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奇异随机偏微分方程中参数MLE的渐近性质 *

张 彩 伢
(浙江大学城市学院计算机与计算科学学院统计系, 杭州, 310015)

摘 要

对一类带有未知参数和小干扰项的奇异随机偏微分方程, 基于连续样本轨道, 给出了参数的极大似然估计, 证明了当干扰项趋于0时, 参数估计量的强相合性和渐近正态性.

关 键 词: 随机偏微分方程, 极大似然估计, 强相合性, 渐近正态性.

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§1. 引 言

众所周知, 偏微分方程可用于随时间和空间变化的复杂系统的建模. 将随机分析的方法引入到偏微分方程中, 便有了随机偏微分方程(Stochastic partial differential equation)(SPDE). 人们利用SPDE建立有关利率、界面动力学、神经心理学和湍流学等随机模型, 不断地取得了显著的成效, 显示了SPDE对于人类认识自然现象和社会现象基本规律的重要性.

有关SPDE的概率理论的研究始于上世纪70年代, 包括不同类型方程的解的存在性、唯一性以及解的性质等问题, Itô (1984), Rozovskii (1990), Ghanem (1999), Balan (2012)等作了较详细的讨论. 有关SPDE的统计推断研究, 包括方程中参数估计问题等, 则始于上世纪90年代初, 较早的研究参见Hübner等(1993), Huebner和Rozovskii (1995), 最近十多年仍有许多学者在讨论(见Prakasa Rao (2000, 2003), Zhang和Lin (2008), Cialenco (2010)及其中的文献).

Prakasa Rao (2003)研究了下列SPDE

$$\mathrm{d}U_\varepsilon(t,x) = b(\theta)(\triangle U_\varepsilon(t,x))\mathrm{d}t + \varepsilon \mathrm{d}W_Q(t,x), \tag{1.1}$$

其中$\triangle = \partial^2/\partial x^2$, Q是$L_2[0,1]$上Wiener过程$W_Q(t,x)$的核协方差算子, θ为未知的参数, ε是干扰项.

设$\mathsf{P}_\theta^\varepsilon$是由$U_\varepsilon(t,x)$在$C[0,T]$上产生的概率测度, Hübner等(1993)证明了$\{\mathsf{P}_\theta^\varepsilon\}$是奇异的. 因此, 我们称模型(1.1)为奇异SPDE.

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当方程(1.1)的解存在时, 其解可以定义为如下形式

$$U_\varepsilon(t,x)=\sum_{i=1}^{\infty}U_{i\varepsilon}(t)e_i(x),$$

其中$\{e_i(x), i\geq 1\}$是算子Q的正交系, $\{U_{i\varepsilon}(t), i\geq 1\}$是$U_\varepsilon(t,x)$的Fourier系数. 基于Fourier系数$U_{i\varepsilon}(t)$在给定时间区间$[0,T]$上的离散观测, Prakasa Rao (2003)先借鉴Bibby和Sørensen (1995)所提出的方法对参数函数$b(\theta)$给出估计, 然后根据变换$\theta=b^{-1}(b(\theta))$给出参数θ的估计量. 在要求$b(\theta)>0$和其他初始条件下, 证明了当Fourier系数的离散观测点数(即样本点数)趋于无穷时, θ估计量的弱相合性和渐近正态性.

本文则基于Fourier系数在给定时间区间$[0,T]$上的连续样本轨道, 对方程中的参数θ直接给出极大似然估计量(MLE), 并在一定的正则条件下(不要求$b(\theta)>0$), 证明了当小干扰项$\varepsilon\uparrow 0$时, MLE的强相合性和渐近正态性.

