用, 而且在大气、地质、海洋生物等领域也有十分重要的应用, 如文献[13]介绍NA概念在风险管理中的重要地位, 作者在文中讨论了几类不同结构随机风险向量的安全性问题, 在边缘分布相同的假定下, 证明了NA结构的风险是凸序意义下最小的, 并以证券市场的具体数据为例, 说明了NA化投资在减小风险中的作用. 因此, 一些学者研究了NA误差下的半参数回归模型(如文献[14–20]等)和非线性回归模型(如文献[21–23]等), 但对于NA误差的半参数EV模型研究甚少. 我们知道小波分析广泛应用于工程和科技领域, 特别是在用电脑处理信号和图像方面. 自十九世纪, 一些学者开始将小波方法应用于统计领域, 如文献[24]和文献[25]用小波的技巧估计了回归函数和密度函数. 基于此, 我们用小波光滑的方法研究模型(1.1), 并得到了估计量的渐近性质.

定义 1.1[26] 称随机变量$X_1,\ldots,X_n\,(n\geq 2)$是NA的, 如果对于任何两个不相交非空子集$A_1$和$A_2$都有$\mathsf{Cov}\,(f_1(X_i,i\in A_1),f_2(X_j,j\in A_2))\leq 0$, 其中$f_1$与$f_2$是任意两个使得协方差存在, 并且对每个变量均非降(或对每个变量均非升)的函数. 称随机变量序列$\{X_j,j\geq 1\}$是NA的, 如果对任意的$n\geq 2$, $X_1,\ldots,X_n$都是NA的.

定义 1.2[27] 称$\{X_i,i\geq 1\}$为ρ混合序列, 如果

$$\rho(n)=\sup_{X\in L^2(F_{-\infty}^0),Y\in L^2(F_n^\infty)}|\mathrm{corr}(X,Y)|\to 0\qquad(n\to\infty),$$

其中F_i^j表示由$\{X_t,i\leq t\leq j\}$生成的σ代数, $L^2(F_i^j)$由具有有限二阶矩的F_i^j可测随机变量所组成.

§2. 估计方法及主要结果

由模型(1.1), 可以得到

$$y_i=X_i\beta+g(t_i)+\varepsilon_i-u_i\beta. \tag{2.1}$$

令$\xi_i=\varepsilon_i-u_i\beta$, 则模型(2.1)变形为

$$y_i=X_i\beta+g(t_i)+\xi_i. \tag{2.2}$$

假定存在定义在$[0,1]$上的某一函数$f(\cdot)$, 有(参考文献[28])

$$x_i=f(t_i)+\eta_i, \tag{2.3}$$

其中$\{\eta_i\}$独立同分布, $\{\eta_i\}$与$\{\varepsilon_i\}$、$\{u_i\}$相互独立, 且$\mathsf{E}\eta_i=0$, $\mathsf{Var}\,(\eta_i)=\sigma_\eta^2<\infty$.

下面采用小波光滑的方法(类似于文献[29])估计β, $g(t)$及σ_i^2. 设有一个给定的刻度函数$\phi(x)\in S_l$ (阶为l的Schwartz空间), 相伴$L^2(R)$的多尺度分析为$\{V_m\}$, 其再生核为

$$E_m(t,s)=2^mE_0(2^mt,2^ms)=2^m\sum_{k\in Z}\phi(2^mt-k)\phi(2^ms-k),$$

其中Z为整数集.

记$A_i=[s_{i-1},s_i]$是$[0,1]$上的分割且$t_i\in A_i$, $1\le i\le n$. 先假定β已知, 由$\mathsf{E}\xi_i=0$有$g(t_i)=\mathsf{E}(y_i-X_i\beta)$, $1\le i\le n$. 因此, 定义$g(\cdot)$的估计为

$$\widehat{g}_0(t,\beta)=\sum_{i=1}^n(y_i-X_i\beta)\int_{A_i}E_m(t,s)\mathrm{d}s. \tag{2.4}$$

由此类似于文献[30–32]中的方法定义β的估计为

$$\widehat{\beta}_n=\arg\min_{\beta}\sum_{i=1}^n[(y_i-X_i\beta-\widehat{g}_0(t,\beta))^2-\sigma_u^2\beta^2]=(\widetilde{X}^T\widetilde{X}-n\sigma_u^2)^{-1}\widetilde{X}^T\widetilde{Y}, \tag{2.5}$$

其中

$$\begin{aligned}
&X=(X_1,\ldots,X_n)^T, && Y=(y_1,\ldots,y_n)^T,\\
&S=(S_{ij})_{n\times n}, && S_{ij}=\int_{A_j}E_m(t_i,s)\mathrm{d}s,\\
&\widetilde{X}=(I-S)X, && \widetilde{Y}=(I-S)Y.
\end{aligned}$$

因此, 定义$g(\cdot)$的估计为

$$\widehat{g}_n(t)=\widehat{g}_0(t,\widehat{\beta}_n)=\sum_{i=1}^n(y_i-X_i\widehat{\beta}_n)\int_{A_i}E_m(t,s)\mathrm{d}s. \tag{2.6}$$

由于$\sigma^2=\mathsf{E}(y_i-x_i\beta-g(t_i))^2$, 所以利用文献[15]中的方法可定义$\sigma^2$的估计为

$$\widehat{\sigma}_n^2=\frac{1}{n}\sum_{i=1}^n(\widetilde{y}_i-\widetilde{X}_i\widehat{\beta}_n)^2-\sigma_u^2\widehat{\beta}_n^2. \tag{2.7}$$

注记 1 当σ_u^2未知时, 我们可以重复测量X来估计σ_u^2, 具体做法参见文献[5].

以下是本文的基本假设:

(A_1) $g(\cdot),f(\cdot)\in H^\alpha$ (阶为α的Sobolev空间), $\alpha>1/2$;

(A_2) $g(\cdot),f(\cdot)\in H^\alpha$满足$\gamma$阶Lipschitz条件, $\gamma>0$;

(A_3) $\phi(\cdot)\in S_l$ (阶为l的Schwartz空间), $l\ge\alpha$; ϕ满足1阶Lipschitz条件且具有紧支撑, 当$\xi\to0$时, $|\widehat{\phi}(\xi)-1|=O(\xi)$, 其中$\widehat{\phi}$为$\phi$的Fourier变换;

(A_4) $s_i\ (i=1,\ldots,n)$和m满足$\max\limits_{1\le i\le n}(s_i-s_{i-1})=O(n^{-1})$, $2^m=O(n^{1/3})$;

(A_5) $\{\xi_i\}$的谱函数$L(w)$满足$0<\delta\le L(w)<\infty$, 其中$w\in(-\pi,\pi]$;

(A$_6$) 存在正整数$p =: p(n)$、$q =: q(n)$和$n' = [n/(p+q)]$, 对于充分大的n有$p+q \le n$, $qp^{-1} \le C < \infty$, 且当$n \to \infty$时, 有

$$q(p+q)^{-1}2^m \to 0, \qquad pn^{-1}2^m \to 0, \qquad n'p\rho^{1/2}(q)(n^{-1}2^m)^{1/2} \to 0.$$

定理 2.1 假定条件(A$_1$)-(A$_4$)都成立, 且设$\sup\limits_{i\ge1}\mathsf{E}|\varepsilon_i|^p < \infty$ $(p>3)$, $\sup\limits_{i\ge1}\mathsf{E}|u_i|^q < \infty$ $(q>2)$, $\mathsf{E}\eta_1^4 < \infty$, $\mathsf{E}u_1^4 < \infty$, 则

$$\widehat{\beta}_n \to \beta, \qquad \text{a.s.},\ n \to \infty.$$

定理 2.2 在定理2.1条件下, 有

$$\widehat{g}_n(t) \to g(t), \qquad \text{a.s.},\ n \to \infty.$$

定理 2.3 在定理2.1条件下, 有

$$\widehat{\sigma}_n^2 \to \sigma^2, \qquad \text{a.s.},\ n \to \infty.$$

定理 2.4 假定条件(A$_1$)-(A$_6$)都成立, 且设$\sup\limits_{i\ge1}\mathsf{E}|\varepsilon_i|^p < \infty$ $(p>3)$, $\sup\limits_{i\ge1}\mathsf{E}|u_i|^q < \infty$ $(q>2)$, $\mathsf{E}\eta_1^4 < \infty$, $\mathsf{E}u_1^4 < \infty$, 则

$$n^{1/2}(\widehat{\beta}_n - \beta) \to^d N(0,\Gamma),$$

其中$\Gamma = \sigma^2(\sigma_\eta^2)^{-1} + \sigma^2\sigma_u^2(\sigma_\eta^2)^{-2} + \sigma_u^2\beta^2(\sigma_\eta^2)^{-1} + \beta^2\mathsf{E}u_1^4(\sigma_\eta^2)^{-2} - \beta^2\sigma_u^4(\sigma_\eta^2)^{-2}$.

定理 2.5 假定条件(A$_1$)-(A$_6$)都成立, 且设$\sup\limits_{i\ge1}\mathsf{E}|\xi_i|^{2+\delta} < \infty\,(\delta>0)$, 令

$$\Gamma_n^2 = \mathsf{Var}\Big(\sum_{i=1}^n \xi_i \int_{A_i} E_m(t,s)\mathrm{d}s\Big),$$

则

$$\frac{\widehat{g}_n(t) - \mathsf{E}(\widehat{g}_n(t))}{\Gamma_n} \to^d N(0,1).$$

定理 2.6 在定理2.4条件下, 有

$$n^{1/2}(\widehat{\sigma}_n^2 - \sigma^2) \to^d N(0,\Lambda),$$

其中$\Lambda = \mathsf{E}[(\varepsilon - u\beta)^2 - (\sigma_u^2\beta^2 + \sigma^2)]^2$.

