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综述报告

Practical Criterion for Uniqueness of Q-Processes *

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Abstract

The note begins with a short story on seeking for a practical sufficiency theorem for the uniqueness of time-continuous Markov jump processes, starting around 1977. The general result was obtained in 1985 for the processes with general state spaces. To see the sufficient conditions are sharp, a dual criterion for non-uniqueness was obtained in 1991. This note is restricted however to the discrete state space (then the processes are called Q-processes or Markov chains), for which the sufficient conditions just mentioned are showing at the end of the note to be necessary. Some examples are included to illustrate that the sufficient conditions either for uniqueness or for non-uniqueness are not only powerful but also sharp.

Keywords: Criterion, uniqueness, Markov chain, Markov jump process.

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Let E be a countable set with elements $i, j, k, \ldots$. A matrix $Q = (q_{ij} : i, j \in E)$ is called a Q-matrix if its non-diagonals are nonnegative and $\sum\limits_{j\in E} q_{ij} \le 0$ for every $i \in E$. Throughout this note, we restrict ourselves to the special case that the Q-matrix is totally stable $q_i := -q_{ii} < \infty$ and conservative $q_i = \sum\limits_{j\neq i} q_{ij}$ for every $i \in E$. It is called bounded if $\sup\limits_{i\in E} q_i < \infty$. For a given Q-matrix $Q = (q_{ij})$ on E, a sub-Markovian semigroup $\{P(t) = (p_{ij}(t) : i, j \in E)\}_{t\ge 0}$ is called a Q-process if

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t)\Big|_{t=0} = Q \qquad \text{(pointwise)}.$$

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The Q-processes may not be unique in general, but there always exists the minimal one, due to Feller (1940, Theorem 1), denoted by $P^{\min}(t) = \big(p_{ij}^{\min}(t) : i, j \in E\big)$. For more than half-century ago, some criteria for the uniqueness were known.

Theorem 1 The Q-process is unique (equivalently, the minimal process $P^{\min}(t)$ is not explosive) iff one of the following equivalent conditions holds:

(C1) $\sum\limits_{j\in E} p_{ij}^{\min}(t) = 1$ for every $i \in E$ and $t \geq 0$.

(C2) $\sum\limits_{n=1}^{\infty} q_{X^{\min}(\tau_n)}^{-1} = \infty$, P_i-a.s., where τ_n is the nth jump time of the minimal process $\{X^{\min}(t) : t \geq 0\}$ corresponding to $P^{\min}(t)$.

(C3) The equation

$$(\lambda I - Q)u = 0, \qquad 0 \leq u \leq 1, \tag{1}$$

has only zero solution for some (equivalently, for all) $\lambda > 0$.

Criterion (C1) goes back to Feller (1940). Criterion (C2) is due to Dobrushin (1952). Criterion (C3) is due to Feller (1957) and Reuter (1957). Refer also to Chung (1967; Part II, § 19, Theorem 1), or Gikhman and Skorokhod (1975; Chap. 3, § 2, Theorems 3 and 4).

The earlier Criterion (C1) often requires a further effort in practice, rather than a direct application. In particular, the proof of the powerful sufficiency theorem (Theorem 2 below) is based on it.

Criterion (C2) is effective in some cases. For instance in the simplest case that $M := \sup\limits_{i\in E} q_i < \infty$, since

$$\sum_{n=1}^{\infty} q_{X^{\min}(\tau_n)}^{-1} \geq \sum_{n=1}^{\infty} M^{-1} = \infty,$$

we obtain the uniqueness of the processes. For pure birth process (i.e., $q_{i,i+1} > 0$ and $q_{ij} = 0$ for all $j \neq i$, $i, j \geq 0$), Criterion (C2) says that the process is unique iff

$$\sum_{n=1}^{\infty} \frac{1}{q_{n,n+1}} = \infty. \tag{2}$$

Besides, if the minimal process is recurrent, then the term q_k^{-1} will appears infinitely often in the summation, hence the process should be unique according to the criterion.

Criterion (C3) is more effective once equation (1) is solvable. More precisely, it is the case if the exit boundary consists at most a single point, for instance the pure birth processes, the birth-death processes or more general the single birth processes (i.e., for $j > i \geq 0$, $q_{ij} > 0$ iff $j = i + 1$; for $0 \leq j < i$, q_{ij} is nonnegative but free). We will come back this story soon.