§2. 假设条件与主要结果

对方程(1.1), 我们假定参数空间Θ为R上的闭区间, 真值θ_0是Θ的一内点, 并考虑特殊的协方差算子$Q=(I-\Delta)^{-1}$, 其中I是单位算子, 则Q的特征根为$q_i=(1+(\pi i)^2)^{-1}=:(1+\lambda_i)^{-1}$, $i\geq 1$, 相应的特征向量为$e_i=\sin(i\pi x)$, $i\geq 1$. 设E是由正交系$\{e_1,\ldots,e_N\}$产生的子空间, $U_\varepsilon^N(t,x)$是$U_\varepsilon(t,x)$在空间E上的投影, 则有

$$U_\varepsilon^N(t,x)=\sum_{i=1}^{N}U_{i\varepsilon}(t)e_i(x),$$

其中Fourier系数$U_{i\varepsilon}(t)$满足下列Ornstein-Uhlenbeck方程

$$\mathrm{d}U_{i\varepsilon}(t)=-b(\theta)\lambda_iU_{i\varepsilon}(t)\mathrm{d}t+\frac{\varepsilon}{\sqrt{\lambda_i+1}}\mathrm{d}W_i(t),\qquad 1\leq i\leq N, \tag{2.1}$$

且满足初始条件

$$U_{i\varepsilon}(0)=v_i,\qquad v_i=\int_0^1\frac{1}{\sqrt{\lambda_i+1}}g(x)e_i(x)\mathrm{d}x,\qquad 1\leq i\leq N.$$

当$\varepsilon=0$时, 方程(2.1)变成如下常微分方程

$$\mathrm{d}U_{i0}(t)=-b(\theta)\lambda_iU_{i0}(t)\mathrm{d}t,\qquad 1\leq i\leq N. \tag{2.2}$$

注记 1 以上理论可参见Rozovskii (1990)的介绍.

现在, 我们来讨论基于Fourier系数$U_{i\varepsilon}(t)$, $1\leq i\leq N$在给定时间区间$[0,T]$上的连续样本轨道, 给出有关方程(1.1)中参数θ的估计问题.

首先, 由Girsanov定理(Girsanov, 1960), 方程(2.1)对应的对数似然函数为

$$L_{i\varepsilon}(\theta)=\frac{1}{\varepsilon}\int_0^T[b(\theta_0)-b(\theta)]\lambda_i\sqrt{\lambda_i+1}U_{i\varepsilon}(t)\mathrm{d}W_i(t)-\frac{1}{2\varepsilon^2}\int_0^T\{[b(\theta_0)-b(\theta)]\lambda_i\sqrt{\lambda_i+1}U_{i\varepsilon}(t)\}^2\mathrm{d}t. \tag{2.3}$$

根据Laredo (1990)理论知, 规划问题$\sup\limits_{\theta\in\Theta}L_{i\varepsilon}(\theta)$的解存在, 因此, 我们可求得方程(2.1)中参数θ的极大似然估计(MLE), 记为$\widehat{\theta}_{i\varepsilon}$, 则有如下主要结果.

定理 2.1 在上述假定条件下, 并设

(I) 对任一$\theta\neq\theta_0$, $b(\theta)\neq b(\theta_0)$;

(II) 对任一$\theta\in\Theta$, $b(\theta)$存在有界一次导数, 记为$b_1(\theta)$;

(III) 存在Θ上的有界测度μ和函数$b_2(\theta)$使得

$$b_1(\theta_1)-b_1(\theta_2)=\int_{\theta_2}^{\theta_1}b_2(\alpha)\mu(\mathrm{d}\alpha)\qquad 对任意\ \theta_1,\theta_2\in\Theta;$$

(IV) 方程(2.2)的解$U_{i0}^{\theta}(t)$满足

$$0<\int_0^T[U_{i0}^{\theta}(t)]^2\mathrm{d}t<+\infty\quad \text{a.s.}\qquad 对任意\ \theta\in\Theta,$$

则估计量$\widehat{\theta}_{i\varepsilon}$满足强相合性, 即有

$$\widehat{\theta}_{i\varepsilon}\to\theta_0\quad \text{a.s.}\qquad 当\ \varepsilon\to 0\ 时.$$

