应用概率统计　第三十卷
第六期　2014年12月

Chinese Journal of Applied Probability
and Statistics Vol.30 No.6 Dec. 2014

The Optimal Dividend and Capital Injection Strategies in the Classical Risk Model with Randomized Observation Periods *

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Abstract

This paper considers the optimal dividend and capital injection strategies in the classical risk model with randomized observation periods. Assume that ruin is prohibited. We aim to maximise the expected discounted dividend payments minus the expected penalised discounted capital injections. We derive the associated Hamilton-Jacobi-Bellman (HJB) equation and prove the verification theorem. The optimal control strategy and the optimal value function are obtained under the assumption that the claim sizes are exponentially distributed.

Keywords: Dividend, capital injection, Hamilton-Jacobi-Bellman equation.

AMS Subject Classification: 62P05, 91B30, 91B70.

§1. Introduction

Finding an optimal dividend strategy has been an important problem in actuarial sciences since "dividend" was considered as a criterion for measuring the stability of an insurance company. De Finetti (1957) first proposed this criterion and solved the problem in a discrete time random walk model by stochastic control theory, and showed that the optimal dividend strategy is a barrier strategy. In recent years, many researches on the issue of maximization of the dividend payments until ruin have been produced. Avanzi (2009) and Albrecher and Thonhauser (2009) are two comprehensive reviews before 2009.

In most of the literature, the surplus processes are continuously observed and dividend can be paid at any time when the surplus is positive. But Albrecher et al. (2011a, 2013) argued that it was more reasonable that companies checked the balance on a periodic basis and then decided whether to pay dividends to the shareholders, which implies that lump sum dividends can only be paid at some randomized observation times. The periodic

*The project was supported by the National Natural Science Foundation of China (11201006).
Received July 8, 2014. Revised September 28, 2014.
doi: 10.3969/j.issn.1001-4268.2014.06.009

dividend problems with Brownian risk model was studied by Albrecher et al. (2011b), and with Lévy model by Albrecher et al. (2011c). Avanzi et al. (2013) derived the integro-differential equations for the Laplace transform of the ruin time and the expected present value of dividends until ruin in the dual model under the assumption that the time intervals between dividend decisions were Erlang distributed and dividends were paid according to a barrier strategy. Avanzi et al. (2014) consider the optimal periodic dividend strategies in the dual model with diffusion. One can refer to Wei et al. (2012), Peng et al. (2013), Liu and Chen (2014), Wang and Liu (2014) and Chen et al. (2014) for more literature about randomized observation periods.

Barrier strategy is usually the optimal dividend strategy for the risk model with unrestricted dividend density and hence ruin will occur almost surely. Dickson and Waters (2004) proposed that capital injections can be taken into account to continue the business once the surplus becomes negative. The models with capital injection are Kulenko and Schmidli (2008), Eisenberg and Schmidli (2009, 2011), Dai et al. (2010), Bai and Paulsen (2010, 2012), Yao et al. (2010, 2011, 2014), Peng et al. (2012) and the references therein.

In Kulenko and Schmidli (2008), they studied the optimal dividend strategy in the classical risk model with capital injection and showed that the optimal dividend strategy is a barrier strategy. Motivated by Kulenko and Schmidli (2008), in this paper, we will investigate the optimal dividend and capital injection problems in the case that dividends can only be paid at some randomized observation times.

This paper is organized as follows. In Section 2, the model we studied in this paper is introduced. In Section 3, the associated Hamilton-Jacobi-Bellman (HJB) equation is found and the verification theorem is proved. In Section 4, the explicit solution to the HJB equation is derived when the claim sizes are exponentially distributed, and we construct a candidate strategy and prove that it is the optimal strategy.

§2. The Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$ be a filtered probability space on which all random processes and variables introduced in the following are defined. Suppose that the uncontrolled surplus process is described by