However, the next model stopped our study for several years at the beginning of the study (1977 – 1978) on non-equilibrium particle systems. To state our model, we use operator Ω instead of the matrix Q:

$$\Omega f(i) = \sum_{j \in E} q_{ij}(f_j - f_i), \qquad i \in E.$$

Of course, in this case, $\Omega f = Qf$. For a Markov chain on a countable set E, by a transform, one often assumes that E is simply the set $\mathbb{Z}_+ = \{0, 1, \ldots\}$. However, such a transform ignores the original geometry of E and may not be convenient in multidimensional case. To state our model, we need some notation. Let $i = (i_u : u \in S)$ and define its updates $i^{u\pm}$ and $i^{u,v}$ as follows:

$$i_w^{u\pm} = \begin{cases} i_u \pm 1 & w = u; \\ i_w & w \neq u, \end{cases} \qquad i_w^{u,v} = \begin{cases} i_u - 1 & w = u; \\ i_v + 1 & w = v; \\ i_w & w \neq u, v, \end{cases} \qquad w \in S.$$

Example 1 (Schlögl's second model) Let S be a finite set and $E = \mathbb{Z}_+^S$. Define a Markov chain on E with operator

$$\begin{aligned} \Omega f(i) = & \sum_{u \in S} \big\{ b(i_u)\big[f\big(i^{u+}\big) - f(i)\big] + a(i_u)\big[f\big(i^{u-}\big) - f(i)\big] \big\} \\ & + \sum_{u,v} i_u p(u,v)\big[f\big(i^{u,v}\big) - f(i)\big], \qquad i = (i_u : u \in S) \in E, \end{aligned}$$

where $(p(u,v) : u, v \in S)$ is a "simple" random walk on S, and

$$\begin{aligned} b(k) &= \beta_0 + \beta_2 k(k-1), \qquad \beta_0,\, \beta_2 > 0, \\ a(k) &= \delta_1 k + \delta_3 k(k-1)(k-2), \qquad \delta_1,\, \delta_3 > 0. \end{aligned}$$

Here in the first sum of Ω, in each vessel u, there is a birth-death process with birth rate $b(k)$ and death rate $a(k)$, respectively. This is called the reaction part of the model. The reactions in different vessels are independent. In the second sum of Ω, a particle from vessel u moves to vessel v. This is called the diffusion part of the model. Thus, it is actually a finite-dimensional reaction-diffusion processes. Replacing the finite S with $S = \mathbb{Z}^d$, we obtain formally an operator of infinite-dimensional reaction-diffusion process which is a typical model from the non-equilibrium statistical physics. Even though the large systems are quite popular today, in that period, it was rather unusual to study such a non-equilibrium system. Our original program is to rebuild the mathematical ground of non-equilibrium statistical physics (cf. Chen (2004; Part IV). An earlier paper on this topic appeared in 1985 (see Chen, 1985)). For this, the model is meaningful only if it

is ergodic in every finite dimension. Thus, the finite dimensional model consists the first doorsill of our program.

In 1983, the author and Yan (see Yan and Chen, 1986), using a comparison technique, overcame this doorsill, based on a systemic study on the single birth processes. To which, we obtained explicit criteria not only for uniqueness but also for ergodicity and so on. This goes back to Yan and Chen (1986), Chen (1986a). Refer to Chen (2004) for updates and to Chen and Zhang (2014) for a unified treatment. After two more years, using an approximating approach, we obtained a powerful sufficiency theorem as stated below.

Theorem 2 (Uniqueness criterion) Let $Q = (q_{ij})$ be a Q-matrix on a countable set E. Then the corresponding Q-process is unique iff the following two conditions hold simultaneously.

(U1) There exist $E_n \uparrow E$ as $n \uparrow \infty$ and a nonnegative function φ such that $\sup\limits_{i\in E_n} q_i < \infty$ and $\lim\limits_{n\to\infty} \inf\limits_{i\notin E_n} \varphi_i = \infty$.

(U2) There exists a constant $c \in \mathbb{R}$ such that $Q\varphi \le c\varphi$.