定理 2.2 在定理2.1的条件下, 估计量$\widehat{\theta}_{i\varepsilon}$满足渐近正态性

$$\varepsilon^{-1}(\widehat{\theta}_{i\varepsilon}-\theta_0)\xrightarrow{\mathfrak{D}}N(0,b_i^{-2})\qquad 当\ \varepsilon\to 0\ 时,$$

其中

$$b_i^2=\lambda_i^2(\lambda_i+1)b_1^2(\theta_0)\int_0^T[U_{i0}^{\theta_0}(s)]^2\mathrm{d}s.$$

定义

$$\widehat{\theta}_{\varepsilon}=\Big(\sum_{i=1}^N b_i^2\widehat{\theta}_{i\varepsilon}\Big)\Big/\Big(\sum_{i=1}^N b_i^2\Big).$$

则由$\widehat{\theta}_{i\varepsilon}$, $1\le i\le N$的独立性, 易得以下定理.

定理 2.3 在定理2.1的条件下, 有

$$\widehat{\theta}_{\varepsilon}\to\theta_0\quad \text{a.s.}\qquad 当\ \varepsilon\to 0\ 时,$$

$$\varepsilon^{-1}(\widehat{\theta}_{\varepsilon}-\theta_0)\xrightarrow{\mathfrak{D}}N(0,b^{-2})\qquad 当\ \varepsilon\to 0\ 时,$$

其中$b^2=\sum\limits_{i=1}^N b_i^2$.

注记 2 根据以上定理可知, $\widehat{\theta}_{i\varepsilon}$, $1 \le i \le N$和$\widehat{\theta}_\varepsilon$都是$\theta$的渐近无偏估计, 但$\widehat{\theta}_\varepsilon$比每一$\widehat{\theta}_{i\varepsilon}$, $1 \le i \le N$具有更小的方差.

§3. 定理的证明

为证明定理2.1和定理2.2, 需先给出下列引理.

引理 3.1 设可测函数$\varphi(t)$和$\alpha(t)$, 对某$L > 0$, 满足

$$\varphi(t) \le \alpha(t) + L\int_0^t \varphi(s)\mathrm{d}s.$$

则有

$$\varphi(t) \le \alpha(t) + \int_0^t \mathrm{e}^{L(t-s)}\alpha(s)\mathrm{d}s. \tag{3.1}$$

证明: 参见Gihman和Skorohod (1972)书中P.41证明. □

下文中, 我们记方程(2.1)、(2.2)的解分别为$U_{i\varepsilon}(t)$和$U_{i0}(t)$.

引理 3.2 在定理2.1的条件下, 对任一连续函数$f(x)$, 有

$$\lim_{\varepsilon\to 0}\int_0^T f(U_{i\varepsilon}(t))\mathrm{d}t = \int_0^T f(U_{i0}(t))\mathrm{d}t \qquad \text{a.s.} \tag{3.2}$$

对任一Lipschitz连续函数$f(x)$, 有

$$\lim_{\varepsilon\to 0}\int_0^T f(U_{i\varepsilon}(t))\mathrm{d}W_i(t) = \int_0^T f(U_{i0}(t))\mathrm{d}W_i(t) \qquad \text{a.s.} \tag{3.3}$$

证明: 由(2.1)知, 对任一给定的$\theta \in \Theta$和正数$\varepsilon_1 \ne \varepsilon_2$, 有

$$U_{i\varepsilon_1}(t) - U_{i\varepsilon_2}(t) = \int_0^t (-b(\theta)\lambda_i)(U_{i\varepsilon_1}(s) - U_{i\varepsilon_2}(s))\mathrm{d}s + \int_0^t \frac{\varepsilon_1 - \varepsilon_2}{\sqrt{\lambda_i + 1}}\mathrm{d}W_i(t),$$