$$U(t) = x + ct - \sum_{i=1}^{N_1(t)} Y_i, \qquad t \geq 0, \tag{2.1}$$

where $x \geq 0$ is the initial surplus, c is the constant premium rate, $\{N_1(t); t \geq 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and the claim sizes $\{Y_i; i = 1, 2, \ldots\}$ form a sequence of independent and identically distributed positive random variables (r.v.s) with generic r.v. Y, whose distribution function is $P(y)$ and density function is $p(y)$. Assume that the net profit condition $c > \lambda\mathsf{E}[Y]$ holds. Denote the jump times of $\{N_1(t); t \geq 0\}$ by $T_1 < T_2 < \cdots$. Let $\{N_2(t); t \geq 0\}$ be a homogeneous Poisson process with intensity $\gamma > 0$ and $S_1 < S_2 < \cdots$ be the jump times. Suppose that dividends can only be paid at times $\{S_i; i = 1, 2, \ldots\}$, and denote the corresponding amounts by $\{L_i; i = 1, 2, \ldots\}$. Let $L(t)$ be the accumulated dividends paid up to time t, then we have $L(t) = \sum\limits_{i=1}^{N_2(t)} L_i$. In addition, we assume that $\{N_1(t); t \geq 0\}$, $\{N_2(t); t \geq 0\}$ and $\{Y_i; i = 1, 2, \ldots\}$ are mutually independent.

Let $H(t)$ be the accumulated capital injections paid up to time t. A control strategy is described by $(L, H) = (L(t), H(t))$. Assume that dividends and capital injections are discounted at a constant force of interest δ. The controlled surplus process associated with (L, H) is described by

$$U_{(L,H)}(t) = U(t) - L(t) + H(t), \qquad t \geq 0. \tag{2.2}$$

Supposing ruin is not permitted, we denote Π the set of all the control strategies (L, H) such that: $L(t)$ with $L(0-) = 0$ and $H(t)$ with $H(0-) = 0$ are adapted càdlàg and non-decreasing processes; and $\mathsf{P}_x[U_{(L,H)}(t) \geq 0 \text{ for all } t \geq 0] = 1$, where P_x is the probability corresponding to the law of $\{U_{(L,H)}(t); t \geq 0\}$ with $U_{(L,H)}(0) = x$. The performance function of a control strategy $(L, H) \in \Pi$ is defined as

$$V_{(L,H)}(x) = \mathsf{E}_x\Big[\sum_{i=1}^{\infty} \exp(-\delta S_i) L_i - k \int_0^{\infty} \exp(-\delta t) \mathrm{d}H(t)\Big], \tag{2.3}$$

where $k > 1$ is a penalising factor and E_x is the expectation corresponding to the law of $\{U_{(L,H)}(t); t \geq 0\}$ with $U_{(L,H)}(0) = x$.

We want to find the optimal value function

$$V(x) = \sup_{(L,H)\in\Pi} V_{(L,H)}(x) = V_{(L^*,H^*)}(x) \qquad \text{for } x \geq 0, \tag{2.4}$$

where (L^*, H^*) is the optimal control strategy.

§3. Hamilton-Jacobi-Bellman Equation

In this section, we derive the HJB equation and prove the verification theorem. We first state a proposition of $V(x)$.

Lemma 3.1 The function $V(x)$ is increasing, Lipschitz continuous, and therefore absolutely continuous.

Proof Using the same strategy for two different initial capitals shows that $V(x)$ is increasing with x. For initial capital x, injecting capital $\triangle x > 0$ and then following a control strategy (L, H), we have

$$V(x) \geq V_{(L,H)}(x+\triangle x) - k\triangle x,$$

and hence

$$V(x+\triangle x) - V(x) \leq k\triangle x,$$

which shows that $V(x)$ is Lipschitz continuous, and therefore absolutely continuous. $\square$

Remark 1 From Lemma 3.1, we know that $V(x)$ is differentiable and $V'(x) \leq k$ almost surely.

Because of discounting, the capital injection only occurs at the times when the surplus becomes negative by a claim, hence we have $H^*(t) = \max\big\{-\inf\limits_{0\leq s\leq t}(U(t) - L^*(t)), 0\big\}$, and we can define

$$V(u) = V(0) + ku \tag{3.1}$$

for $u < 0$. Similar to Fleming and Soner (2005), we derive the HJB equation associated with (2.4) as

$$\max_{0\leq l\leq x}\{\gamma[l + V(x-l)]\} + \mathcal{A}V(x) = 0, \tag{3.2}$$

where

$$\mathcal{A}V(x) = \lambda\int_0^\infty V(x-y)\mathrm{d}P(y) - (\lambda+\delta+\gamma)V(x) + cV'(x). \tag{3.3}$$

The verification theorem is stated as follows.