Certainly, for Schlögl's model for instance, in condition (U2), it is more convenient to use $\Omega\varphi$ instead of $Q\varphi$. Besides, an important fact should be very helpful in practice: if φ satisfies the conditions with $c \ge 0$, then so does $M + \varphi$ for every constant $M \ge 0$. In particular, a local modification of Q does not interfere the conclusion.

From Chen (2004; Parts I and II), it is now clear that a large part of the theory of Q-processes can be generalized to the so-called Markov jump processes on general state space. To save the space, we will not really go to the last subject but it is worth to mention the extension. We now use the codes "GS" and "DS" to distinguish the "general state space" and the "discrete state space", respectively. The sufficient part of the last theorem first appeared in Chen (1986a; Theorem 2.37 (GS)) and Chen (1986b; Theorem (16) (GS)). Because it is regarded as one of the author's favourite contributions to the theory of Markov jump processes, this result was then introduced several times in the author's publications: Chen (1991; Theorem 1.11 (DS)), Chen and Yan (1991; Theorem 3.9 (GS)), Chen (2004; Theorem 2.25 (GS)), Chen (1997; Theorem 2.1 (DS)), Chen (2005; Theorem 9.4 (DS)), and Chen and Mao (2007; Theorem 2.9 (DS)).

Theorem 2 is often accompanied in the publications just listed by the next simpler result.

Corollary 1 Suppose that there exist a function $\varphi \ge q$ and a constant $c \in \mathbb{R}$ such that $Q\varphi \le c\varphi$ on E. Then the Q-process is unique.

Proof Set $E_n = \{i \in E : q_i \le n\}$. If $M := \sup\limits_{i\in E} q_i < \infty$, then for large enough n,

we have $E_n = E$ and so $\inf\limits_{k\notin E_n} q_k = \infty$ by standard convention $\inf\limits_{\emptyset}\varphi = \infty$. In this case, condition (U2) is trivial with $\varphi = 1 + M$. If $M = \infty$, then $\inf\limits_{k\notin E_n} q_k \geq n \to \infty$ as $n \to \infty$. Combining this with (U2), the conclusion follows from Theorem 2. □

Corollary 1 is almost explicit since one can simply specify $\varphi = 1 + q$. This enables us to use it easier in practice. However, such a specification makes the assumption becomes a little stronger. We will come back this point later.

Let us make some remarks about the conditions in Theorem 2. Condition (U2) is a relax of the equation in (1): finding a solution to an inequality is easier than finding a solution to the corresponding equality. Criterion (C3) says that there is only trivial bounded solution to the equation (1). Conversely, if a solution of the equation is fixed at some point, say θ, such that $\varphi_\theta = 1$, then the solution φ should be unbounded. This leads to the condition $\lim\limits_{n\to\infty}\inf\limits_{k\notin E_n}\varphi_k = \infty$ in (U1). Using this idea, we prove that the assumptions in Theorem 2 are necessary for single birth processes (see Chen, 2004; Remark 3.20). The reason we allow some subset of E_n to be infinite is to rule out some region of E, on which $\sup\limits_{i\in E_n} q_i < \infty$. The key in the proof of this result is an economic approximation by bounded Q-processes. Certainly, the necessity shows that the assumptions of the theorem are sharp, and is valuable as illustrated by Chen (1986b; Theorem (25)). However, it does not mean that the inverse of the conditions can be used in practice to show the non-uniqueness of the processes. Hence, we went to an opposite way proving the following criterion (see Chen, 2004; Theorem 2.27 (GS), its proof in 2nd edition uses Lemma 5.18 rather than Lemma 5.15).

Theorem 3 (Non-uniqueness criterion) For a given Q-matrix Q on a countable set E, the Q-processes are not unique if for some (equivalently, for all) $c > 0$, there is a bounded function φ with $\sup\limits_{k\in E}\varphi_k > 0$ such that $Q\varphi \geq c\varphi$. Conversely, these conditions plus $\varphi \geq 0$ are also necessary.

