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Differentiability and Asymptotic Properties of Gerber-Shiu Function Associated with Absolute Ruin Time for a Risk Model with Random Premium Income *

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Abstract

In this paper, the differentiability and asymptotic properties of Gerber-Shiu expected discounted penalty function (Gerber-Shiu function for short) associated with the absolute ruin time are investigated, where the risk model is given by classical risk model with additional random premium incomes. The additional random premium income process is specified by a compound Poisson process. A couple of integro-differential equations satisfied by Gerber-Shiu function are derived, several sufficient conditions which guarantee the second-order or third-order differentiability of Gerber-Shiu function are provided. Based on the differentiability results, when the individual claim and premium income are both exponential distribution, the previous integro-differential equations can be deduced into a third-order constant ordinary differential equation (ODE for short). With the standard techniques on ODE, we find the asymptotic behavior of absolute ruin probability when the initial surplus tends to infinity.

Keywords: Absolute ruin time, Gerber-Shiu function, random premium income, differentiability, asymptotic property.

AMS Subject Classification: 60K05, 62P05, 90A46.

§1. Introduction

In the classical risk model, the premium rate is constant and the ruin time is defined as the first time that the surplus drops below zero. However, under safety loading assumption, the surplus will be positive soon or later, thus the insurer can maintain the practice by a loan. Consequently, debit interest has to be taken by the insurer for covering the negative by the loan. This leads to the concept of absolute ruin, which occurs when the premiums received are not sufficient to make the interest payments on the debt. Gerber (1971)

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considered the probability of absolute ruin in the compound Poisson model when the debt and credit interest rates are the same. A closed-form solution is given in the case of an exponential claim amount distribution. Substantial refinements and generalizations are given by Dassios and Embrechts (1989). From then on, the absolute ruin problem has attracted a lot of interest in the study of risk theory. Cai and Yang (2014) studied the decomposition of absolute ruin probability under a compound Poisson risk model with diffusion. For other literatures concentrating on this topic, see Gerber and Yang (2007), Yang et al. (2008) and references therein. A unified approach to cope with this quantities is to study the Gerber-Shiu expected discounted penalty function, which is introduced by Gerber and Shiu (1997). Now, the Gerber-Shiu function has become a standard concept to study ruin theory in all kinds of risk models, see Yuan and Hu (2008), Zhu and Yang (2008, 2009) and the references therein for detailed discussions.

Another extension on classical risk model is to take the randomness of premium income into account. Melnikov (2004) studied the ruin probabilities in the risk model with stochastic premium incomes and all capital of an insurer with such surplus process was invested in stock market. Stepped literatures can be found in Pan and Wang (2009), Wei et al. (2008), Xu et al. (2014) and Yu (2013) et al.. In view of this, it is natural to extend the study of Gerber-Shiu penalty function associated with absolute ruin time for risk model with fixed premium income rate to the one with random premium income and debit interest. In this paper, we focus on the case that the premium incomes follow a compound Poisson process. It is worthy of mentioning that Albrecher et al. (2010) studied the Gerber-Shiu function associated ruin time for a model with both aggregate premium process and aggregate claim process are compound Poisson process, therefore, the problem discussed in this paper can be viewed as a generalization of Albrecher et al. (2010) to the case of absolute ruin time. As results, several sufficient conditions which guarantee the second-order or third-order differentiability of Gerber-Shiu function are presented. Based on the differentiability results, when the individual premium and individual claim are both exponential distribution, the previous integro-differential equations can be transformed into a third-order ODE with constant coefficients. With standard techniques on ODE, we find the asymptotic estimation of absolute ruin probability when the initial surplus tends to infinity.

The rest of this paper is organized as follows. Section 2 presents an introduction to the model and the problem. Section 3 presents the sufficient conditions that guarantee the differentiability of the Gerber-Shiu function. Section 4 discusses the asymptotic behavior of the ruin probability when the initial surplus goes to infinity.