Theorem 3.1 Let $f(x):[0,\infty)\to[0,\infty)$ with $f'(0+) \leq k$ be a continuously differentiable, increasing and concave function.

(i) If $f(x)$ satisfies

$$\max_{0\leq l\leq x}\{\gamma[l + f(x-l)]\} + \mathcal{A}f(x) \leq 0 \tag{3.4}$$

with the definition $f(u) = f(0) + ku$ for $u < 0$, then we have

$$f(x) \geq V(x). \tag{3.5}$$

(ii) If $f(x)$ satisfies

$$\max_{0\leq l\leq x}\{\gamma[l + f(x-l)]\} + \mathcal{A}f(x) = 0 \tag{3.6}$$

with the definition $f(u) = f(0) + ku$ for $u < 0$, then we have

$$f(x) = V(x). \tag{3.7}$$

Proof Denote $\mathcal{M}_1 = \{T_i; i = 1, 2, \ldots\}$ and $\mathcal{M}_2 = \{S_i; i = 1, 2, \ldots\}$. For any $(L, H) \in \Pi$, from generalized Itô formula, we have

$$\begin{aligned} \mathrm{e}^{-\delta t} f(U_{(L,H)}(t)) = {} & f(x) - \delta \int_0^t \mathrm{e}^{-\delta s} f(U_{(L,H)}(s-))\mathrm{d}s \\ & + c\int_0^t \mathrm{e}^{-\delta s} f'(U_{(L,H)}(s-))\mathrm{d}s + \Omega_1(t) + \Omega_2(t) + \Omega_3(t), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \Omega_1(t) &= \sum_{s\in\mathcal{M}_1, s\le t} \mathrm{e}^{-\delta s}[f(U_{(L,H)}(s) + \triangle U_{(L,H)}(s)) - f(U_{(L,H)}(s))], \\ \Omega_2(t) &= \sum_{s\in\mathcal{M}_2, s\le t} \mathrm{e}^{-\delta s}[f(U_{(L,H)}(s-) - \triangle L(s)) - f(U_{(L,H)}(s-))] \end{aligned}$$

and

$$\Omega_3(t) = \sum_{s\le t} \mathrm{e}^{-\delta s}[f(U_{(L,H)}(s)) - f(U_{(L,H)}(s) - \triangle H(s))].$$

Defining

$$\begin{aligned} \Phi_1(t) &= \Omega_1(t) - \lambda\int_0^t \mathrm{e}^{-\delta s}\int_0^\infty [f(U_{(L,H)}(s) - y) - f(U_{(L,H)}(s))]\mathrm{d}P(y)\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{-\delta s}[f(U_{(L,H)}(s) + \triangle U_{(L,H)}(s)) - f(U_{(L,H)}(s))]\mathrm{d}N_1(s) \\ &\quad - \lambda\int_0^t \mathrm{e}^{-\delta s}\int_0^\infty [f(U_{(L,H)}(s) - y) - f(U_{(L,H)}(s))]\mathrm{d}P(y)\mathrm{d}s, \\ \Phi_2(t) &= \Omega_2(t) - \gamma\int_0^t \mathrm{e}^{-\delta s}[f(U_{(L,H)}(s-) - \triangle L(s)) - f(U_{(L,H)}(s-))]\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{-\delta s}[f(U_{(L,H)}(s-) - \triangle L(s)) - f(U_{(L,H)}(s-))]\mathrm{d}N_2(s) \\ &\quad - \gamma\int_0^t \mathrm{e}^{-\delta s}[f(U_{(L,H)}(s-) - \triangle L(s)) - f(U_{(L,H)}(s-))]\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} \Phi_3(t) &= \sum_{s\in\mathcal{M}_2,\ s\le t} \mathrm{e}^{-\delta s}\triangle L(s) - \gamma\int_0^t \mathrm{e}^{-\delta s}\triangle L(s)\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{-\delta s}\triangle L(s)\mathrm{d}N_2(s) - \gamma\int_0^t \mathrm{e}^{-\delta s}\triangle L(s)\mathrm{d}s, \end{aligned}$$

we know that $\Phi_1(t)$, $\Phi_2(t)$ and $\Phi_3(t)$ are martingales with zero-expectation. Hence

$$\begin{aligned}
&\mathrm{e}^{-\delta t}f(U_{(L,H)}(t))-f(x)\\
=&\int_0^t \mathrm{e}^{-\delta s}[\gamma(f(U_{(L,H)}(s-)-\triangle L(s))+\triangle L(s))+\mathcal{A}f(U_{(L,H)}(s-))]\mathrm{d}s\\
&-\sum_{s\in\mathcal{M}_2,s\le t}\mathrm{e}^{-\delta s}\triangle L(s)+\sum_{s\le t}\mathrm{e}^{-\delta s}\int_{-\triangle H(s)}^0 f'(U_{(L,H)}(s)+y)\mathrm{d}y\\
&+\Phi_1(t)+\Phi_2(t)+\Phi_3(t).
\end{aligned}\tag{3.9}$$