注记 2 如果$u_i = 0$, 则模型(1.1)变为NA误差下(随机设计)半参数回归模型(见文献[14]), 从而由上面的结果容易得到文献[14]中估计量$\widehat{\beta}_n$, $\widehat{g}_n(t)$和$\widehat{h}_n(u)$的强相合性.

§3. 主要结果的证明

为了证明本文的主要结论, 我们先不加证明地给出一些引理. 下文证明过程中C表示任意正常数, 即使在同一式子中也可能取不同值.

引理 3.1[24] 若条件(A_3)成立, 则有

(I) $|E_0(t,s)|\le C_k/(1+|t-s|)^k$, $|E_m(t,s)|\le 2^m C_k/(1+2^m|t-s|)^k$ (其中$k\in N$, $C_k\in R$);

(II) $\sup\limits_{0\le s\le 1}|E_m(t,s)|=O(2^m)$;

(III) $\sup\limits_t\int_0^1|E_m(t,s)|\mathrm{d}s\le C$, $\int_{A_i}|E_m(t,s)|\mathrm{d}s=O(2^m/n)$.

引理 3.2[33] 若当$1/2<\alpha<3/2$时$\tau_m=2^{-m(\alpha-1/2)}$, 当$\alpha=3/2$时$\tau_m=\sqrt{m}\cdot 2^{-m}$, 当$\alpha>3/2$时$\tau_m=2^{-m}$. 且若条件($A_1$)-($A_4$)都成立, 则

$$\sup_t\Big|f(t)-\sum_{i=1}^n\Big(\int_{A_i}E_m(t,s)\mathrm{d}s\Big)f(t_i)\Big|=O(n^{-\gamma})+O(\tau_m),$$
$$\sup_t\Big|g(t)-\sum_{i=1}^n\Big(\int_{A_i}E_m(t,s)\mathrm{d}s\Big)g(t_i)\Big|=O(n^{-\gamma})+O(\tau_m).$$

引理 3.3[14] 若$\{\varepsilon_i,i\ge1\}$是NA序列, 假设条件(A_3)和(A_4)成立, 且$\sup\limits_{i\ge1}\mathsf{E}|\varepsilon_i|^p<\infty$ ($p>3$), 则

$$\sup_{0\le t\le1}\Big|\sum_{i=1}^n\varepsilon_i\int_{A_i}E_m(t,s)\mathrm{d}s\Big|=O(n^{-1/3}\log n),\qquad \text{a.s..}$$

引理 3.4[34] 若$\{\varepsilon_i,i\ge1\}$是一个强混合序列, 且满足$\mathsf{E}\varepsilon_i=0$和当$p>2$时有$\sup\limits_{i\ge1}\mathsf{E}|\varepsilon_i|^p<\infty$. 又假设$\sum\limits_{n=1}^{\infty}\Big(\sum\limits_{i=1}^{\infty}a_{ni}^2\log n\Big)^{p/2}<\infty$和$\sum\limits_{n=1}^{\infty}\alpha(n)^{(p-2)/p}<\infty$成立, 则

$$\sum_{i=1}^{\infty}a_{ni}\varepsilon_i=o(1),\qquad \text{a.s.,}$$

其中$\alpha(n)$为混合系数, $\{a_{ni},i=1,2,\ldots\}$为实数序列.

引理 3.5[35] 若$\{b_n,n\ge1\}$是一个非降正实数序列, $\{\varepsilon_i,i\ge1\}$是均值为零的NA序列, 有$\sum\limits_{n=1}^{\infty}\sigma_n^2/b_n^2<\infty$, 其中$\sigma_n^2=\mathsf{Var}\,(\varepsilon_n)$. 假设$0<b_n\uparrow\infty$. 则

$$\sum_{i=1}^n\frac{\varepsilon_i}{b_n}=o(1),\qquad \text{a.s..}$$

引理 3.6[36] (Marcinkiewicz强大数定律) 设$\{X_n,n\in N\}$是独立同分布的随机变量, 若$\mathsf{E}|X_1|^p<+\infty$, 则对某个有限常数$a$, 有

$$\frac{1}{n^{1/p}}\sum_{k=1}^n(X_k-a)\to0,\qquad \text{a.s.,}$$

其中, 当$0<p<1$时, a可取任意实数; 当$1\le p<2$时, $a=\mathsf{E}X_1$.

引理 3.7[37] 设$\{X_i, i\in N\}$为ρ混合序列, $\mathsf{E}X_i=0$, $\mathsf{E}|X_i|^q<\infty$, $q\ge 2$, $\rho(1)<1$, 记$S_n=\sum\limits_{i=1}^n X_i$, 则存在仅依赖于$\rho(\cdot)$和$q$的正整数$C$, 使$\forall\, n\ge 1$有

$$\mathsf{E}|S_n|^q\le C\Big\{\sum_{i=1}^n \mathsf{E}|X_i|^q+\Big(\sum_{i=1}^n \mathsf{E}|X_i|^2\Big)^{q/2}\Big\}.$$

引理 3.8[38] 设$\{X_i, i\in N\}$为ρ混合序列, p, q为两个正整数, 记$\eta_l=\sum\limits_{j=(l-1)(p+q)+1}^{(l-1)(p+q)+p} X_j$ $(1\le l\le k)$, 则有

$$\Big|\mathsf{E}\exp\Big(it\sum_{l=1}^k\eta_l\Big)-\prod_{l=1}^k \mathsf{E}\exp(it\eta_l)\Big|\le C|t|\rho^{1/2}(q)\sum_{l=1}^k\|\eta_l\|_2.$$

定理2.1的证明: 令$\widetilde{\xi}=(I-S)\xi$, $\widetilde{g}=(I-S)g$, 则

$$\widehat{\beta}_n-\beta=(n^{-1}\widetilde{X}^T\widetilde{X}-\sigma_u^2)^{-1}(n^{-1}\widetilde{X}^T\widetilde{g}+n^{-1}\widetilde{X}^T\widetilde{\xi}+\sigma_u^2\beta). \tag{3.1}$$

首先, 证明

$$n^{-1}\widetilde{X}^T\widetilde{X}-\sigma_u^2\to\sigma_\eta^2, \qquad \text{a.s.},\ n\to\infty. \tag{3.2}$$

事实上,

$$\begin{aligned}
n^{-1}\widetilde{X}^T\widetilde{X} &= \frac{1}{n}\sum_{i=1}^n\Big(f(t_i)+\eta_i+u_i-\sum_{j=1}^n S_{ij}(f(t_j)+\eta_j+u_j)\Big)^2\\
&= \frac{1}{n}\sum_{i=1}^n\Big(f(t_i)-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}sf(t_j)\Big)^2\\
&\quad +2\cdot\frac{1}{n}\sum_{i=1}^n\Big(f(t_i)-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}sf(t_j)\Big)\Big(\eta_i-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\\
&\quad +2\cdot\frac{1}{n}\sum_{i=1}^n\Big(f(t_i)-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}sf(t_j)\Big)\Big(u_i-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}su_j\Big)\\
&\quad +2\cdot\frac{1}{n}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\Big(u_i-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}su_j\Big)\\
&\quad +\frac{1}{n}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)^2+\frac{1}{n}\sum_{i=1}^n\Big(u_i-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}su_j\Big)^2\\
&=: U_1+2U_2+2U_3+2U_4+U_5+U_6.
\end{aligned}\tag{3.3}$$

由引理3.2得

$$|U_1|\le\max_{1\le i\le n}|f(t_i)-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}sf(t_j)|^2\to 0, \qquad \text{a.s.}. \tag{3.4}$$

由文献[29]引理4(i)的(2.4)式易知

$$\sup_t \Big|\sum_{i=1}^n \eta_i \int_{A_i} E_m(t,s)\mathrm{d}s\Big| = o(1), \qquad \text{a.s.},\ n\to\infty. \tag{3.5}$$

$$\sup_t \Big|\sum_{i=1}^n u_i \int_{A_i} E_m(t,s)\mathrm{d}s\Big| = o(1), \qquad \text{a.s.},\ n\to\infty. \tag{3.6}$$

由强大数定律、(3.5)式及引理3.2有

$$\begin{aligned}|U_2| &\le \Big(\max_{1\le i\le n}\Big|f(t_i)-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}sf(t_j)\Big|\Big)\\ &\quad\cdot\Big(\frac{1}{n}\sum_{i=1}^n|\eta_i|+\max_{1\le i\le n}\Big|\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big|\Big)\\ &= o(1)(\mathsf{E}|\eta_i|+o(1))\to 0, \qquad \text{a.s.}.\end{aligned} \tag{3.7}$$

类似于(3.7)式的证明, 易得

$$U_3 = o(1)(\mathsf{E}|u_i|+o(1))\to 0, \qquad \text{a.s.}. \tag{3.8}$$

由强大数定律及(3.5)式、(3.6)式得

$$\begin{aligned}|U_4| &= \frac{1}{n}\sum_{i=1}^n(\eta_i-o(1))(u_i-o(1))\\ &= \frac{1}{n}\sum_{i=1}^n(\eta_i u_i-u_io(1)-\eta_io(1)+o(1))\to 0, \qquad \text{a.s.}.\end{aligned} \tag{3.9}$$

由(3.5)式及$\mathsf{E}\eta_i=0$, $\mathsf{Var}\,(\eta_i)=\sigma_\eta^2$, 有

$$\begin{aligned}U_5 &= \frac{1}{n}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)^2\\ &= \frac{1}{n}\sum_{i=1}^n\Big(\eta_i^2-2\eta_i\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j+\Big(\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)^2\Big)\\ &\to \sigma_\eta^2, \qquad \text{a.s.}.\end{aligned} \tag{3.10}$$

同理有

$$U_6\to\sigma_u^2, \qquad \text{a.s.},\ n\to\infty. \tag{3.11}$$

由(3.3)、(3.4)、(3.7)、(3.8)、(3.9)、(3.10)、(3.11)式即得

$$n^{-1}\widetilde{X}^T\widetilde{X}\to\sigma_\eta^2+\sigma_u^2, \qquad \text{a.s.},\ n\to\infty. \tag{3.12}$$

由(3.12)式即得(3.2)式.