则

$$\begin{aligned}
&\sup_{0\le t\le T}|U_{i\varepsilon_1}(t) - U_{i\varepsilon_2}(t)|^2 \\
&\le C_1\Big\{\sup_{0\le t\le T}\Big[\int_0^t |U_{i\varepsilon_1}(s) - U_{i\varepsilon_2}(s)|\mathrm{d}s\Big]^2 + \sup_{0\le t\le T}\Big[\int_0^t |\varepsilon_1 - \varepsilon_2|\mathrm{d}W_i(t)\Big]^2\Big\} \\
&\le C_2\Big\{\int_0^T [U_{i\varepsilon_1}(s) - U_{i\varepsilon_2}(s)]^2\mathrm{d}s + \Big[\int_0^T |\varepsilon_1 - \varepsilon_2|\mathrm{d}W_i(s)\Big]^2\Big\},
\end{aligned}$$

其中$C_1=C_1(\theta,\lambda_i)>0$, $C_2=C_2(\theta,\lambda_i,T)>0$是有限的常量. 因此由随机积分性质可得

$$\begin{aligned}
&\mathsf{E}\sup_{0\le t\le T}|U_{i\varepsilon_1}(t)-U_{i\varepsilon_2}(t)|^2\\
&\le C_2\Big[\int_0^T\mathsf{E}\sup_{0\le s\le t}|U_{i\varepsilon_1}(s)-U_{i\varepsilon_2}(s)|^2\mathrm{d}t+\int_0^T(\varepsilon_1-\varepsilon_2)^2\mathrm{d}t\Big]\\
&=C_2T(\varepsilon_1-\varepsilon_2)^2+C_2\int_0^T\mathsf{E}\sup_{0\le s\le t}|U_{i\varepsilon_1}(s)-U_{i\varepsilon_2}(s)|^2\mathrm{d}t,
\end{aligned}\tag{3.4}$$

又由(3.1), 我们有

$$\begin{aligned}
\mathsf{E}\sup_{0\le t\le T}|U_{i\varepsilon_1}(t)-U_{i\varepsilon_2}(t)|^2&\le C_2T(\varepsilon_1-\varepsilon_2)^2+\int_0^T\mathrm{e}^{C_2(T-t)}C_2t(\varepsilon_1-\varepsilon_2)^2\mathrm{d}t\\
&\le[C_2T+C_2T^2\mathrm{e}^{C_2T}](\varepsilon_1-\varepsilon_2)^2\\
&=:M_1(\varepsilon_1-\varepsilon_2)^2,
\end{aligned}$$

其中$M_1>0$为有限的常数. 故

$$\lim_{\varepsilon_1\to\varepsilon_2}\sup_{0\le t\le T}|U_{i\varepsilon_1}(t)-U_{i\varepsilon_2}(t)|=0\qquad\text{a.s.}$$

我们可取适当的正数ε_0, 使得对任意的$\varepsilon\in(0,\varepsilon_0)$和连续函数$f(\cdot)$, $U_{i\varepsilon}(t)$和$f(U_{i\varepsilon}(t))$都是有界的. 则根据积分有界收敛定理, 即得(3.2)成立.

若$f(x)$是Lipschitz连续函数, 则存在常数$L>0$, 使得

$$|f(x)-f(y)|\le L|x-y|\qquad\text{对任意 } x,y\in(-\infty,+\infty).$$

再次利用随机积分性质并结合(3.4)得

$$\begin{aligned}
\mathsf{E}\Big|\int_0^T f(U_{i\varepsilon_1}(t))-f(U_{i\varepsilon_2}(t))\mathrm{d}W_i(t)\Big|^2&=\int_0^T\mathsf{E}[f(U_{i\varepsilon_1}(t))-f(U_{i\varepsilon_2}(t))]^2\mathrm{d}t\\
&\le L^2\cdot\int_0^T\mathsf{E}[U_{i\varepsilon_1}(t)-U_{i\varepsilon_2}(t)]^2\mathrm{d}t\\
&\le L^2T\cdot\mathsf{E}\sup_{0\le t\le T}[U_{i\varepsilon_1}(t)-U_{i\varepsilon_2}(t)]^2\\
&\le L^2TM_1\cdot(\varepsilon_1-\varepsilon_2)^2\\
&=:M_2(\varepsilon_1-\varepsilon_2)^2,
\end{aligned}$$