If the condition (3.4) holds, we have

$$\begin{aligned}
f(x)\ge&\ \mathsf{E}_x\Big[\sum_{s\in\mathcal{M}_2,s\le t}\mathrm{e}^{-\delta s}\triangle L(s)-\sum_{s\le t}\mathrm{e}^{-\delta s}\int_{-\triangle H(s)}^0 f'(U_{(L,H)}(s)+y)\mathrm{d}y\Big]\\
&+\mathsf{E}_x[\mathrm{e}^{-\delta t}f(U_{(L,H)}(t))]\\
\ge&\ \mathsf{E}_x\Big[\sum_{s\in\mathcal{M}_2,s\le t}\mathrm{e}^{-\delta s}\triangle L(s)-k\sum_{s\le t}\mathrm{e}^{-\delta s}\triangle H(s)\Big]+\mathsf{E}_x[\mathrm{e}^{-\delta t}f(U_{(L,H)}(t))]\\
\ge&\ \mathsf{E}_x\Big[\sum_{s\in\mathcal{M}_2,s\le t}\mathrm{e}^{-\delta s}\triangle L(s)-k\sum_{s\le t}\mathrm{e}^{-\delta s}\triangle H(s)\Big]+\mathrm{e}^{-\delta t}f(0).
\end{aligned}$$

Letting $t\to\infty$, we get $f(x)\ge V_{(L,H)}(x)$ for all $(L,H)\in\Pi$, and hence (3.5) holds.

If the condition (3.6) holds, we let $H^*(t)=\max\big\{-\inf\limits_{0\le s\le t}(U(t)-L^*(t)),0\big\}$ and $L_i^*=l(U_{(L^*,H^*)}(T_i-))$, where $l(x)$ satisfies

$$\gamma[l(x)+f(x-l(x))]\}+\mathcal{A}f(x)=0.$$

Then we have

$$\begin{aligned}
\sum_{s\le t}\mathrm{e}^{-\delta s}\int_{-\triangle H^*(s)}^0 f'(U_{(L^*,H^*)}(s)+y)\mathrm{d}y&=\sum_{s\le t}\mathrm{e}^{-\delta s}\int_{-\triangle H^*(s)}^0 f'(y)\mathrm{d}y\\
&=k\sum_{s\le t}\mathrm{e}^{-\delta s}\triangle H^*(s).
\end{aligned}$$

It is easy to see that $U_{(L^*,H^*)}(t)\le x+ct$, which together with the facts that $f(x)$ is a concave function and $f'(0+)\le k$, implies that $\lim\limits_{t\to\infty}\mathsf{E}_x\big[\mathrm{e}^{-\delta t}f(U_{(L^*,H^*)}(t))\big]=0$. Letting $(L,H)=(L^*,H^*)$ in (3.9), taking expectation and then letting $t\to\infty$ yield $f(x)=V_{(L^*,H^*)}(x)$, and therefore (3.7) holds. The proof is completed. $\square$

§4. Analysis for Exponential Claims

In this section, supposing $P(y)=1-\mathrm{e}^{-\beta y}$, $y>0$, $\beta>0$, we investigate the optimal control strategy and the optimal value function.