其次, 可以证明

$$n^{-1}\widetilde{X}^T\widetilde{g}\to 0, \qquad \text{a.s.},\ n\to\infty. \tag{3.13}$$

事实上,

$$\begin{aligned}
n^{-1}\widetilde{X}^T\widetilde{g} &= \frac{1}{n}\sum_{i=1}^n\Big(X_i-\sum_{j=1}^n S_{ij}X_j\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&= \frac{1}{n}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n S_{ij}\eta_j\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&\quad+\frac{1}{n}\sum_{i=1}^n\Big(f(t_i)-\sum_{j=1}^n S_{ij}f(t_j)\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&\quad+\frac{1}{n}\sum_{i=1}^n\Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&=: J_1+J_2+J_3. \qquad (3.14)
\end{aligned}$$

类似于(3.7)式的证明, 易得

$$J_1\to 0,\quad \text{a.s.},\qquad J_3\to 0,\quad \text{a.s.}. \qquad (3.15)$$

由引理3.2得

$$\begin{aligned}
|J_2| &\le \max_{1\le i\le n}\Big|\Big(f(t_i)-\sum_{j=1}^n S_{ij}f(t_j)\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\Big|\\
&= O(n^{-2\gamma})+O(\tau_m^2)\to 0,\qquad \text{a.s.}. \qquad (3.16)
\end{aligned}$$

故由(3.14)、(3.15)、(3.16)式即得(3.13)式.

最后, 可以证明

$$n^{-1}\widetilde{X}^T\widetilde{\xi}\to -\sigma_u^2\beta,\qquad \text{a.s.},\ n\to\infty. \qquad (3.17)$$

事实上,

$$\begin{aligned}
n^{-1}\widetilde{X}^T\widetilde{\xi} &= \frac{1}{n}\sum_{i=1}^n\Big(X_i-\sum_{j=1}^n S_{ij}X_j\Big)\Big(\xi_i-\sum_{k=1}^n S_{ik}\xi_k\Big)\\
&= \frac{1}{n}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n S_{ij}\eta_j\Big)\Big(\xi_i-\sum_{k=1}^n S_{ik}\xi_k\Big)\\
&\quad+\frac{1}{n}\sum_{i=1}^n\Big(f(t_i)-\sum_{j=1}^n S_{ij}f(t_j)\Big)\Big(\xi_i-\sum_{k=1}^n S_{ik}\xi_k\Big)\\
&\quad+\frac{1}{n}\sum_{i=1}^n\Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)\Big(\xi_i-\sum_{k=1}^n S_{ik}\xi_k\Big)\\
&=: T_1+T_2+T_3. \qquad (3.18)
\end{aligned}$$

而

$$\begin{aligned}
T_1 &= \frac{1}{n}\sum_{i=1}^n\Big(\eta_i\xi_i-\Big(\sum_{j=1}^n S_{ij}\eta_j\Big)\xi_i-\Big(\sum_{k=1}^n S_{ik}\xi_k\Big)\eta_i\Big)+\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}\eta_j\Big)\Big(\sum_{k=1}^n S_{ik}\xi_k\Big)\\
&=: T_1^{(1)}-T_1^{(2)}-T_1^{(3)}+T_1^{(4)}. \qquad (3.19)
\end{aligned}$$

因为$\{\eta_i\}$与$\{\xi_i\}$相互独立, 且$\mathsf{E}\eta_i=0$, $\mathsf{E}\xi_i=0$, 所以$\mathsf{Cov}\,(\eta_i\xi_i,\eta_j\xi_j)=0$, 从而$\{\eta_i\xi_i,i=1,2,\ldots,n\}$是$\rho$混合随机序列. 由文献[39]知, $0\le\alpha(n)\le\rho(n)/4=0$, 即$\rho$混合随机序列是$\alpha$混合序列.

令$a_{ni}=n^{-2/3}\log^{-1}n$, $\alpha(n)=0$, 由引理3.4得

$$T_1^{(1)}\to 0,\qquad \text{a.s..}\tag{3.20}$$

因为$\{\varepsilon_i,i=1,2,\ldots,n\}$是NA序列, 所以$\{\varepsilon_i^+,i=1,2,\ldots,n\}$和$\{\varepsilon_i^-,i=1,2,\ldots,n\}$均为NA序列. 由引理3.5知,

$$\frac{1}{n}\sum_{i=1}^n\varepsilon_i^+=o(1),\quad \text{a.s.},\qquad \frac{1}{n}\sum_{i=1}^n\varepsilon_i^-=o(1),\quad \text{a.s..}$$

又有$|\varepsilon_i|=\varepsilon_i^++\varepsilon_i^-$, 所以

$$\frac{1}{n}\sum_{i=1}^n|\varepsilon_i|=o(1),\qquad \text{a.s.},\ n\to\infty.\tag{3.21}$$

因为

$$\begin{aligned}T_1^{(2)}&=\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)(\varepsilon_i-u_i\beta)\\&=\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\varepsilon_i-\beta\cdot\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)u_i\\&=:T_1^{(21)}-\beta\cdot T_1^{(22)},\end{aligned}\tag{3.22}$$

由(3.5)式及(3.21)式得

$$|T_1^{(21)}|\le\max_{1\le i\le n}\Big|\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big|\cdot\Big(\frac{1}{n}\sum_{i=1}^n|\varepsilon_i|\Big)\to 0,\qquad \text{a.s..}\tag{3.23}$$

由强大数定律及(3.5)式有

$$|T_1^{(22)}|\le\max_{1\le i\le n}\Big|\sum_{j=1}^n\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big|\cdot\Big(\frac{1}{n}\sum_{i=1}^n|u_i|\Big)=o(1)\mathsf{E}|u_i|\to 0,\qquad \text{a.s..}\tag{3.24}$$

由(3.22)、(3.23)、(3.24)式知

$$T_1^{(2)}\to 0,\qquad \text{a.s..}\tag{3.25}$$

因为

$$\begin{aligned}T_1^{(3)}&=\frac{1}{n}\sum_{i=1}^n\Big(\sum_{k=1}^n\int_{A_k}E_m(t_i,s)\mathrm{d}s\xi_k\Big)\eta_i\\&=\frac{1}{n}\sum_{i=1}^n\Big(\sum_{k=1}^n\int_{A_k}E_m(t_i,s)\mathrm{d}s\varepsilon_k\Big)\eta_i-\beta\cdot\frac{1}{n}\sum_{i=1}^n\Big(\sum_{k=1}^n\int_{A_k}E_m(t_i,s)\mathrm{d}su_k\Big)\eta_i\\&=:T_1^{(31)}-\beta\cdot T_1^{(32)},\end{aligned}\tag{3.26}$$

由强大数定律及引理3.3有

$$|T_1^{(31)}| \leq \max_{1\leq i\leq n}\Big|\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}s\varepsilon_k\Big|\cdot\Big(\frac{1}{n}\sum_{i=1}^{n}|\eta_i|\Big)=o(1)\mathsf{E}|\eta_i|\to 0,\qquad \text{a.s.}. \tag{3.27}$$

由强大数定律及(3.6)式有

$$|T_1^{(32)}| \leq \max_{1\leq i\leq n}\Big|\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}su_k\Big|\cdot\Big(\frac{1}{n}\sum_{i=1}^{n}|\eta_i|\Big)=o(1)\mathsf{E}|\eta_i|\to 0,\qquad \text{a.s.}. \tag{3.28}$$

由(3.26)、(3.27)、(3.28)式得

$$T_1^{(3)}\to 0,\qquad \text{a.s.}. \tag{3.29}$$

因为

$$\begin{aligned}
T_1^{(4)} &= \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\Big(\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}s\xi_k\Big)\\
&= \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\Big(\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}s\varepsilon_k\Big)\\
&\quad -\beta\cdot\frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\Big(\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}su_k\Big)\\
&=: T_1^{(41)}-\beta\cdot T_1^{(42)},
\end{aligned} \tag{3.30}$$

由引理3.3及(3.5)式有

$$|T_1^{(41)}| \leq \max_{1\leq i\leq n}\Big|\Big(\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}s\varepsilon_k\Big)\cdot\Big(\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\Big|\to 0,\qquad \text{a.s.}. \tag{3.31}$$

由(3.5)式及(3.6)式有

$$|T_1^{(42)}| \leq \max_{1\leq i\leq n}\Big|\Big(\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}su_k\Big)\cdot\Big(\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}s\eta_j\Big)\Big|\to 0,\qquad \text{a.s.}. \tag{3.32}$$