We remark that three results (Theorems 2, 3 and Corollary 1), we have talked so far are specialized from their original case in GS to the one in DS. Theorems 2 and 3 are somehow the extensions of Criterion (C3) in two opposite directions. As we will see soon that the extended theorems are much effective than the original Criterion (C3). Using two opposite sufficiency results instead of a single criterion is often meaningful. For instance, for recurrence, we have a criterion (see Chen, 2004; Proposition 4.21) which is accompanied with more practical criteria (see Chen, 2004; Theorems 4.24 and 4.25) for the recurrence and transiency, respectively. As a companion to Chen (2004; Theorem 4.25), refer to Meyn and Tweedie (2009; Theorem 8.0.2) and Hairer (2010; Proposition 1.3) or more recent criteria. Next, for ergodicity and nonergodicity, refer to Chen (2004;

Theorem 4.45 (1)) and Kim and Lee (2008; Theorem 1), respectively. For various stability speeds/principal eigenvalues, in Chen (2005), we have not only the classical variational formula, but also dual variational formulas to describe their lower and upper bounds, respectively.

It is interesting that there is now a direct way to prove the necessity of Theorem 2 in the context of DS based on a recent result by Spieksma (2014).

Theorem 4 Everything is the same as in Theorem 2 except (U1) is replaced by

(U1)′ In the original (U1), assume in addition that each E_n is finite and ignore "$\sup_{i\in E_n} q_i < \infty$".

It is now the position to illustrate by examples the power of our results and compare conditions (U1) and (U1)′.

The next two examples show that in Theorem 2, the condition "$\lim_{n\to\infty} \varphi_n = \infty$" is not necessary, which is however necessary in a criterion for recurrence used in the proof of Theorem 4 (see its proof below).

Example 2 Let E be a countable set and $Q = (q_{ij})$ be a bounded conservative Q-matrix on E. Then assumptions of Theorem 2 hold but its test function φ can be bounded.

Proof (a) Simply set $E_n \equiv E$ (may be infinite) for every $n \geq 1$ and $\varphi_i \equiv 1$. Then it is obvious that $0 = Q\varphi \leq \varphi$ and $\lim_n \inf_{i\notin E_n} \varphi_i = \infty$ since $\inf_{\emptyset} \varphi = \infty$ by the standard convention. Hence by Theorem 2, the process is unique. As we have seen before, Corollary 1 is also applicable in such a trivial case.

(b) Knowing that the process is unique, then by Theorem 4, there should exist a φ satisfying (U1)′, as well as (U2). The problem is that the resulting φ is not explicitly known when E is infinite. In this sense, Theorem 4 is theoretic correct but not practical in such simplest case. □

Example 3 Let $E = \mathbb{Z}_+$ and $Q^{(1)}$ be a bounded conservative Q-matrix on E. Denote its test function by $\varphi^{(1)} \equiv 1$ as in the last example. Next, let $Q^{(2)}$ be a conservative Q-matrix on E satisfying the assumptions of Theorem 2 with a sequence of finite subsets $\{E_n\}_{n\geq 1}$ and a test function $\varphi^{(2)}$. Finally, we construct a new Q as follows: on the odd numbers in E, we use the transition mechanism of $Q^{(1)}$, and on the even numbers in E, we adopt the one of $Q^{(2)}$. Define $\varphi = \varphi^{(1)}$ on the odd numbers and $\varphi = \varphi^{(2)}$ on the even numbers. Then the assumptions of Theorem 2 hold but its test function φ_n has no limit as $n \to \infty$: $\overline{\lim}_{n\to\infty} \varphi_n = \infty$ and $\underline{\lim}_{n\to\infty} \varphi_n = 1$.

Proof First, note that for the original $Q^{(2)}$ on E, because each E_n is a finite subset

of E, the condition $\lim\limits_{n\to\infty}\inf\limits_{k\notin E_n}\varphi_k^{(2)}=\infty$ is equivalent to $\lim\limits_{n\to\infty}\varphi_n^{(2)}=\infty$. Therefore, we have

$$\overline{\lim_{n\to\infty}}\,\varphi_n=\lim_{n\to\infty}\varphi_n^{(2)}=\infty,\qquad \underline{\lim_{n\to\infty}}\,\varphi_n=\lim_{n\to\infty}\varphi_n^{(1)}=1.$$