其中$M_2>0$为有限的常数. 因此(3.3)得证. □

引理 3.3 在定理2.1的条件下, 存在有限的常数$M>0$使得

$$\lim_{\varepsilon\to0}\sup_{\theta\in\Theta}\int_0^T[b(\theta)-b(\theta_0)]U_{i\varepsilon}(t)\mathrm{d}W_i(t)\le M\qquad\text{a.s.}$$

证明: 首先由定理2.1的条件(I), (II)和Cauchy-Schwarz不等式, 得

$$\begin{aligned}\Big\{\int_0^T[b(\theta)-b(\theta_0)]U_{i\varepsilon}(t)\mathrm{d}W_i(t)\Big\}^2&=\Big\{\int_0^T\int_{\theta_0}^{\theta}b_1(\alpha)U_{i\varepsilon}(t)\mathrm{d}\alpha\mathrm{d}W_i(t)\Big\}^2\\&=\Big\{\int_{\theta_0}^{\theta}b_1(\alpha)\int_0^T U_{i\varepsilon}(t)\mathrm{d}W_i(t)\mathrm{d}\alpha\Big\}^2\\&\le\int_{\theta_0}^{\theta}b_1^2(\alpha)\mathrm{d}\alpha\int_{\theta_0}^{\theta}\Big\{\int_0^T U_{i\varepsilon}(t)\mathrm{d}W_i(t)\Big\}^2\mathrm{d}\alpha,\end{aligned}$$

即有

$$\begin{aligned}\mathsf{E}\sup_{\theta\in\Theta}\Big\{\int_0^T[b(\theta)-b(\theta_0)]U_{i\varepsilon}(t)\mathrm{d}W_i(t)\Big\}^2&\le\mathsf{E}\int_\Theta b_1^2(\alpha)\mathrm{d}\alpha\int_\Theta\Big\{\int_0^T U_{i\varepsilon}(t)\mathrm{d}W_i(t)\Big\}^2\mathrm{d}\alpha\\&=\int_\Theta b_1^2(\alpha)\mathrm{d}\alpha\cdot\int_\Theta\int_0^T\mathsf{E}U_{i\varepsilon}^2(t)\mathrm{d}t\mathrm{d}\alpha.\end{aligned}$$

又由(3.2)可得

$$\lim_{\varepsilon\to0}\mathsf{E}\sup_{\theta\in\Theta}\Big\{\int_0^T[b(\theta)-b(\theta_0)]U_{i\varepsilon}(t)\mathrm{d}W_i(t)\Big\}^2\le\int_\Theta b_1^2(\alpha)\mathrm{d}\alpha\cdot\int_\Theta\int_0^T\mathsf{E}U_{i0}^2(t)\mathrm{d}t\mathrm{d}\alpha.$$

因此, 在定理2.1的条件下, 存在有限的常数$M>0$使得

$$\lim_{\varepsilon\to0}\sup_{\theta\in\Theta}\int_0^T[b(\theta)-b(\theta_0)]U_{i\varepsilon}(t)\mathrm{d}W_i(t)\le M\qquad\text{a.s.}\qquad\square$$

现在, 我们来证明定理2.1.

定理2.1的证明: 首先由(2.3), 我们可将$\varepsilon^2L_{i\varepsilon}(\theta)$分解成两部分, 第一部分是随机积分项, 第二部分是普通积分项, 即

$$\begin{aligned}\varepsilon^2L_{i\varepsilon}(\theta)=\varepsilon&\int_0^T(b(\theta_0)-b(\theta))\lambda_i\sqrt{\lambda_i+1}U_{i\varepsilon}(t)\mathrm{d}W_i(t)\\&-\frac12\int_0^T[b(\theta)-b(\theta_0)]^2\lambda_i^2(1+\lambda_i)U_{i\varepsilon}^2(t)\mathrm{d}t.\end{aligned}\tag{3.5}$$

由引理3.3, 得

$$\lim_{\varepsilon\to0}\sup_{\theta\in\Theta}\varepsilon\int_0^T(b(\theta_0)-b(\theta))\lambda_i\sqrt{\lambda_i+1}U_{i\varepsilon}(t)\mathrm{d}W_i(t)=0\qquad\text{a.s.}\tag{3.6}$$

因此, 我们只需对普通积分项进行讨论.