由(3.30)、(3.31)、(3.32)式得

$$T_1^{(4)}\to 0,\qquad \text{a.s.}. \tag{3.33}$$

从而由(3.19)、(3.20)、(3.25)、(3.29)、(3.33)式得

$$T_1\to 0,\qquad \text{a.s.},\ n\to\infty. \tag{3.34}$$

因为

$$\begin{aligned}
T_2 &= \frac{1}{n}\sum_{i=1}^{n}\Big(f(t_i)-\sum_{j=1}^{n}S_{ij}f(t_j)\Big)\Big(\varepsilon_i-\sum_{k=1}^{n}S_{ik}\varepsilon_k\Big)\\
&\quad -\beta\cdot\frac{1}{n}\sum_{i=1}^{n}\Big(f(t_i)-\sum_{j=1}^{n}S_{ij}f(t_j)\Big)\Big(u_i-\sum_{k=1}^{n}S_{ik}u_k\Big)\\
&=: T_2^{(1)}-\beta\cdot T_2^{(2)},
\end{aligned} \tag{3.35}$$

由引理3.2、引理3.3及(3.21)式有

$$
\begin{aligned}
|T_2^{(1)}| &\leq \max_{1\leq i\leq n}\Big|f(t_i)-\sum_{j=1}^{n}S_{ij}f(t_j)\Big|\cdot\Big(\frac{1}{n}\sum_{i=1}^{n}|\varepsilon_i|+\max_{1\leq k\leq n}\Big|\sum_{k=1}^{n}S_{ik}\varepsilon_k\Big|\Big)\\
&= (O(n^{-\gamma})+O(\tau_m))\cdot(o(1)+O(n^{-1/3}\log n))\to 0,\qquad \text{a.s..} \qquad (3.36)
\end{aligned}
$$

类似于(3.7)式的证明, 易得

$$
T_2^{(2)} = o(1)(\mathsf{E}|u_i|+o(1))\to 0,\qquad \text{a.s..} \qquad (3.37)
$$

故由(3.35)、(3.36)、(3.37)式得

$$
T_2\to 0,\qquad \text{a.s.},\ n\to\infty. \qquad (3.38)
$$

因为

$$
\begin{aligned}
T_3 &= \frac{1}{n}\sum_{i=1}^{n}\Big(u_i-\sum_{j=1}^{n}S_{ij}u_j\Big)\Big((\varepsilon_i-u_i\beta)-\sum_{k=1}^{n}S_{ik}(\varepsilon_k-u_k\beta)\Big)\\
&= \frac{1}{n}\sum_{i=1}^{n}\Big(u_i-\sum_{j=1}^{n}S_{ij}u_j\Big)\Big(\varepsilon_i-\sum_{k=1}^{n}S_{ik}\varepsilon_k\Big)\\
&\quad -\beta\cdot\frac{1}{n}\sum_{i=1}^{n}\Big(u_i-\sum_{j=1}^{n}S_{ij}u_j\Big)\Big(u_i-\sum_{k=1}^{n}S_{ik}u_k\Big)\\
&=: T_3^{(1)}-\beta\cdot T_3^{(2)}, \qquad (3.39)
\end{aligned}
$$

由强大数定律、引理3.3、(3.6)式及(3.21)式有

$$
\begin{aligned}
|T_3^{(1)}| &\leq \Big(\frac{1}{n}\sum_{i=1}^{n}|u_i|+\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)\mathrm{d}su_j\Big|\Big)\\
&\quad\cdot\Big(\frac{1}{n}\sum_{i=1}^{n}|\varepsilon_i|+\max_{1\leq k\leq n}\Big|\sum_{k=1}^{n}\int_{A_k}E_m(t_i,s)\mathrm{d}s\varepsilon_k\Big|\Big)\\
&= (\mathsf{E}|u_i|+o(1))o(1)\to 0,\qquad \text{a.s..} \qquad (3.40)
\end{aligned}
$$

由(3.6)式及$\mathsf{E}u_i=0$, $\mathsf{E}u_i^2=\sigma_u^2$, 有

$$
\begin{aligned}
|T_3^{(2)}| &= \Big|\frac{1}{n}\sum_{i=1}^{n}\Big(u_i-\sum_{j=1}^{n}S_{ij}u_j\Big)\Big(u_i-\sum_{k=1}^{n}S_{ik}u_k\Big)\Big|\\
&= \Big|\frac{1}{n}\sum_{i=1}^{n}\Big(u_i^2-\sum_{j=1}^{n}S_{ij}u_ju_i-\sum_{k=1}^{n}S_{ik}u_ku_i+\sum_{j=1}^{n}S_{ij}u_j\sum_{k=1}^{n}S_{ik}u_k\Big)\Big|\\
&\leq \Big|\frac{1}{n}\sum_{i=1}^{n}u_i^2\Big|+\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}S_{ij}u_ju_i\Big|\\
&\quad+\max_{1\leq i\leq n}\Big|\sum_{k=1}^{n}S_{ik}u_ku_i\Big|+\max_{1\leq i\leq n}\Big|\sum_{j=1}^{n}S_{ij}u_j\sum_{k=1}^{n}S_{ik}u_k\Big|\\
&\to \sigma_u^2,\qquad \text{a.s..} \qquad (3.41)
\end{aligned}
$$

从而由(3.39)、(3.40)、(3.41)式得

$$T_3 \to -\sigma_u^2\beta, \qquad \text{a.s.},\ n\to\infty. \tag{3.42}$$

从而由(3.18)、(3.34)、(3.38)、(3.42)式即得(3.17)式.

至此, 综合(3.1)、(3.2)、(3.13)、(3.17)式定理得证. □

定理2.2的证明:

$$\begin{aligned}
\sup_t|\widehat{g}_n(t)-g(t)| &= \sup_t\Big|\widehat{g}_0(t,\beta)-g(t)+\sum_{j=1}^n X_j(\beta-\widehat{\beta}_n)\int_{A_j}E_m(t,s)\mathrm{d}s\Big| \\
&\le \sup_t\Big|\sum_{j=1}^n g(t_j)\int_{A_j}E_m(t,s)\mathrm{d}s-g(t)\Big| \\
&\quad+\sup_t\Big|\sum_{j=1}^n \xi_j\int_{A_j}E_m(t,s)\mathrm{d}s\Big| \\
&\quad+|\widehat{\beta}_n-\beta|\sup_t\Big|\sum_{j=1}^n f(t_j)\int_{A_j}E_m(t,s)\mathrm{d}s\Big| \\
&\quad+|\widehat{\beta}_n-\beta|\sup_t\Big|\sum_{j=1}^n \eta_j\int_{A_j}E_m(t,s)\mathrm{d}s\Big| \\
&\quad+|\widehat{\beta}_n-\beta|\sup_t\Big|\sum_{j=1}^n u_j\int_{A_j}E_m(t,s)\mathrm{d}s\Big| \\
&=: K_1+K_2+K_3+K_4+K_5.
\end{aligned} \tag{3.43}$$

由引理3.2得

$$K_1=O(n^{-\gamma})+O(\tau_m)\to 0, \qquad \text{a.s.},\ n\to\infty. \tag{3.44}$$

因为

$$\begin{aligned}
K_2 &= \sup_t\Big|\sum_{j=1}^n \xi_j\int_{A_j}E_m(t,s)\mathrm{d}s\Big| \\
&\le \sup_t\Big|\sum_{j=1}^n \varepsilon_j\int_{A_j}E_m(t,s)\mathrm{d}s\Big|+\beta\sup_t\Big|\sum_{j=1}^n u_j\int_{A_j}E_m(t,s)\mathrm{d}s\Big| \\
&=: K_2^{(1)}+\beta\cdot K_2^{(2)},
\end{aligned} \tag{3.45}$$

所以由引理3.3和(3.6)式分别得

$$K_2^{(1)}\to 0, \qquad \text{a.s.}, \tag{3.46}$$

$$K_2^{(2)}\to 0, \qquad \text{a.s.}. \tag{3.47}$$

由(3.45)、(3.46)、(3.47)式得

$$K_2\to 0, \qquad \text{a.s.},\ n\to\infty. \tag{3.48}$$

由定理2.1及引理3.1有

$$\begin{aligned} K_3 &\leq |\widehat{\beta}_n-\beta|\cdot\sup_{t_j}|f(t_j)|\cdot\sup_t\int_0^1|E_m(t,s)|\mathrm{d}s \\ &\leq C|\widehat{\beta}_n-\beta|\to 0,\qquad \text{a.s., } n\to\infty. \end{aligned}\tag{3.49}$$

由定理2.1及(3.5)式有

$$K_4\to 0,\qquad \text{a.s., } n\to\infty.\tag{3.50}$$

由定理2.1及(3.6)式有

$$K_5\to 0,\qquad \text{a.s., } n\to\infty.\tag{3.51}$$

至此, 综合(3.43)、(3.44)、(3.48)、(3.49)、(3.50)、(3.51)式定理得证. □

定理2.3的证明: 　由(2.7)式及$\widetilde{y}_i=\widetilde{X}_i\beta+\widetilde{g}(t_i)+\widetilde{\xi}_i$有

$$\begin{aligned} \widehat{\sigma}_n^2 &= \frac{1}{n}\sum_{i=1}^n(\widetilde{X}_i(\beta-\widehat{\beta}_n))^2+\frac{1}{n}\sum_{i=1}^n\widetilde{g}^2(t_i)+\frac{1}{n}\sum_{i=1}^n\widetilde{\xi}_i^2 \\ &\quad +2\frac{1}{n}\sum_{i=1}^n\widetilde{X}_i(\beta-\widehat{\beta}_n)\widetilde{g}(t_i)+2\frac{1}{n}\sum_{i=1}^n\widetilde{X}_i(\beta-\widehat{\beta}_n)\widetilde{\xi}_i \\ &\quad +2\frac{1}{n}\sum_{i=1}^n\widetilde{g}(t_i)\widetilde{\xi}_i-\sigma_u^2\widehat{\beta}_n^2 \\ &=: I_1+I_2+I_3+2I_4+2I_5+2I_6-\sigma_u^2\widehat{\beta}_n^2. \end{aligned}\tag{3.52}$$