§2. Model and Problem

In this paper, the risk process with stochastic premium income is specified as

$$U(t)=u+pt+\sum_{i=1}^{N_1(t)}X_i-\sum_{i=1}^{N_2(t)}Y_i,\qquad t\geq 0,\tag{2.1}$$

where $U(t)$ is insurer's surplus at time t, $u\geq 0$ is the insurer's initial surplus, p is the constant premium income rate, $\sum\limits_{i=1}^{N_1(t)}X_i$ is the extra stochastic premiums received up to time t, $N_1(t)$ is a Poisson process with intensity λ_1, denotes the number of extra premiums arrived up to time t. $\sum\limits_{i=1}^{N_2(t)}Y_i$ is the total claims by time t, $N_2(t)$ is a Poisson process with intensity λ_2, denotes the number of claims arrived up to time t.

To proceed our explorations, we make the following assumptions. The individual premium amounts $X_1,X_2,\ldots$ are independent, identically distributed positive random variables with common distribution $F(x)=\mathsf{P}(X_1\leq x)$, $F(0)=0$, $\mathsf{E}X_1=\mu_1$. The individual claim amounts $Y_1,Y_2,\ldots$ are independent, identically distributed positive random variables with common distribution $G(y)=\mathsf{P}(Y_1\leq y)$, $G(0)=0$, $\mathsf{E}Y_1=\mu_2$. The safety loading condition holds: $pt+\mathsf{E}\big(\sum\limits_{i=1}^{N_1(t)}X_i\big)>\mathsf{E}\big(\sum\limits_{i=1}^{N_2(t)}Y_i\big)$, or equivalently, $p+\lambda_1\mu_1>\lambda_2\mu_2$. Processes $\{N_1(t),t\geq 0\}$, $\{X_i,i\geq 0\}$, $\{N_2(t),t\geq 0\}$ and $\{Y_i,i\geq 0\}$ are mutually independent.

We assume that when the surplus is negative or the insurer is on deficit, the insurer could borrow an amount of money equal to the deficits with a debit interest force $\delta>0$. Meanwhile, the insurer will repay debts continuously from her premium income. When the mean premium income can not cover the debit, we define that the absolute ruin occurs. That is to say once the negative surplus is below $-c/\delta$, where $c=p+\lambda_1\mu_1$, we say that absolute ruin occurs at this situation.

Denote by $U_\delta(t)$ the surplus at time t of the insurer with the debit interest force δ, the dynamic of $U_\delta(t)$ is specified as

$$\mathrm{d}U_\delta(t)=\delta U_\delta(t)I(U_\delta(t)<0)\mathrm{d}t+\mathrm{d}Y(t)-\mathrm{d}Z(t),\tag{2.2}$$

where $Y(t)=\sum\limits_{i=1}^{N_1(t)}X_i$, $Z(t)=\sum\limits_{i=1}^{N_2(t)}Y_i$ and $I(C)$ denotes the indictor function of an event C throughout this paper. Denote the absolute ruin time of the surplus process $\{U_\delta(t),t\geq 0\}$ by T_δ, i.e.

$$T_\delta=\inf\{t\geq 0:U_\delta(t)\leq -c/\delta\}\tag{2.3}$$

with the convention that $T_\delta=\infty$ if $U_\delta(t)>-c/\delta$ for all $t\geq 0$. Let $U_\delta(T_\delta^-)$ and $|U_\delta(T_\delta)|$ denote the surplus immediately before absolute ruin time and the deficit at absolute ruin time respectively. Note that the deficit at absolute ruin time is at least c/δ and the surplus immediately before absolute ruin time is in the range of $(-c/\delta, \infty)$.

Remark 1 In classical risk model or model with constant premium income rate, "absolute ruin" is a easygoing concept since once the premium income does not cover the debit, the ruin will occurs definitely. However, under a risk model with stochastic premium income process, it is not easy to verify such deterministic behavior. Here, we adopt the idea presented in Gerber and Yang (2007), where absolute ruin problem is studied under the risk model with diffusion and the "absolute ruin time" is defined as the time that the risk process "does not make profit in average", i.e. the expected total profit of insurer is negative. Thus, the absolute ruin time and absolute ruin probability in this paper is similar to the ones studied in Gerber and Yang (2007).