定义

$$K_{i\varepsilon}(\theta)=\int_0^T[b(\theta)-b(\theta_0)]^2U_{i\varepsilon}^2(t)\mathrm{d}t.$$

根据(3.2), 对任一给定的$\theta \in \Theta$, 有

$$\lim_{\varepsilon \to 0} |K_{i\varepsilon}(\theta) - K_{i0}(\theta)| = 0 \qquad \text{a.s.} \tag{3.7}$$

利用定理2.1中的条件(I)和(III)并结合引理3.2, 我们可取正数ε_0和M使得, 对任意$\theta_1, \theta_2 \in \Theta$和$\varepsilon \in [0, \varepsilon_0]$, 有

$$|K_{i\varepsilon}(\theta_1) - K_{i\varepsilon}(\theta_2)| \leq M|\theta_1 - \theta_2| \qquad \text{a.s.} \tag{3.8}$$

因Θ是闭区间, 根据有限覆盖定理, 对任意的$\delta > 0$, 存在有限个点$\theta_1, \ldots, \theta_n$使得

$$\Theta \subset \bigcup_{j=1}^{n} \{\theta : |\theta - \theta_j| < \delta\}.$$

结合(3.7)和(3.8), 可推得

$$\begin{aligned} &\lim_{\varepsilon \to 0} \sup_{\theta \in \Theta} |K_{i\varepsilon}(\theta) - K_{i0}(\theta)| \\ \leq\ & \lim_{\varepsilon \to 0} \sup_{1 \leq j \leq n} |K_{i\varepsilon}(\theta_j) - K_{i0}(\theta_j)| + \lim_{\varepsilon \to 0} \sup_{|\theta_1 - \theta_2| < \delta} |K_{i\varepsilon}(\theta_1) - K_{i\varepsilon}(\theta_2)| \\ &+ \sup_{|\theta_1 - \theta_2| < \delta} |K_{i0}(\theta_1) - K_{i0}(\theta_2)| \\ \leq\ & 2M\delta \qquad \text{a.s.} \end{aligned}$$

再由δ的任意性, 我们有

$$\lim_{\varepsilon \to 0} \sup_{\theta \in \Theta} |K_{i\varepsilon}(\theta) - K_{i0}(\theta)| = 0 \qquad \text{a.s.} \tag{3.9}$$

另外, 由定理的条件可知, θ_0是规划问题$\inf\limits_{\theta \in \Theta} K_{i0}(\theta)$的唯一解. 同时, 由(3.8)易证, $K_{i\varepsilon}(\theta)$和$K_{i0}(\theta)$都关于θ连续. 因此, 结合(3.5), (3.6)和(3.9)可知, 规划问题$\sup\limits_{\theta \in \Theta} L_{i\varepsilon}(\theta)$的解$\widehat{\theta}_{i\varepsilon}$满足

$$\lim_{\varepsilon \to 0} \widehat{\theta}_{i\varepsilon} = \theta_0 \qquad \text{a.s.} \qquad \square$$

接下去, 我们来证明$\widehat{\theta}_{i\varepsilon}$的渐近正态性.