To show the assumptions in Theorem 2 hold, simply let $E_0=\{\text{odd integers}\}$, and let $E_n\,(n\geq 1)$ be the union of E_0 and the natural modification of the original E_n used for $Q^{(2)}$. Then the resulting $E_n\uparrow E$ as $n\to\infty$, $\sup\limits_{k\in E_n}q_k<\infty$ for each $n\geq 0$, and

$$\lim_{n\to\infty}\inf_{k\notin E_n}\varphi_k=\lim_{n\to\infty}\inf_{k\notin E_n}\varphi_k^{(2)}=\lim_{n\to\infty}\varphi_n^{(2)}=\infty.$$

Finally, because of the independence of $Q^{(1)}$ and $Q^{(2)}$, $\varphi^{(1)}$ and $\varphi^{(2)}$, the condition $Q\varphi\leq\max\{c_2,1\}\varphi$ on the set of odd numbers follows from

$$Q^{(1)}\varphi^{(1)}\leq\varphi^{(1)}\qquad\text{on}\ \ E;$$

and the same condition on the set of even numbers follows from

$$Q^{(2)}\varphi^{(2)}\leq c_2\varphi^{(2)}\qquad\text{on}\ \ E.$$

We have thus obtained the required conclusion.

As mentioned in the last proof, in the present situation, we do not know how to use Theorem 4. □

Note that the last matrix Q is reducible. However, we can add a connection between 0 and 1 to produce an irreducible version of the example. This is not essential since a local modification does not interfere the uniqueness problem. Furthermore, one may replace the set {odd integers} or {even integers} by any infinite subset of E, but not E itself, the set of primer numbers for instance. The conclusion of Example 3 remains the same by an obvious modification.

The point is that some E_n is allowed to be infinite in (U1) but not in (U1)′.

Example 4 The pure birth process is unique iff (2) holds. In particular, set $q_{n,n+1}=$ the nth primer, then Theorem 2 is suitable but Corollary 1 fails.

Proof Note that $q_k=q_{k,k+1}$ for $k\geq 0$.

(a) If $\sum\limits_k q_k^{-1}=\infty$, set $E_n=\{0,1,\ldots,n\}$ and

$$\varphi_k=1+\sum_{1\leq j\leq k-1}\frac{1}{q_j}\to\infty\qquad\text{as}\ k\to\infty.$$

Then $Q\varphi\leq\varphi$ and so Theorem 2 gives us the uniqueness of the processes. In the particular case that $q_{n,n+1}=n+1$, the above φ has order $\log n$. However, we can also choose $\varphi_n=1+n$ and apply Theorem 2. This shows that there are some freedom in choosing φ.

(b) If $M := \sum_k q_k^{-1} < \infty$, set E_n as above and

$$\varphi_k = \frac{1}{2} + \sum_{1 \le j \le k-1} \frac{1}{q_j} - M \in \Big[\frac{1}{2} - M, \frac{1}{2}\Big].$$

Then $\sup_k \varphi_k = 1/2 > 0$, $Q\varphi \ge \varphi$, and so by Theorem 3, the processes are not unique. We remark that it would be awful to use the necessity in Theorems 2 or 4 to prove this non-uniqueness property.

(c) The last assertion is due to J.L. Zheng (cf. Chen, 1986a; Example 2.3.12; or Chen, 2004; Example 2.26). □

Proof of the uniqueness for Example 1 For $i \in E = \mathbb{Z}_+^S$, define its level by $|i| = \sum_{u \in S} i_u$ and set $E_n = \{i \in E : |i| \le n\}$ for $n \ge 1$.

(a) Next, define $\varphi(i) = 1 + |i|$. Then it is clear that $\lim_{n\to\infty} \inf_{k \notin E_n} \varphi(k) = \infty$. Because the diffusions do not change the levels, we have

$$\Omega\varphi(|i|) = \sum_{u \in S} [b(i_u) - a(i_u)] = \sum_{u \in S} \big[\alpha_0 - \alpha_1 i_u + \alpha_2 i_u^2 - \alpha_3 i_u^3\big]$$

for some positive $\{\alpha_k\}_{k=0}^3$. Next, since

$$\sum_{u \in S} i_u^2 \le |i|^2, \qquad \frac{1}{|S|} \sum_{u \in S} i_u^3 \ge \Big(\frac{|i|}{|S|}\Big)^3 \text{ (Jensen's inequality)},$$

where $|S|$ is the cardinality of S (finite but arbitrary), we have

$$\Omega\varphi(|i|) \le \alpha_0' - \alpha_1'|i| + \alpha_2'|i|^2 - \alpha_3'|i|^3$$

for some positive $\{\alpha_k'\}_{k=0}^3$. Now, because the right-hand side becomes negative for large enough $|i|$, it is clear that $\Omega\varphi(|i|) \le c\varphi(|i|)$ for every $i \in E$ and large enough c. The assertion now follows from Theorem 2. Hopefully, we have seen the role played by the geometry of E. The proof shows the power of our result. A good sufficiency result may be more effective than a criterion.