The Gerber-Shiu function associated to the absolute ruin time is defined as

$$\Phi(u) = \mathsf{E}\big(\mathrm{e}^{-\alpha T_\delta}\omega(U_\delta(T_\delta^-), |U_\delta(T_\delta)|)I(T_\delta < \infty)|U_\delta(0) = u\big), \tag{2.4}$$

where $\omega(x_1, x_2)$, $x_1 > -c/\delta$, $x_2 \geq c/\delta$ is a bounded, non-negative function and denotes the penalty due at absolute ruin. One should note that $\Phi(u)$ has different sample paths for $u \geq 0$ and $-c/\delta < u < 0$. Hence, to distinguish the two situations, write $\Phi(u) = \Phi_+(u)$ for $u \geq 0$ and $\Phi(u) = \Phi_-(u)$ for $-c/\delta < u < 0$.

§3. Differentiability and Integro-Differential Equations

In this section, we concentrate on the integral equations that satisfied by $\Phi_+(u)$ and $\Phi_-(u)$ firstly, and then we present sufficient conditions for differentiability of $\Phi_+(u)$ and $\Phi_-(u)$ respectively. The following Theorem 3.1 and Theorem 3.2 are basic and the corresponding proofs are just directly application of the renewal techniques, thus we omit them here.

Theorem 3.1 For $u \geq 0$,

$$\begin{aligned}(\lambda_1 + \lambda_2 + \alpha)\Phi_+(u) = \lambda_1 \int_0^\infty \Phi_+(u+x)\mathrm{d}F(x) + \lambda_2\Big[\int_0^u \Phi_+(u-y)\mathrm{d}G(y) \\ + \int_u^{u+c/\delta} \Phi_-(u-y)\mathrm{d}G(y) + A(u)\Big] + p\Phi_+'(u),\end{aligned} \tag{3.1}$$

where $A(u) = \int_{u+c/\delta}^\infty \omega(u, y-u)\mathrm{d}G(y)$.

Theorem 3.2 $\Phi_-(u)$ satisfies the following integro-differential equation

$$\begin{aligned}(\lambda_1+\lambda_2+\alpha)\Phi_-(u) = (u\delta+p)\Phi_-'(u) + \lambda_1\Big[\int_0^{-u}\Phi_-(u+x)\mathrm{d}F(x) \\ + \int_{-u}^{\infty}\Phi_+(u+x)\mathrm{d}F(x)\Big] + \lambda_2\Big[\int_0^{u+c/\delta}\Phi_-(u-y)\mathrm{d}G(y) \\ + \int_{u+c/\delta}^{\infty}\omega(u,y-u)\mathrm{d}G(y)\Big].\end{aligned} \tag{3.2}$$

Theorem 3.3 Suppose that $F(x)$ has density function $f(y)$, $G(y)$ has density function $g(y)$ and

(1) $\omega(x,y)$ in Equation (2.4) is bounded in $x > -c/\delta$, $y \geq c/\delta$;

(2) $f(x)$ and $g(y)$ are twice continuously differentiable on $[0,\infty)$ with $\int_0^\infty |f'(x)|\mathrm{d}x < \infty$, $\int_0^\infty |f''(x)|\mathrm{d}x < \infty$, $\int_0^\infty |g'(y)|\mathrm{d}y < \infty$ and $\int_0^\infty |g''(y)|\mathrm{d}y < \infty$;

(3) $A(y)$ is twice continuously differentiable on $[0,\infty)$ and both $A'(y)$ and $A''(y)$ are bounded on $[0,\infty)$.

Then $\Phi_+(u)$ is third continuously differentiable in $u \geq 0$ and both $\Phi_+'(u)$, $\Phi_+''(u)$ and $\Phi_+'''(u)$ are bounded in $u \geq 0$.