定理2.2的证明: 定义

$$h(\theta) = \int_0^1 \Big\{ \int_{\theta_0}^{\theta_0 + \gamma(\theta - \theta_0)} b_2(\beta)\mu(d\beta) \Big\} \mathrm{d}\gamma.$$

则在定理2.1的条件下, 有

$$b(\theta) - b(\theta_0) = [h(\theta) + b_1(\theta_0)](\theta - \theta_0).$$

由(2.3), 我们又可将$L_{i\varepsilon}(\theta)$分解为

$$L_{i\varepsilon}(\theta) = [A_{i\varepsilon} + A_{i\varepsilon}(\theta)][\varepsilon^{-1}(\theta_0 - \theta)] - \frac{1}{2}[B_{i\varepsilon} + B_{i\varepsilon}(\theta)][\varepsilon^{-1}(\theta - \theta_0)]^2, \tag{3.10}$$

其中

$$A_{i\varepsilon}=\lambda_i\sqrt{\lambda_i+1}b_1(\theta_0)\int_0^T U_{i\varepsilon}(t)\mathrm{d}W_i(t),$$

$$B_{i\varepsilon}=\lambda_i^2(\lambda_i+1)b_1^2(\theta_0)\int_0^T U_{i\varepsilon}^2(t)\mathrm{d}t,$$

$$A_{i\varepsilon}(\theta)=\lambda_i\sqrt{\lambda_i+1}\int_0^T h(\theta)U_{i\varepsilon}(t)\mathrm{d}W_i(t),$$

$$B_{i\varepsilon}(\theta)=\lambda_i^2(\lambda_i+1)\int_0^T [h^2(\theta)+2h(\theta)b_1(\theta_0)]U_{i\varepsilon}^2(t)\mathrm{d}t.$$

下面我们将分别对这四项进行讨论.

首先, 根据随机积分中心极限定理(参见Mckean, 1969), 有

$$A_{i\varepsilon}\xrightarrow{\mathfrak{D}}N(0,b_i^2)\qquad \text{当 }\varepsilon\to 0\text{ 时},\tag{3.11}$$

其中

$$b_i^2=\lambda_i^2(\lambda_i+1)b_1^2(\theta_0)\int_0^T[U_{i0}^{\theta_0}(s)]^2\mathrm{d}s.$$

又由引理3.2可得

$$B_{i\varepsilon}\to b_i^2\quad \text{a.s.}\qquad \text{当 }\varepsilon\to 0\text{ 时}.\tag{3.12}$$

现在, 我们来考虑第三项$A_{i\varepsilon}(\theta)$. 注意到$h(\theta)$的定义并利用Cauchy-Schwarz不等式, 有

$$|A_{i\varepsilon}(\widehat{\theta}_{i\varepsilon})|^2\le\mu\{[\theta_0,\widehat{\theta}_{i\varepsilon}]\}\int_\Theta\Big\{\int_0^T b_2(\beta)\lambda_i\sqrt{1+\lambda_i}U_{i\varepsilon}(t)\mathrm{d}W_i(t)\Big\}^2\mu(\mathrm{d}\beta),$$

因此, 仿照引理3.3的证明方法, 我们可取常数$\varepsilon_0>0$和$M>0$使得对任意$\varepsilon\in[0,\varepsilon_0]$, 有

$$\mathsf{E}[A_{i\varepsilon}(\widehat{\theta}_{i\varepsilon})]^2\le M\mu\{[\theta_0,\widehat{\theta}_{i\varepsilon}]\}.$$

由测度μ的连续性和定理2.1的结果, $\lim\limits_{\varepsilon\to0}\widehat{\theta}_{i\varepsilon}=\theta_0$ a.s., 可得

$$A_{i\varepsilon}(\widehat{\theta}_{i\varepsilon})\to 0\quad \text{a.s.}\qquad \text{当 }\varepsilon\to 0\text{ 时}.$$

利用同样的技巧, 可以证明

$$B_{i\varepsilon}(\widehat{\theta}_{i\varepsilon})\to 0\quad \text{a.s.}\qquad \text{当 }\varepsilon\to 0\text{ 时}.$$

令

$$\widetilde{\theta}_{i\varepsilon}=\theta_0-\varepsilon B_{i\varepsilon}^{-1}A_{i\varepsilon}.$$