由强大数定律和定理2.1有

$$\begin{aligned} |I_1| &\leq |\beta-\widehat{\beta}_n|^2\cdot\Big|\frac{1}{n}\sum_{i=1}^n(\widetilde{f}^2(t_i)+\widetilde{\eta}_i^2+\widetilde{u}_i^2)\Big| \\ &\quad +|\beta-\widehat{\beta}_n|^2\cdot\Big|2\frac{1}{n}\sum_{i=1}^n\widetilde{f}(t_i)\widetilde{\eta}_i+2\frac{1}{n}\sum_{i=1}^n\widetilde{f}(t_i)\widetilde{u}_i+2\frac{1}{n}\sum_{i=1}^n\widetilde{\eta}_i\widetilde{u}_i\Big| \\ &\leq o^2(1)\cdot\Big(\max_{1\leq i\leq n}\widetilde{f}^2(t_i)+\sigma_\eta^2+\sigma_u^2\Big)+o^2(1)\cdot o(1)\to 0,\qquad \text{a.s..} \end{aligned}\tag{3.53}$$

由引理3.1及引理3.2有

$$\sup_t|I_2|=(O(n^{-2\gamma})+O(\tau_m^2))\cdot o(1)\to 0,\qquad \text{a.s..}\tag{3.54}$$

下面将证明

$$I_3-\sigma^2\to\sigma_u^2\beta^2,\qquad \text{a.s., } n\to\infty.\tag{3.55}$$

由于

$$\begin{aligned} |I_3-\sigma^2| &= \Big|\frac{1}{n}\sum_{i=1}^n \widetilde{\xi}_i^2-\sigma^2\Big| \\ &\le \Big|\frac{1}{n}\sum_{i=1}^n \Big[\Big(\varepsilon_i-\sum_{j=1}^n S_{ij}\varepsilon_j\Big)^2-\sigma^2\Big]\Big| \\ &\quad +2\Big|\frac{1}{n}\sum_{i=1}^n \Big(\varepsilon_i-\sum_{j=1}^n S_{ij}\varepsilon_j\Big)\Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)\beta\Big| \\ &\quad +\Big|\frac{1}{n}\sum_{i=1}^n \Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)^2\beta^2\Big| \\ &=: I_3^{(1)}+2I_3^{(2)}+I_3^{(3)}. \end{aligned} \tag{3.56}$$

因为

$$\begin{aligned} I_3^{(1)} &\le \Big|\frac{1}{n}\sum_{i=1}^n(\varepsilon_i^2-\sigma^2)\Big|+2\Big|\frac{1}{n}\sum_{i=1}^n\varepsilon_i\Big(\sum_{j=1}^n S_{ij}\varepsilon_j\Big)\Big|+\Big|\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}\varepsilon_j\Big)^2\Big| \\ &=: I_3^{(11)}+2I_3^{(12)}+I_3^{(13)}. \end{aligned} \tag{3.57}$$

令$\varsigma_i=\varepsilon_i^2-\sigma^2=\varepsilon_i^2-\mathsf{E}\varepsilon_i^2=(\varepsilon_i^+)^2-(\mathsf{E}\varepsilon_i^+)^2-((\varepsilon_i^-)^2-(\mathsf{E}\varepsilon_i^-)^2)=\varsigma_i^+-\varsigma_i^-$, 则$\{\varsigma_i^+,i\ge1\}$和$\{\varsigma_i^-,i\ge1\}$都是NA序列, 且$\mathsf{E}\varsigma_i^\pm=0$, $\mathsf{Var}\,(\varsigma_i^\pm)<\infty$. 则由(3.21)有

$$I_3^{(11)}\to0,\qquad \text{a.s.}. \tag{3.58}$$

由引理3.3及(3.21)式有

$$I_3^{(12)}\le\max_{1\le i\le n}\Big|\frac{1}{n}\sum_{j=1}^n S_{ij}\varepsilon_j\Big|\cdot\Big(\frac{1}{n}\sum_{i=1}^n|\varepsilon_i|\Big)=o(1)\cdot o(1)\to0,\qquad \text{a.s.}. \tag{3.59}$$

由引理3.3有

$$I_3^{(13)}\le\max_{1\le i\le n}\Big|\sum_{j=1}^n S_{ij}\varepsilon_j\Big|^2=o^2(1)\to0,\qquad \text{a.s.}. \tag{3.60}$$

由(3.57)、(3.58)、(3.59)、(3.60)式得

$$I_3^{(1)}\to0,\qquad \text{a.s.}. \tag{3.61}$$

由于

$$\begin{aligned} I_3^{(2)} &\le \beta\Big|\frac{1}{n}\sum_{i=1}^n\Big(\varepsilon_iu_i-\Big(\sum_{j=1}^n S_{ij}\varepsilon_j\Big)u_i-\Big(\sum_{j=1}^n S_{ij}u_j\Big)\varepsilon_i\Big| \\ &\quad +\beta\Big|\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}\varepsilon_j\Big)\Big(\sum_{j=1}^n S_{ij}u_j\Big)\Big| \\ &\le \beta\Big|\frac{1}{n}\sum_{i=1}^n\varepsilon_iu_i\Big|+\beta\Big|\frac{1}{n}\sum_{i=1}^n u_i\Big(\sum_{j=1}^n S_{ij}\varepsilon_j\Big)\Big|+\beta\Big|\frac{1}{n}\sum_{i=1}^n\varepsilon_i\Big(\sum_{j=1}^n S_{ij}u_j\Big)\Big| \\ &\quad +\beta\Big|\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}\varepsilon_j\Big)\Big(\sum_{j=1}^n S_{ij}u_j\Big)\Big| \\ &=: \beta I_3^{(21)}+\beta I_3^{(22)}+\beta I_3^{(23)}+\beta I_3^{(24)}. \end{aligned} \tag{3.62}$$

类似于(3.20)式的证明, 有

$$I_3^{(21)} \to 0, \qquad \text{a.s..} \tag{3.63}$$

类似于(3.27)式的证明, 有

$$I_3^{(22)} = o(1) \cdot \mathsf{E}|u_i| \to 0, \qquad \text{a.s..} \tag{3.64}$$

类似于(3.23)式的证明, 有

$$I_3^{(23)} \to 0, \qquad \text{a.s..} \tag{3.65}$$

类似于(3.31)式的证明, 有

$$I_3^{(24)} \to 0, \qquad \text{a.s..} \tag{3.66}$$

由(3.62)、(3.63)、(3.64)、(3.65)、(3.66)式得

$$I_3^{(2)} \to 0, \qquad \text{a.s..} \tag{3.67}$$

由强大数定律及(3.6)式有

$$\begin{aligned} I_3^{(3)} &= \Big|\frac{1}{n}\sum_{i=1}^n u_i^2 - 2\frac{1}{n}\sum_{i=1}^n u_i\Big(\sum_{j=1}^n S_{ij}u_j\Big) + \frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}u_j\Big)^2\Big| \cdot \beta^2 \\ &\le \Big|\frac{1}{n}\sum_{i=1}^n u_i^2\beta^2\Big| + 2\Big|\frac{1}{n}\sum_{i=1}^n u_i\Big(\sum_{j=1}^n S_{ij}u_j\Big)\beta^2\Big| + \Big|\frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}u_j\Big)^2\beta^2\Big| \\ &= \Big|\frac{1}{n}\sum_{i=1}^n u_i^2\beta^2\Big| + 2\cdot o(1)\cdot \mathsf{E}|u_i|\beta^2 + o^2(1)\beta^2 \to \sigma_u^2\beta^2, \qquad \text{a.s..} \end{aligned} \tag{3.68}$$

由(3.56)、(3.61)、(3.67)及(3.68)式得(3.55)式成立.

由柯西不等式、(3.53)式及(3.54)式有

$$I_4^2 \le \frac{1}{n}\sum_{i=1}^n(\widetilde{X}_i(\beta - \widehat{\beta}_n))^2 \cdot \frac{1}{n}\sum_{i=1}^n \widetilde{g}^2(t_i) = I_1 \cdot I_2 \to 0, \qquad \text{a.s..} \tag{3.69}$$

由柯西不等式、(3.53)式及(3.55)式有

$$I_5^2 \le \frac{1}{n}\sum_{i=1}^n(\widetilde{X}_i(\beta - \widehat{\beta}_n))^2 \cdot \frac{1}{n}\sum_{i=1}^n \widetilde{\xi}_i^2 = I_1 \cdot I_3 \to 0, \qquad \text{a.s..} \tag{3.70}$$

由柯西不等式、(3.54)式及(3.55)式有

$$I_6^2 \le \frac{1}{n}\sum_{i=1}^n \widetilde{g}^2(t_i) \cdot \frac{1}{n}\sum_{i=1}^n \widetilde{\xi}_i^2 = I_2 \cdot I_3 \to 0, \qquad \text{a.s..} \tag{3.71}$$