Let us find a continuously differentiable, increasing and concave solution $V(x)$ to (3.2) with (3.1). Suppose that there exists some point $b>0$ such that $V'(x)>1$ for $x<b$ and $V'(x)\le 1$ for $x\ge b$. Then a candidate of the optimal dividend strategy is

$$L_i^*=\begin{cases}0, & U_{(L^*,H^*)}(S_i-)<b;\\ U_{(L^*,H^*)}(S_i-)-b, & U_{(L^*,H^*)}(S_i-)\ge b,\end{cases}\tag{4.1}$$

and (3.2) is translated into

$$\begin{aligned}&cV'(x)-(\lambda+\delta)V(x)+\lambda\beta\mathrm{e}^{-\beta x}\int_0^x V(z)\mathrm{e}^{\beta z}\mathrm{d}z\\&+\lambda\Big(V(0)-\frac{k}{\beta}\Big)\mathrm{e}^{-\beta x}=0,\qquad x<b\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}&cV'(x)-(\lambda+\delta+\gamma)V(x)+\lambda\beta\mathrm{e}^{-\beta x}\int_0^x V(z)\mathrm{e}^{\beta z}\mathrm{d}z\\&+\lambda\Big(V(0)-\frac{k}{\beta}\Big)\mathrm{e}^{-\beta x}+\gamma[x+V(b)-b]=0,\qquad x\ge b.\end{aligned}\tag{4.3}$$

Applying the operator $(\mathrm{d}/\mathrm{d}x+\beta)$ to (4.2) and (4.3) respectively yields

$$cV''(x)+(\beta c-\lambda-\delta)V'(x)-\beta\delta V(x)=0\tag{4.4}$$

and

$$cV''(x)+(\beta c-\lambda-\delta-\gamma)V'(x)-\beta(\delta+\gamma)V(x)+\beta\gamma[x+V(b)-b]+\gamma=0.\tag{4.5}$$

Hence $V(x)$ can be denoted as

$$V(x)=\begin{cases}A_1\mathrm{e}^{r_1x}+A_2\mathrm{e}^{r_2x}, & x<b;\\ B\mathrm{e}^{sx}+\dfrac{\gamma}{\delta+\gamma}(x-b)+\dfrac{\gamma}{\delta}B\mathrm{e}^{sb}+\dfrac{(\beta c-\lambda)\gamma}{\beta\delta(\delta+\gamma)}, & x\ge b\end{cases}\tag{4.6}$$

for some constants A_1, A_2 and B, where

$$r_1=\frac{-(\beta c-\lambda-\delta)+\sqrt{(\beta c-\lambda-\delta)^2+4\beta c\delta}}{2c},$$
$$r_2=\frac{-(\beta c-\lambda-\delta)-\sqrt{(\beta c-\lambda-\delta)^2+4\beta c\delta}}{2c}$$

and

$$s=\frac{-(\beta c-\lambda-\delta-\gamma)-\sqrt{(\beta c-\lambda-\delta-\gamma)^2+4\beta c(\delta+\gamma)}}{2c}.$$

It is easy to see that $V(x)$ should satisfy $V(b-)=V(b+)$ and $V'(b-)=V'(b+)=1$, which give

$$A_1\mathrm{e}^{r_1b}+A_2\mathrm{e}^{r_2b}=\frac{\delta+\gamma}{\delta}B\mathrm{e}^{sb}+\frac{(\beta c-\lambda)\gamma}{\beta\delta(\delta+\gamma)}, \tag{4.7}$$

$$A_1r_1\mathrm{e}^{r_1b}+A_2r_2\mathrm{e}^{r_2b}=1, \tag{4.8}$$

$$Bs\mathrm{e}^{sb}+\frac{\gamma}{\delta+\gamma}=1. \tag{4.9}$$

Solving (4.7)-(4.9), we obtain

$$A_1=\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_2}{r_1-r_2}\mathrm{e}^{-r_1b}, \tag{4.10}$$

$$A_2=-\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_1}{r_1-r_2}\mathrm{e}^{-r_2b}, \tag{4.11}$$

$$B=\frac{\delta}{(\delta+\gamma)s}\mathrm{e}^{-\mu b}<0. \tag{4.12}$$

Letting $x\to 0+$ in (4.2) gives

$$c(A_1r_1+A_2r_2)-\delta(A_1+A_2)=\frac{\lambda k}{\beta}. \tag{4.13}$$

Plugging (4.10) and (4.11) into (4.13), we obtain an equation for b:

$$\begin{aligned}&\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_2}{r_1-r_2}(cr_1-\delta)\mathrm{e}^{-r_1b}\\ -&\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_1}{r_1-r_2}(cr_2-\delta)\mathrm{e}^{-r_2b}=\frac{\lambda k}{\beta}.\end{aligned} \tag{4.14}$$

Lemma 4.1 gives the condition of the existence of a positive b which satisfies (4.14).