(b) It is also possible to use Corollary 1 to prove the required assertion, simply choose $\varphi(i) = \gamma\Big(1 + \sum_{u \in S} i_u^3\Big)$. First, choose γ large enough so that $\varphi \ge q$. Next, choose c large enough so that $\Omega\varphi \le c\varphi$. □

It is worthy to mention that in accompany to Theorem 2, we also have a similar, practical sufficiency result for (exponential) ergodicity. Refer to Chen (1989; Theorem 3 (GS)), Chen (1991; Theorem 1.18 (DS)), Chen (2004; Corollary 4.49 (DS) and Theorem 14.1 (GS)).

In the past nearly 30 years, Theorem 2 and Corollary 1 have very successful applications. A list of the literature was collected in Chen (2005; §9.2). Certainly, the results used a lot by the author (in Chen (2004) for instance). In particular, it was used at the first step to construct a large class of infinite-dimensional processes (see Chen, 2004; §13.2), 15 models are included in Chen (2004; §13.4). Corollary 1 with some extension was used by Song (1988) in a quite earlier stage for Markov decision processes moving from bounded to unbounded situation. It is now quite often to see the influence of the study on Markov jump processes to the theory of Markov decision processes. Based on Chen (1986b), Theorem 2 was collected into Anderson (1991; Corollary 2.2.16), its originality was unfortunately ignored, even though the original paper (see Chen; 1986b) is included in the references of the book. For some corrections and comments on the last book, refer to Chen (1996). Very recently, Theorem 2 (GS) is applied by Chen and Ma (2014) to genetic study having continuous state space. Finally, we mention that the results have already extended to the time-inhomogeneous case by Zheng and Zheng (1987) and Zheng (1993) using the martingale approach.

Before going to the proofs, note that equation (1) is equivalent to

$$\Pi(\lambda)u = u, \qquad 0 \le u \le 1 \ \text{ on } \ E, \qquad \lambda > 0, \tag{3}$$

where

$$\Pi(\lambda) = \Big(\frac{(1-\delta_{ij})q_{ij}}{\lambda + q_i} : i, j \in E\Big).$$

Here the matrix $\Pi(\lambda)$ is sub-stochastic. We introduce a fictitious state Δ and define on the enlarged state space $E_\Delta = E \cup \{\Delta\}$ a new transition probability matrix

$$\Pi_{ij}^{\Delta}(\lambda) = \begin{cases} \Pi_{ij}(\lambda) & \text{if } i, j \in E; \\ \dfrac{\lambda}{\lambda + q_i} & \text{if } i \in E, \ j = \Delta; \\ p_j & \text{if } i = \Delta, \ j \in E, \end{cases}$$

where $(p_j : j \in E)$ is a positive probability measure on E. The enlarged transition probability matrix is irreducible even the original one may be not.

Lemma 1 The equation (1) has zero solution only iff so does the equation

$$\Pi^{\Delta}(\lambda)(u\mathbb{1}_E) = u, \qquad 0 \le u \le 1 \ \text{ on } \ E_\Delta, \qquad \lambda > 0. \tag{4}$$

Thus, the original Q-process is unique iff the $\Pi^{\Delta}(\lambda)$-chain is recurrent.

Proof Noting that $u_\Delta = \sum_{k \in E} p_k u_k$, it is clear that $u_\Delta = 0$ iff $u_k = 0$ for all $k \in E$ since $p_k > 0$ for all $k \in E$. Equation (4) restricted to E coincides with (3) and then (1). This proves the first assertion.