至此, 综合(3.52)、(3.53)、(3.54)、(3.55)、(3.69)、(3.70)及(3.71)式定理得证. □

定理2.4的证明: 令$\Delta_n=n^{-1}\widetilde{X}^T\widetilde{X}-\sigma_u^2$, 有

$$
\begin{aligned}
n^{1/2}(\widehat{\beta}_n-\beta) &= n^{-1/2}\Delta_n^{-1}(\widetilde{X}^T\widetilde{g}+\widetilde{X}^T\widetilde{\xi}+n\sigma_u^2\beta)\\
&= n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n\Big(x_i-\sum_{j=1}^n S_{ij}x_j\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&\quad+n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n\Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&\quad+n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n\Big(x_i-\sum_{j=1}^n S_{ij}x_j\Big)\Big(\varepsilon_i-\sum_{k=1}^n S_{ik}\varepsilon_k\Big)\\
&\quad+n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n\Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)\Big(\varepsilon_i-\sum_{k=1}^n S_{ik}\varepsilon_k\Big)\\
&\quad-n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n\Big(x_i-\sum_{j=1}^n S_{ij}x_j\Big)\Big(u_i-\sum_{k=1}^n S_{ik}u_k\Big)\beta\\
&\quad-n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n\Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)\Big(u_i-\sum_{k=1}^n S_{ik}u_k\Big)\beta\\
&\quad+n^{1/2}\Delta_n^{-1}\sigma_u^2\beta\\
&=: Q_1+Q_2+Q_3+Q_4-Q_5-Q_6+Q_7. \qquad (3.72)
\end{aligned}
$$

事实上,

$$
\begin{aligned}
Q_1 &= \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(f(t_i)-\sum_{j=1}^n S_{ij}f(t_j)\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&\quad+\Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n S_{ij}\eta_j\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&=: \Delta_n^{-1}\cdot Q_{11}+\Delta_n^{-1}\cdot Q_{12}. \qquad (3.73)
\end{aligned}
$$

由引理3.2得

$$
\begin{aligned}
Q_{11} &\le \sqrt{n}\sup_{1\le i\le n}\Big|\Big(f(t_i)-\sum_{j=1}^n S_{ij}f(t_j)\Big)\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\Big|\\
&= \sqrt{n}\cdot[O(n^{-2\gamma})+O(\tau_m^2)]\to 0. \qquad (3.74)
\end{aligned}
$$

因为

$$
\begin{aligned}
Q_{12} &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\eta_i\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\\
&\quad-\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)\Big)\eta_j\\
&=: Q_{121}-Q_{122}, \qquad (3.75)
\end{aligned}
$$

则有

$$
\mathsf{Var}\,(Q_{121})=\frac{\sigma_\eta^2}{n}\sum_{i=1}^n\Big(g(t_i)-\sum_{k=1}^n S_{ik}g(t_k)\Big)^2, \qquad (3.76)
$$

由引理3.2有

$$|\mathsf{Var}\,(Q_{121})| \le \sup_{1\le i\le n} \Big|\Big(g(t_i) - \sum_{k=1}^{n} S_{ik} g(t_k)\Big)^2\Big| \to 0, \tag{3.77}$$

再由Chebychev不等式及(3.77)式, 有

$$\mathsf{P}(|Q_{121}| > \varepsilon) \le \frac{\mathsf{Var}\,(Q_{121})}{\varepsilon^2} \to 0, \tag{3.78}$$

即

$$Q_{121} = o_p(1). \tag{3.79}$$

由引理3.1、引理3.2有

$$\begin{aligned}
\mathsf{Var}\,(Q_{122}) &= \frac{1}{n}\mathsf{Var}\,\Big(\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} S_{ij}\Big(g(t_i) - \sum_{k=1}^{n} S_{ik} g(t_k)\Big)\Big)\eta_j\Big) \\
&= \frac{\sigma_\eta^2}{n}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n} S_{ij}\Big(g(t_i) - \sum_{k=1}^{n} S_{ik} g(t_k)\Big)\Big)^2 \\
&\le \frac{\sigma_\eta^2}{n}\cdot \max_{1\le i\le n}\Big|g(t_i) - \sum_{k=1}^{n} S_{ik} g(t_k)\Big|^2 \cdot \max_{1\le j\le n}\Big|\sum_{i=1}^{n} S_{ij}\Big| \cdot n \max_{1\le i\le n}\Big|\sum_{j=1}^{n} S_{ij}\Big| \\
&= [O(n^{-2\gamma}) + O(\tau_m^2)]\cdot C \to 0,
\end{aligned} \tag{3.80}$$

再由Chebychev不等式及(3.80)式, 有

$$\mathsf{P}(|Q_{122}| > \varepsilon) \le \frac{\mathsf{Var}\,(Q_{122})}{\varepsilon^2} \to 0, \tag{3.81}$$

即

$$Q_{122} = o_p(1). \tag{3.82}$$

由(3.75)、(3.79)、(3.80)式有

$$Q_{12} = o_p(1). \tag{3.83}$$

从而由(3.2)、(3.73)、(3.74)、(3.83)式有

$$Q_1 = o_p(1). \tag{3.84}$$

类似于Q_{12}的证明, 有

$$Q_2 = o_p(1). \tag{3.85}$$

因为

$$\begin{aligned}
Q_3 &= \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(f(t_i) - \sum_{j=1}^{n} S_{ij} f(t_j)\Big)\Big(\varepsilon_i - \sum_{k=1}^{n} S_{ik}\varepsilon_k\Big) - \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} S_{ij}\eta_j\Big)\varepsilon_i \\
&\quad - \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\eta_i - \sum_{j=1}^{n} S_{ij}\eta_j\Big)\sum_{k=1}^{n} S_{ik}\varepsilon_k + \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta_i\varepsilon_i \\
&=: \Delta_n^{-1}\cdot Q_{31} - \Delta_n^{-1}\cdot Q_{32} - \Delta_n^{-1}\cdot Q_{33} + \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta_i\varepsilon_i,
\end{aligned} \tag{3.86}$$

由引理3.2、引理3.3及引理3.5有

$$\begin{aligned}|Q_{31}| &\le \max_{1\le i\le n}\Big|f(t_i)-\sum_{j=1}^n S_{ij}f(t_j)\Big|\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\varepsilon_i-\sum_{k=1}^n S_{ik}\varepsilon_k\Big)\\ &\le \Big|n^{-5/6}\sum_{i=1}^n\Big(\varepsilon_i-\sum_{k=1}^n S_{ik}\varepsilon_k\Big)\Big|\le\Big|\sum_{i=1}^n\frac{\varepsilon_i}{n^{5/6}}\Big|+n^{1/6}\Big|\sum_{k=1}^n S_{ik}\varepsilon_k\Big|\\ &= o_p(1).\end{aligned}\tag{3.87}$$

由引理3.5及(3.5)式有

$$\begin{aligned}Q_{32} &\le \max_{1\le i\le n}\Big|\sum_{j=1}^n S_{ij}\eta_j\Big|\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\varepsilon_i\\ &= n^{-1/3}\cdot n^{-1/2}\sum_{i=1}^n\varepsilon_i=\sum_{i=1}^n\frac{\varepsilon_i}{n^{5/6}}=o_p(1).\end{aligned}\tag{3.88}$$

因为

$$Q_{33}\le\max_{1\le i\le n}\Big|\sum_{k=1}^n S_{ik}\varepsilon_k\Big|\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n S_{ij}\eta_j\Big),\tag{3.89}$$

而

$$\mathsf{E}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n\eta_i\Big)^2=\frac{1}{n}\sum_{i=1}^n\mathsf{E}\eta_i^2=\sigma_\eta^2,\tag{3.90}$$

又由引理3.1有

$$\begin{aligned}\mathsf{E}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n\sum_{j=1}^n S_{ij}\eta_j\Big)^2 &= \frac{\sigma_\eta^2}{n}\sum_{j=1}^n\Big(\sum_{i=1}^n S_{ij}\Big)^2\le\frac{\sigma_\eta^2}{n}\max_{1\le j\le n}\Big|\sum_{i=1}^n S_{ij}\Big|\cdot n\sup_{1\le i\le n}\sum_{j=1}^n|S_{ij}|\\ &\le C\cdot\frac{\sigma_\eta^2}{n}\cdot n\cdot o_p\Big(\frac{2^m}{\sqrt{n}}\Big)=o_p(1),\end{aligned}\tag{3.91}$$

由引理3.3、(3.89)、(3.90)、(3.91)式有

$$Q_{33}=o_p(1).\tag{3.92}$$

由(3.2)、(3.86)、(3.87)、(3.88)、(3.92)式有

$$Q_3=n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n\eta_i\varepsilon_i+o_p(1).\tag{3.93}$$

因为

$$\begin{aligned}Q_4 &= \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n u_i\varepsilon_i-\Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}u_j\Big)\varepsilon_i\\ &\quad-\Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(u_i-\sum_{j=1}^n S_{ij}u_j\Big)\sum_{k=1}^n S_{ik}\varepsilon_k\\ &=: \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n u_i\varepsilon_i-\Delta_n^{-1}\cdot Q_{41}-\Delta_n^{-1}\cdot Q_{42},\end{aligned}\tag{3.94}$$

类似于Q_{32}的证明, 有

$$Q_{41} = o_p(1), \tag{3.95}$$

类似于Q_{33}的证明, 有

$$Q_{42} = o_p(1), \tag{3.96}$$

由(3.2)、(3.94)、(3.95)、(3.96)式有

$$Q_4 = n^{-1/2}\Delta_n^{-1}\sum_{i=1}^n u_i\varepsilon_i + o_p(1). \tag{3.97}$$