Proof Introduce

$$k_1(u) = \int_0^\infty \Phi_+(u+x)f(x)\mathrm{d}x = \int_u^\infty \Phi_+(t)f(t-u)\mathrm{d}t, \tag{3.3}$$

$$k_2(u) = \int_0^u \Phi_+(u-y)g(y)\mathrm{d}y = \int_0^u \Phi_+(t)g(u-t)\mathrm{d}t, \tag{3.4}$$

$$k_3(u) = \int_u^{u+c/\delta} \Phi_-(u-y)g(y)\mathrm{d}y = \int_{-c/\delta}^0 \Phi_-(t)g(u-t)\mathrm{d}t, \tag{3.5}$$

and rewrite Equation(3.1) as

$$\Phi_+'(u) = \frac{\lambda_1+\lambda_2+\alpha}{p}\Phi_+(u) - \frac{\lambda_1}{p}k_1(u) - \frac{\lambda_2}{p}(k_2(u)+k_3(u)+A(u)). \tag{3.6}$$

We first show that $\Phi_+'(u)$ is continuous and bounded in $u \geq 0$. Note that $\omega(x,y) \leq M$ for some constant M large enough, naturally, we have $0 \leq \Phi_+(u) \leq M\mathsf{P}(T_\delta < \infty) \leq M$ and $0 \leq \Phi_-(u) \leq M\mathsf{P}(T_\delta < \infty) \leq M$. By assumption (1), it follows that $\Phi_+(u)$ and $\Phi_-(u)$ are bounded and consequently, it is easy to see that $k_1(u)$, $k_2(u)$ and $k_3(u)$ are continuous since $f(u)$ and $g(u)$ is continuous and $\Phi_+(u)$ and $\Phi_-(u)$ are bounded. Furthermore, $k_1(u)$, $k_2(u)$ and $k_3(u)$ are bounded. Thus, by Equation (3.6), $\Phi_+'(u)$ is continuous and bounded. We next prove that $\Phi_+(u)$ is twice continuously differentiable. Since $\Phi_+(u)$ is continuous and $f(u)$ is continuously differentiable in $u \geq 0$, we have that $k_1(u)$ is continuously differentiable when $u \geq 0$ and

$$k_1'(u) = -\Phi_+(u)f(0+) - \int_u^{\infty} \Phi_+(t)f'(t-u)\mathrm{d}t. \tag{3.7}$$

Together with assumption (2) and note that the $\Phi_+(u)$ is bounded, we have

$$|k_1'(u)| \leq M_1 \qquad \text{for some constants } M_1 > 0. \tag{3.8}$$

With the same discussion, we have both $k_2'(u)$ and $k_3'(u)$ are continuously differentiable and

$$|k_2'(u)| \leq M_2 \qquad \text{for some constants } M_2 > 0, \tag{3.9}$$

$$|k_3'(u)| \leq M_3 \qquad \text{for some constants } M_3 > 0. \tag{3.10}$$

Equation (3.8) and Equation (3.10) imply that $\Phi_+''(u)$ is continuous, bounded and is specified as

$$\Phi_+''(u) = \frac{\lambda_1+\lambda_2+\alpha}{p}\Phi_+'(u) - \frac{\lambda_1}{p}k_1'(u) - \frac{\lambda_2}{p}(k_2'(u)+k_3'(u)+A'(u)). \tag{3.11}$$

We now prove that $\Phi_+(u)$ is third continuously differentiable. By Equation (3.8) and Equation (3.11), it follows that

$$k_1''(u) = -\Phi_+'(u)f(0+) + \Phi_+(u)f'(0+) + \int_u^{\infty} \Phi_+(t)f''(t-u)\mathrm{d}t. \tag{3.12}$$

Together with assumption (3) and note that $\Phi_+(u)$ and $\Phi_+'(u)$ are bounded, we have

$$|k_1''(u)| \leq M_4 \qquad \text{for some constants } M_4 > 0. \tag{3.13}$$

With a similar discussion, we can have $k_2''(u)$ and $k_3''(u)$ are bounded and the following equations holds

$$\Phi_+'''(u) = \frac{\lambda_1+\lambda_2+\alpha}{p}\Phi_+''(u) - \frac{\lambda_1}{p}k_1''(u) - \frac{\lambda_2}{p}(k_2''(u)+k_3''(u)+A''(u)) \tag{3.14}$$

is continuous and bounded. □

Theorem 3.4 Suppose that $F(x)$ has density function $f(y)$, $G(y)$ has density function $g(y)$ and