则由(3.11)和(3.12)知, 当$\varepsilon\to 0$时,

$$\widetilde{\theta}_{i\varepsilon}\to\theta_0\quad \text{a.s.}\qquad \text{和}\qquad \varepsilon^{-1}(\widetilde{\theta}_{i\varepsilon}-\theta_0)\xrightarrow{\mathfrak{D}}N(0,b_i^{-2}).\tag{3.13}$$

根据$L(\widehat{\theta}_{i\varepsilon})=\sup\limits_{\theta\in\Theta}L_{i\varepsilon}(\theta)\geq L(\widetilde{\theta}_{i\varepsilon})$, 并由(3.10)可推得

$$
\begin{aligned}
&[(\varepsilon^{-1}(\widehat{\theta}_{i\varepsilon}-\theta_0))^2-(\varepsilon^{-1}(\widetilde{\theta}_{i\varepsilon}-\theta_0))^2]+2A_{i\varepsilon}B_{i\varepsilon}^{-1}[(\varepsilon^{-1}(\widehat{\theta}_{i\varepsilon}-\theta_0))-(\varepsilon^{-1}(\widetilde{\theta}_{i\varepsilon}-\theta_0))]\\
\leq{}& 2B_{i\varepsilon}^{-1}(R_\varepsilon(\widehat{\theta}_{i\varepsilon})-R_\varepsilon(\widetilde{\theta}_{i\varepsilon})),
\end{aligned}
$$

其中

$$
R_\varepsilon(\theta)=A_{i\varepsilon}(\theta)(\varepsilon^{-1}(\theta_0-\theta))-\frac{1}{2}B_{i\varepsilon}(\theta)(\varepsilon^{-1}(\theta_0-\theta))^2.
$$

注意到$A_{i\varepsilon}B_{i\varepsilon}^{-1}=-\varepsilon^{-1}(\widetilde{\theta}_{i\varepsilon}-\theta_0)$, 则有

$$
[(\varepsilon^{-1}(\widehat{\theta}_{i\varepsilon}-\theta_0))-(\varepsilon^{-1}(\widetilde{\theta}_{i\varepsilon}-\theta_0))]^2\leq 2B_{i\varepsilon}^{-1}(R_\varepsilon(\widehat{\theta}_{i\varepsilon})-R_\varepsilon(\widetilde{\theta}_{i\varepsilon})).
$$

根据以上证明可知, 当$\varepsilon\to 0$时, $R_\varepsilon(\widehat{\theta}_{i\varepsilon})\to 0$ a.s.且$R_\varepsilon(\widetilde{\theta}_{i\varepsilon})\to 0$ a.s., 从而

$$
|\varepsilon^{-1}(\widehat{\theta}_{i\varepsilon}-\theta_0)-\varepsilon^{-1}(\widetilde{\theta}_{i\varepsilon}-\theta_0)|\to 0 \qquad \text{a.s.}
$$

因此, $\varepsilon^{-1}(\widehat{\theta}_{i\varepsilon}-\theta_0)$具有和$\varepsilon^{-1}(\widetilde{\theta}_{i\varepsilon}-\theta_0)$相同的渐近分布. 由(3.13), 即得

$$
\varepsilon^{-1}(\widehat{\theta}_{i\varepsilon}-\theta_0)\xrightarrow{\mathfrak{D}}N(0,b_i^{-2}) \qquad \text{当 } \varepsilon\to 0 \text{ 时}.
$$

定理证毕. □

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Asymptotic Properties about MLE of the Parameter in Some Singular Stochastic Partial Differential Equation

Zhang Caiya

(*Department of Statistics, School of Computer and Computing Science, Zhejiang University City College, Hangzhou, 310015*)

In this paper, the singular stochastic partial differential equation with an unknown parameter and a small noise is studied. The maximum likelihood estimator of the parameter based on the continuous observation of the Fourier coefficients is proposed. The strong convergence and asymptotic normality of the estimator are established as the noise tends to zero.

Keywords: Stochastic partial differential equation, maximum likelihood estimator, strong consistency, asymptotic normality.

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