Lemma 4.1 If $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]<\lambda k/\beta$, the equation (4.14) has unique positive root.

Proof Let

$$\begin{aligned}g(x)=&\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_2}{r_1-r_2}(cr_1-\delta)\mathrm{e}^{-r_1x}\\ &-\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_1}{r_1-r_2}(cr_2-\delta)\mathrm{e}^{-r_2x}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}&1-\Big(\frac{1}{s}+\frac{(\beta c-\lambda)\gamma}{\beta\delta(\delta+\gamma)}\Big)r_1\\ =&\Big(\frac{\sqrt{(\beta c-\lambda-\delta)^2+4\beta c\delta}}{2\beta\delta}+\frac{\sqrt{(\beta c-\lambda-\delta-\gamma)^2+4\beta c(\delta+\gamma)}}{2\beta(\delta+\gamma)}-\frac{(\beta c-\lambda)\gamma}{2\beta\delta(\delta+\gamma)}\Big)r_1\\ >&0.\end{aligned} \tag{4.15}$$

As the function $y = (\sqrt{(\beta c-\lambda-x)^2+4\beta cx}+\beta c-\lambda)/(2\beta x)$ is a decreasing function in $(0,\infty)$, we have

$$1-\Big(\frac{1}{s}+\frac{(\beta c-\lambda)\gamma}{\beta\delta(\delta+\gamma)}\Big)r_2=\Big(\frac{\sqrt{(\beta c-\lambda-\delta-\gamma)^2+4\beta c(\delta+\gamma)}}{2\beta(\delta+\gamma)}+\frac{\beta c-\lambda}{2\beta(\delta+\gamma)}-\frac{\sqrt{(\beta c-\lambda-\delta)^2+4\beta c\delta}}{2\beta\delta}-\frac{\beta c-\lambda}{2\beta\delta}\Big)r_2>0. \tag{4.16}$$

Since $cr_1>\delta$, then $g''(x)>0$, hence $g'(x)\geq g'(0)=\delta(1-cs/(\delta+\gamma))>0$. Therefore the equation (4.14) has unique positive root iff $g(0)=c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]<\lambda k/\beta$. □

Lemma 4.2 If $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]<\lambda k/\beta$, then $V'(0)\leq k$, where $V(x)$ is defined by (4.6).

Proof If $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]<\lambda k/\beta$, then $b>0$. Let

$$h(x)=\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_2}{r_1-r_2}\Big[\Big(c-\frac{\lambda}{\beta}\Big)r_1-\delta\Big]\mathrm{e}^{-r_1x}-\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_1}{r_1-r_2}\Big[\Big(c-\frac{\lambda}{\beta}\Big)r_2-\delta\Big]\mathrm{e}^{-r_2x}.$$

Since $r_1<\beta\delta/(\beta c-\lambda)$, we know that $V(x)$ is increasing in $[0,\infty)$. Hence

$$h(b)>h(0)=\frac{(\beta c-\lambda)\delta}{\beta(\delta+\gamma)}-\frac{\delta}{s}\geq 0.$$

Using (4.14), we get

$$V'(0)=\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_2}{r_1-r_2}r_1\mathrm{e}^{-r_1b}-\frac{1-[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))]r_1}{r_1-r_2}r_2\mathrm{e}^{-r_2b}\leq k. \qquad \square$$

If $V'(x)<1$ for all $x\geq 0$, then a candidate of the optimal dividend strategy should be

$$L_i^*=\max\{U_{(L^*,H^*)}(S_i-),0\} \tag{4.17}$$

and $V(x)$ should satisfy

$$cV'(x)-(\lambda+\delta+\gamma)V(x)+\lambda\beta\mathrm{e}^{-\beta x}\int_0^x V(z)\mathrm{e}^{\beta z}\mathrm{d}z+\lambda\Big(V(0)-\frac{k}{\beta}\Big)\mathrm{e}^{-\beta x}+\gamma[x+V(0)]=0,\qquad x\geq 0. \tag{4.18}$$

Hence $V(x)$ can be denoted as

$$V(x)=C\mathrm{e}^{sx}+\frac{\gamma}{\delta+\gamma}x+\frac{\gamma}{\delta}C+\frac{(\beta c-\lambda)\gamma}{\beta\delta(\delta+\gamma)} \tag{4.19}$$

for constant C. Plugging (4.19) into (4.18) and then letting $x \to 0+$, we obtain

$$C = \frac{\lambda[k(\delta+\gamma)-\gamma]}{\beta(\delta+\gamma)(cs-\delta-\gamma)} < 0. \tag{4.20}$$

Theorem 4.1 If $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))] < \lambda k/\beta$, $V(x)$ defined by (4.6) is a twice differentiable, increasing and concave solution to (3.2) with (3.1). If $c-\delta \cdot[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))] \geq \lambda k/\beta$, $V(x)$ defined by (4.19) is a twice differentiable, increasing and concave solution to (3.2) with (3.1).