因为

$$\begin{aligned} Q_5 &= \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(f(t_i)-\sum_{j=1}^n S_{ij}f(t_j)\Big)\Big(u_i-\sum_{k=1}^n S_{ik}u_k\Big)\beta \\ &\quad +\Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n S_{ij}\eta_j\Big)\Big(u_i-\sum_{k=1}^n S_{ik}u_k\Big)\beta \\ &=: \Delta_n^{-1}\cdot Q_{51}+\Delta_n^{-1}\cdot Q_{52}, \end{aligned} \tag{3.98}$$

类似于Q_{12}的证明, 有

$$Q_{51} = o_p(1). \tag{3.99}$$

因为

$$\begin{aligned} Q_{52} &= \frac{1}{\sqrt{n}}\sum_{i=1}^n\eta_i u_i\beta-\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\sum_{j=1}^n S_{ij}\eta_j\Big)u_i\beta \\ &\quad -\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\eta_i-\sum_{j=1}^n S_{ij}\eta_j\Big)\Big(\sum_{k=1}^n S_{ik}u_k\Big)\beta \\ &=: \frac{1}{\sqrt{n}}\sum_{i=1}^n\eta_i u_i\beta-Q_{521}-Q_{522}, \end{aligned} \tag{3.100}$$

由引理3.6及(3.5)式有

$$Q_{521} \le \beta\cdot\max_{1\le i\le n}\Big|\sum_{j=1}^n S_{ij}\eta_j\Big|\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^n u_i = \beta\cdot\frac{1}{n^{5/6}}\sum_{i=1}^n u_i\to 0, \tag{3.101}$$

即

$$Q_{521} = o_p(1), \tag{3.102}$$

类似于Q_{33}的证明, 有

$$Q_{522} = o_p(1), \tag{3.103}$$

由(3.100)、(3.102)、(3.103)式有

$$Q_{52} = \frac{1}{\sqrt{n}}\sum_{i=1}^n\eta_i u_i\beta + o_p(1), \tag{3.104}$$

由(3.2)、(3.98)、(3.99)、(3.104)式有

$$Q_5 = n^{-1/2}\Delta_n^{-1}\sum_{i=1}^{n}\eta_i u_i\beta + o_p(1). \tag{3.105}$$

因为

$$\begin{aligned} Q_6 &= \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}u_i^2\beta - \Delta_n^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}S_{ij}u_j\Big)u_i\beta \\ &\quad - \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(u_i - \sum_{j=1}^{n}S_{ij}u_j\Big)\Big(\sum_{k=1}^{n}S_{ik}u_k\Big)\beta \\ &=: \Delta_n^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}u_i^2\beta - \Delta_n^{-1}\cdot Q_{61} - \Delta_n^{-1}\cdot Q_{62}, \end{aligned} \tag{3.106}$$

类似于Q_{521}的证明, 有

$$Q_{61} = o_p(1), \tag{3.107}$$

类似于Q_{33}的证明, 有

$$Q_{62} = o_p(1), \tag{3.108}$$

由(3.2)、(3.106)、(3.107)、(3.108)式有

$$Q_6 = n^{-1/2}\Delta_n^{-1}\sum_{i=1}^{n}u_i^2\beta + o_p(1). \tag{3.109}$$

所以由(3.72)、(3.84)、(3.85)、(3.93)、(3.97)、(3.105)、(3.109)式有

$$n^{1/2}(\widehat{\beta}_n - \beta) = n^{-1/2}\Delta_n^{-1}\sum_{i=1}^{n}(\eta_i\varepsilon_i + u_i\varepsilon_i - \eta_i u_i\beta - u_i^2\beta + \sigma_u^2\beta) + o_p(1).$$

记$\zeta_i = \eta_i\varepsilon_i + u_i\varepsilon_i - \eta_i u_i\beta - u_i^2\beta + \sigma_u^2\beta$, 因为$\{\varepsilon_i\}$、$\{u_i\}$与$\{\eta_i\}$相互独立, 且$\mathsf{E}\varepsilon_i = 0$, $\mathsf{E}u_i = 0$, $\mathsf{E}\eta_i = 0$, 所以$\mathsf{Cov}\,(\zeta_i, \zeta_j) = 0$, 从而$\{\zeta_i, i = 1, 2, \ldots, n\}$是$\rho$混合随机序列.

注意到

$$\begin{aligned} P_n &=: \sum_{i=1}^{n}n^{-1/2}\Delta_n^{-1}\zeta_i =: \sum_{i=1}^{n}V_{ni} = \sum_{m=1}^{n'}\upsilon_{nm} + \sum_{m=1}^{n'}\upsilon'_{nm} + \upsilon''_{nn'+1} \\ &=: P'_n + P''_n + P'''_n, \end{aligned} \tag{3.110}$$

其中$\upsilon_{nm} = \sum_{i=k_m}^{k_m+p-1}V_{ni}$, $\upsilon'_{nm} = \sum_{j=l_m}^{l_m+q-1}V_{nj}$, $\upsilon''_{nn'+1} = \sum_{k=n'(p+q)+1}^{n}V_{nk}$, $k_m = (m-1)(p+q)+1$, $l_m = (m-1)(p+q)+p+1$, $m = 1, 2, \ldots, n'$, 且p、q、n'满足条件(A_6).

要证明此定理, 我们只需证明以下二个式子成立即可,

$$P''_n + P'''_n \to^P 0, \tag{3.111}$$

$$P'_n \to^d N(0, \Gamma). \tag{3.112}$$

由条件(A_6)及(3.2)式有

$$\begin{aligned}\mathsf{E}(P_n'')^2 &\leq C\sum_{m=1}^{n'}\sum_{j=l_m}^{l_m+q-1}(n^{-1/2}\Delta_n^{-1})^2 \leq Cn'qn^{-1}\to 0,\\ \mathsf{E}(P_n''')^2 &\leq C\sum_{k=n'(p+q)+1}^{n}(n^{-1/2}\Delta_n^{-1})^2 \leq C(n-n'(p+q))n^{-1}\\ &\leq C(1+qp^{-1})pn^{-1}\to 0,\end{aligned}$$

由Chebychev不等式, 对$\forall\,\varepsilon>0$有

$$\begin{aligned}\mathsf{P}(|P_n''+P_n'''|>2\varepsilon) &\leq \mathsf{P}(|P_n''|>\varepsilon)+\mathsf{P}(|P_n'''|>\varepsilon)\\ &\leq \frac{\mathsf{E}(P_n'')^2}{\varepsilon^2}+\frac{\mathsf{E}(P_n''')^2}{\varepsilon^2}\to 0,\end{aligned}$$

由ε的任意性知(3.111)式成立.

令$p_n^2=\sum\limits_{m=1}^{n'}\mathsf{Var}\,(\upsilon_{nm})$, $F_n=\sum\limits_{1\leq i<j\leq n'}\mathsf{Cov}\,(\upsilon_{ni},\upsilon_{nj})$, 则有

$$p_n^2=\mathsf{E}(P_n')^2-2F_n,\qquad \mathsf{E}(P_n^2)=\Gamma, \tag{3.113}$$

由于$\mathsf{E}(P_n')^2=\mathsf{E}[P_n-(P_n''+P_n''')]^2=\Gamma+\mathsf{E}(P_n''+P_n''')^2-2\mathsf{E}[P_n(P_n''+P_n''')]$, 从而有

$$|\mathsf{E}(P_n')^2-\Gamma|=|\mathsf{E}(P_n''+P_n''')^2-2\mathsf{E}[P_n(P_n''+P_n''')]|\to 0, \tag{3.114}$$

由引理3.1及(3.2)式有

$$\begin{aligned}|F_n| &\leq \sum_{1\leq i<j\leq n'}\sum_{k=k_i}^{k_i+p-1}\sum_{l=k_j}^{k_j+p-1}|\mathsf{Cov}\,(V_{nk},V_{nl})|\\ &\leq C\sum_{1\leq i<j\leq n'}\sum_{k=k_i}^{k_i+p-1}\sum_{l=k_j}^{k_j+p-1}n^{-1}|\mathsf{Cov}\,(\zeta_k,\zeta_l)|\\ &\leq C\sum_{1\leq i<j\leq n'}\sum_{k=k_i}^{k_i+p-1}\sum_{l=k_j}^{k_j+p-1}n^{-1}\rho(l-k)\|\zeta_k\|_2\|\zeta_l\|_2\\ &\leq C\sum_{i=1}^{n'-1}\sum_{k=k_i}^{k_i+p-1}n^{-1}\sum_{j=i+1}^{n'}\sum_{l=k_j}^{k_j+p-1}\rho(l-k)\\ &\leq C\sum_{k=1}^{n}n^{-1}\sum_{j=q}^{\infty}\rho(j)=C\sum_{j=q}^{\infty}\rho(j)\to 0\qquad(q\to\infty),\end{aligned} \tag{3.115}$$

从而由(3.113)、(3.114)、(3.115)式有

$$\mathsf{E}(P_n')^2\to\Gamma,\qquad p_n^2\to\Gamma. \tag{3.116}$$

为了建立P_n'的渐近正态性, 假设$\{\mu_{nm},m=1,2,\ldots,n'\}$是独立随机变量序列, 且$\mu_{nm}$与$\upsilon_{nm}$ $(m=1,2,\ldots,n')$有相同的分布, 则有$\mathsf{E}\mu_{nm}=0$, $\mathsf{Var}\,(\mu_{nm})=\mathsf{Var}\,(\upsilon_{nm})$. 令$G_{nm}=\mu_{nm}/p_n$,