To prove the second assertion, it suffices to note that $\Pi^{\Delta}(\lambda)$-chain is recurrent iff equation (4) has only trivial solution. The last result comes from Yan and Chen (1986), Chen (1986a; Lemma 12.1.27), or Chen (2004; Lemma 4.51). We remark here that the regularity assumption used in the cited references can be replaced by the minimal process, due to the equivalence of recurrence of the minimal process and its embedded chain. Refer to Chen (1986a; Lemma 12.3.1), or Chen (2004; Theorem 4.34). □

Proof of Theorem 4 When $|E| < \infty$, the conclusion is trivial and the assumptions hold for the specific $E_n \equiv E$ and $\varphi_i \equiv 1$ as seen from proof (a) of Example 2. Hence we may assume that $E = \mathbb{Z}_+$. Since each E_n is finite, the condition $\lim\limits_{n} \inf\limits_{k \notin E_n} \varphi_k = \infty$ becomes $\lim\limits_{n \to \infty} \varphi_n = \infty$. In this case, conditions (U1)$'$ and (U2) consist a criterion for the recurrence of the Markov chain $\Pi^{\Delta}(\lambda)$, refer to Chen (2004; Theorem 4.24) and its references within.

We remark that it is at this point, the finiteness of E_n is required and so the present sufficiency proof is not suitable for Theorem 2. At the moment, we do not know how to extend the necessity result of Theorem 4 from DS to GS.

Here is a part of an alternative proof given in Spieksma (2014). Let $P^{\min}(\lambda)$ be the Laplace transform of $P^{\min}(t)$. Using the second successive approximation scheme for the backward Kolmogorov equation (goes back to Feller (1940; Theorem 1)), we obtain

$$P^{\min}(\lambda) = \sum_{n=0}^{\infty} \Pi(\lambda)^n \mathrm{diag}\Big(\frac{1}{\lambda + q}\Big)$$

(cf. Chen, 2004, page 75, line -6). Hence

$$\lambda P^{\min}(\lambda) \text{ column } (1) = \sum_{n=0}^{\infty} \Pi(\lambda)^n \text{ column } \Big(\frac{\lambda}{\lambda + q}\Big).$$

The process is unique iff the left-hand side equals 1 at some/every $i \in E$, the right-hand side is the probabilistic decomposition of the time that the Markov chain $\Pi^{\Delta}(\lambda)$ starts from some $i \in E$, first visits Δ at some step $n \geq 1$, which equals 1 iff the irreducible Markov chain $\Pi^{\Delta}(\lambda)$ is recurrent. We have thus come back to the last lemma. □

Proof of Theorem 2 Here we adopt a circle argument.

(U1)$'$ + (U2) $\Longrightarrow$ (U1) + (U2). This is easy since (U1) is weaker that (U1)$'$.

(U1)+(U2) $\Longrightarrow$ uniqueness. This is the sufficiency part of Theorem 2 and was proved long time ago, even for GS.

Uniqueness $\Longrightarrow$ (U1)$'$ + (U2). This is the necessity part of Theorem 4. □

We remark that a similar phenomena is appeared in Theorem 3, the conditions for sufficiency are weaker than the ones for necessity. As we have seen from Example 4, this is very helpful in practice. However, these conditions are actually equivalent: conditions for necessity $\Longrightarrow$ conditions for sufficiency $\Longrightarrow$ non-uniqueness $\Longrightarrow$ conditions for necessity.

In view of these discussions, one may combine Theorems 2 and 4 into one having the style of Theorem 3.

In conclusion, this note as well as the practice during the past 30 years confirm that the sufficient part of Theorem 2 and Theorem (Criterion) 3 are not only powerful but also sharp, even though at the moment we are still unable to prove the necessity part of Theorem 2 for general state spaces.

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Q过程唯一性的实用判别准则

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首先简要介绍自1977年左右开始的寻找连续时间马尔可夫跳过程实用的唯一性充分条件的故事. 对于一般状态空间的一般性结果是1985年得到的. 为展示这些充分条件的精确性, 于1991年找到非唯一性的对偶判别准则. 本文主要限于离散空间(此时的跳过程也称为Q过程或马尔可夫链). 在这种情况下, 我们将在文末证明上述充分条件也是必要的. 我们还将举例说明不论对于唯一性或非唯一性, 我们的充分条件不仅强有力, 而且精确.

关键词: 判别准则, 唯一性, 马尔可夫链, 马尔可夫跳过程.

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