Proof If $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))] < \lambda k/\beta$, we know that $b>0$ by Lemma 4.1, and $V''(b-)=V''(b+)=[\delta/(\delta+\gamma)]s<0$. From (4.6), (4.15) and (4.16), we have $V'(x)=A_1r_1\mathrm{e}^{r_1x}+A_2r_2\mathrm{e}^{r_2x}>0$ and $V'''(x)=A_1r_1^3\mathrm{e}^{r_1x}+A_2r_2^3\mathrm{e}^{r_2x}>0$ for $x<b$, therefore $V''(x)<0$ for $x<b$, which together with that $V''(x)=Bs^2\mathrm{e}^{sx}<0$ for $x\geq b$ implies that $V(x)$ defined by (4.6) is a twice differentiable, increasing and concave function.

If $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))] \geq \lambda k/\beta$, it is easy to see that $V(x)$ defined by (4.19) is twice differentiable, increasing and concave, we only need to show that $V'(0)\leq 1$. Since

$$\begin{aligned}
&[\lambda k(\delta+\gamma)-\lambda\gamma-\beta c\delta]s\\
\geq&\Big[\beta c(\delta+\gamma)-\beta\delta(\delta+\gamma)\Big(\frac{1}{s}+\frac{(\beta c-\lambda)\gamma}{\beta\delta(\delta+\gamma)}\Big)-\lambda\gamma-\beta c\delta\Big]s\\
=&-\beta\delta(\delta+\gamma),
\end{aligned}$$

we have $V'(0)\leq 1$. The proof is completed. □

Combining Theorem 3.1, Lemma 4.2 with Theorem 4.1, we obtain the following theorem.

Theorem 4.2 Assume that $P(y)=1-\mathrm{e}^{-\beta y}$, $y>0$, $\beta>0$. If $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))] < \lambda k/\beta$, the optimal dividend strategy is given by (4.1); if $c-\delta[1/s+(\beta c-\lambda)\gamma/(\beta\delta(\delta+\gamma))] \geq \lambda k/\beta$, the optimal dividend strategy is given by (4.17). The optimal capital injection strategy is given by $H^*(t)=\max\{-\inf_{0\leq s\leq t}(U(t)-L^*(t)),0\}$. The function $V(x)$ defined by (4.6) and (4.19) are the optimal value functions, respectively.

Example 1 Let $c=5$, $k=1.3$, $\lambda=2$, $\beta=0.5$, $\delta=0.05$ and $\gamma=0.2$, we have $b=2.48$ and

$$V(x)=\begin{cases}12.8138\mathrm{e}^{0.0388x}-4.8335\mathrm{e}^{-0.1288x}, & x<2.48;\\ -1.7101\mathrm{e}^{-0.1851x}+0.8x+9.6935, & x\geq 2.48.\end{cases}$$

Example 2 Let $c = 5$, $k = 1.1$, $\lambda = 2$, $\beta = 0.5$, $\delta = 0.05$ and $\gamma = 0.2$. Since $c - \delta[1/s + (\beta c - \lambda)\gamma/(\beta\delta(\delta + \gamma))] = 4.4702 > \lambda k/\beta = 4.4$, we have $b = 0$. Hence $V(x) = -1.0209\mathrm{e}^{-0.1851x} + 0.8x + 11.9163$.

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具有随机观测周期的经典风险模型中最优分红和注资策略

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本文考虑具有随机观测周期的经典风险模型中最优分红和注资策略. 假设破产是被禁止的. 目的是最大化分红折现总额减去资金注射折现总额的期望值. 得到了HJB方程并证明了验证性定理. 并在指数理赔假设下得到了最优控制策略和最优值函数.

关键词: 分红, 资金注射, HJB方程.

学科分类号: O212.62.