$m=1,2,\ldots,n'$, 则$\{G_{nm}, m=1,2,\ldots,n'\}$是独立的, 且$\mathsf{E}G_{nm}=0$, $\sum\limits_{m=1}^{n'}\mathsf{Var}\,(G_{nm})=1$. 用$\psi_X(t)$表示随机变量$X$的特征函数, 则有

$$\begin{aligned}\Big|\psi_{\sum\limits_{m=1}^{n'}\upsilon_{nm}}(t)-e^{-t^2/2}\Big| &\le \Big|\mathsf{E}\exp\Big(it\sum_{m=1}^{n'}\upsilon_{nm}\Big)-\prod_{m=1}^{n'}\mathsf{E}\exp(it\upsilon_{nm})\Big| \\ &\quad+\Big|\prod_{m=1}^{n'}\mathsf{E}\exp(it\upsilon_{nm})-e^{-t^2/2}\Big| \\ &\le \Big|\mathsf{E}\exp\Big(it\sum_{m=1}^{n'}\upsilon_{nm}\Big)-\prod_{m=1}^{n'}\mathsf{E}\exp(it\upsilon_{nm})\Big| \\ &\quad+\Big|\prod_{m=1}^{n'}\mathsf{E}\exp(itG_{nm})-e^{-t^2/2}\Big| \\ &=: H_1+H_2. \end{aligned}\tag{3.117}$$

由引理3.8、条件(A_6)及(3.2)式有

$$\begin{aligned} H_1 &\le C|t|\rho^{1/2}(q)\sum_{m=1}^{n'}\|\upsilon_{nm}\|_2 \\ &\le C|t|\rho^{1/2}(q)\sum_{m=1}^{n'}\mathsf{E}\Big(\sum_{i=k_m}^{k_m+p-1}|n^{-1/2}\Delta_n^{-1}\zeta_i|^2\Big)^{1/2} \\ &\le C|t|\rho^{1/2}(q)\sum_{m=1}^{n'}\sum_{i=k_m}^{k_m+p-1}(n^{-1})^{1/2} \\ &\le C|t|\rho^{1/2}(q)n'p(n^{-1})^{1/2}\to 0. \end{aligned}\tag{3.118}$$

对于H_2, 要证$H_2\to 0$, 等价于证明$\sum\limits_{m=1}^{n'}G_{nm}\to^d N(0,1)$, 又因为有(3.116)式, 从而只需证明

$$\sum_{m=1}^{n'}\mu_{nm}\to^d N(0,\Gamma),$$

对于某个$\delta>0$, 由引理3.7、条件(A_6)及(3.2)式有

$$\begin{aligned}\sum_{m=1}^{n'}\mathsf{E}|\upsilon_{nm}|^{2+\delta} &\le C\sum_{m=1}^{n'}\Big[\sum_{i=k_m}^{k_m+p-1}\mathsf{E}|V_{ni}|^{2+\delta}+\Big(\sum_{i=k_m}^{k_m+p-1}\mathsf{E}|V_{ni}|^2\Big)^{(2+\delta)/2}\Big] \\ &\le C\sum_{m=1}^{n'}\Big[\sum_{i=k_m}^{k_m+p-1}(n^{-1}\Delta_n^{-2})^{(2+\delta)/2}+\Big(\sum_{i=k_m}^{k_m+p-1}n^{-1}\Delta_n^{-2}\Big)^{(2+\delta)/2}\Big] \\ &\le C\sum_{i=1}^{n}(n^{-1}\Delta_n^{-2})^{(2+\delta)/2}\le Cn^{-\delta/2}\to 0, \end{aligned}\tag{3.119}$$

由(3.116)、(3.119)式有

$$\frac{1}{p_n^{2+\delta}}\sum_{m=1}^{n'}\mathsf{E}|\upsilon_{nm}|^{2+\delta}\to 0,\tag{3.120}$$

从而由Berry-Esseen定理有 $\sum_{m=1}^{n'} \mu_{nm} \to^d N(0,\Gamma)$成立, 即有

$$H_2 \to 0. \tag{3.121}$$

由(3.117)、(3.118)、(3.121)式, 可把P_n'看做是独立不同分布随机变量之和, 又因为有(3.120)式成立, 则由Liapounov中心极限定理有(3.112)式成立, 从而由(3.111)、(3.112)式及Sluker定理得此定理成立. □

定理2.5的证明: 由(2.2)式有

$$\begin{aligned}
\widehat{g}_n(t) &= \sum_{i=1}^{n}(y_i - X_i\widehat{\beta}_n)\int_{A_i} E_m(t,s)\mathrm{d}s \\
&= \sum_{i=1}^{n} X_i(\beta - \widehat{\beta}_n)\int_{A_i} E_m(t,s)\mathrm{d}s + \sum_{i=1}^{n} g(t_i)\int_{A_i} E_m(t,s)\mathrm{d}s \\
&\quad + \sum_{i=1}^{n}\xi_i \int_{A_i} E_m(t,s)\mathrm{d}s.
\end{aligned}$$

从而由定理2.4有

$$\begin{aligned}
\mathsf{E}\widehat{g}_n(t) &= \mathsf{E}\Big(\sum_{i=1}^{n} X_i(\beta - \widehat{\beta}_n)\int_{A_i} E_m(t,s)\mathrm{d}s + \sum_{i=1}^{n} g(t_i)\int_{A_i} E_m(t,s)\mathrm{d}s \\
&\quad + \sum_{i=1}^{n}\xi_i \int_{A_i} E_m(t,s)\mathrm{d}s\Big) \\
&= \sum_{i=1}^{n} g(t_i)\int_{A_i} E_m(t,s)\mathrm{d}s.
\end{aligned}$$

即有

$$\begin{aligned}
\frac{\widehat{g}_n(t) - \mathsf{E}\widehat{g}_n(t)}{\Gamma_n} &= \frac{\sum_{i=1}^{n}(y_i - X_i\widehat{\beta}_n)\int_{A_i} E_m(t,s)\mathrm{d}s - \sum_{i=1}^{n} g(t_i)\int_{A_i} E_m(t,s)\mathrm{d}s}{\Gamma_n} \\
&= \frac{\sum_{i=1}^{n}\xi_i \int_{A_i} E_m(t,s)\mathrm{d}s}{\Gamma_n} - \frac{\sum_{i=1}^{n} X_i(\widehat{\beta}_n - \beta)\int_{A_i} E_m(t,s)\mathrm{d}s}{\Gamma_n} \\
&=: S_{n1} - S_{n2}.
\end{aligned} \tag{3.122}$$

类似于定理2.4的证明, 我们容易证得

$$S_{n1} \to^d N(0,1), \tag{3.123}$$

$$S_{n2} = o_p(1). \tag{3.124}$$

故由(3.123)、(3.124)式及Sluker定理得此定理成立. □

定理2.6的证明: 由(2.7)式及$\widetilde{y}_i = \widetilde{X}_i\beta + \widetilde{g}(t_i) + \widetilde{\xi}_i$有

$$\begin{aligned}
n^{1/2}(\widehat{\sigma}_n^2 - \sigma^2) &= n^{-1/2}\Big[\sum_{i=1}^n (\widetilde{y}_i - \widetilde{X}_i\widehat{\beta}_n)^2 - n\sigma_u^2\widehat{\beta}_n^2 - n\sigma^2\Big] \\
&= n^{-1/2}\sum_{i=1}^n [(\widetilde{y}_i - \widetilde{X}_i\widehat{\beta}_n)^2 - \sigma_u^2\widehat{\beta}_n^2 - \sigma^2] \\
&= n^{-1/2}\sum_{i=1}^n [(\widetilde{X}_i(\beta - \widehat{\beta}_n) + \widetilde{g}(t_i) + \widetilde{\xi}_i)^2 - \sigma_u^2\widehat{\beta}_n^2 - \sigma^2] \\
&= n^{-1/2}\sum_{i=1}^n (\widetilde{X}_i(\beta - \widehat{\beta}_n))^2 + n^{-1/2}\sum_{i=1}^n \widetilde{g}^2(t_i) \\
&\quad + n^{-1/2}\sum_{i=1}^n (\widetilde{\xi}_i^2 - \sigma_u^2\widehat{\beta}_n^2 - \sigma^2) + 2n^{-1/2}\sum_{i=1}^n (\widetilde{X}_i(\beta - \widehat{\beta}_n))\widetilde{g}(t_i) \\
&\quad + 2n^{-1/2}\sum_{i=1}^n (\widetilde{X}_i(\beta - \widehat{\beta}_n))\widetilde{\xi}_i + 2n^{-1/2}\sum_{i=1}^n \widetilde{g}(t_i)\widetilde{\xi}_i \\
&=: M_1 + M_2 + M_3 + 2M_4 + 2M_5 + 2M_6. \qquad (3.125)
\end{aligned}$$

类似于定理2.4的证明, 可得

$$M_i \to^p 0, \qquad i = 1, 2, 4, 5, 6, \qquad (3.126)$$

及

$$M_3 = \frac{1}{\sqrt{n}}\sum_{i=1}^n [(\varepsilon_i - u_i\beta)^2 - \sigma_u^2\beta^2 - \sigma^2] + o_p(1) \to^d N(0, \Lambda). \qquad (3.127)$$

故由(3.125)-(3.127)式有结论成立. □

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Asymptotic Properties of Wavelet Estimators in a Semiparametric EV Model under NA Samples

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In this paper, we research the semiparametric EV model under NA samples. Some estimators of the parameter, nonparameter and the variance function are established by the wavelet smoothing method. Under some general conditions, the strong consistency and the asymptotic normality of wavelet estimators are studied.

Keywords: Semiparametric EV model, wavelet estimate, NA sequence, strong consistency, asymptotic normality.

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