(1) $\omega(x,y)$ in Equation (2.4) is bounded in $x > -c/\delta$, $y \geq c/\delta$;

(2) $f(x)$ and $g(y)$ are twice continuously differentiable on $[0,\infty)$ with $\int_0^{\infty}|f'(x)|\mathrm{d}x < \infty$, $\int_0^{\infty}|f''(x)|\mathrm{d}x < \infty$, $\int_0^{\infty}|g'(y)|\mathrm{d}y < \infty$ and $\int_0^{\infty}|g''(y)|\mathrm{d}y < \infty$;

(3) $A(y)$ is twice continuously differentiable on $(-c/\delta, 0)$ and both $A'(y)$ and $A''(y)$ are bounded on $(-c/\delta, 0)$.

Then $\Phi_-(u)$ is third continuously differentiable in $-c/\delta < u < 0$ and both $\Phi_-'(u)$, $\Phi_-''(u)$ and $\Phi_-'''(u)$ are bounded in $-c/\delta < u < 0$.

Proof Let

$$h_1(u) = \int_0^{-u} \Phi_-(u+x)f(x)\mathrm{d}x = \int_u^0 \Phi_-(t)f(t-u)\mathrm{d}t, \tag{3.15}$$

$$h_2(u) = \int_{-u}^{\infty} \Phi_+(u+x)f(x)\mathrm{d}x = \int_0^{\infty} \Phi_+(t)f(t-u)\mathrm{d}t, \tag{3.16}$$

$$h_3(u) = \int_0^{u+c/\delta} \Phi_-(u-y)g(y)\mathrm{d}y = \int_{-c/\delta}^{u} \Phi_-(t)g(u-t)\mathrm{d}t, \tag{3.17}$$

rewrite Equation (3.2) as

$$\Phi'_-(u) = \frac{\lambda_1+\lambda_2+\alpha}{u\delta+p}\Phi_-(u) - \frac{\lambda_1}{u\delta+p}(h_1(u)+h_2(u)) - \frac{\lambda_2}{u\delta+p}(h_3(u)+A(u)). \tag{3.18}$$

It is easy to see that $\Phi'_-(u)$ is continuous and bounded. We next prove that $\Phi_-(u)$ is twice continuously differentiable. Since $\Phi_-(u)$ is continuous and $f(u)$ is continuously differentiable in $u>0$, we have that $h_1(u)$ is continuously differentiable when $-c/\delta < u < 0$ and

$$h'_1(u) = -\Phi_-(u)f(0+) - \int_u^0 \Phi_-(t)f'(t-u)\mathrm{d}t. \tag{3.19}$$

Together with assumption (2) and note that the $\Phi_-(u)$ is bounded, we have

$$|h'_1(u)| \le N_1 \qquad \text{for some constants } N_1 > 0. \tag{3.20}$$

With the same discussion, we have both $h'_2(u)$ and $h'_3(u)$ are continuously differentiable and

$$|h'_2(u)| \le N_2 \qquad \text{for some constants } N_2 > 0, \tag{3.21}$$

$$|h'_3(u)| \le N_3 \qquad \text{for some constants } N_3 > 0. \tag{3.22}$$

Equation (3.20) to Equation (3.22) imply that $\Phi''_-(u)$ is continuous, bounded and is specified as

$$\begin{aligned}\Phi''_-(u) = {}& \frac{(\lambda_1+\lambda_2+\alpha)\Phi'_-(u) - \lambda_1(h'_1(u)+h'_2(u)) - \lambda_3(h'_3(u)+A'(u))}{u\delta+p} \\ & - \frac{(\lambda_1+\lambda_2+\alpha)\Phi_-(u)\delta - \lambda_1(h_1(u)+h_2(u))\delta - \lambda_2(h_3(u)+A(u))\delta}{(u\delta+p)^2}.\end{aligned} \tag{3.23}$$

Further, similarly, we can show that $\Phi'''_-(u)$ is continuous and bounded in $-c/\delta < u < 0$. □

§4. Exponential Case

In this section we illustrate the impact of stochastic premium income on the Gerber-Shiu function and ruin probability and present an explicit expression for the solution of integro-differential equation obtained in Section 3 when both individual premium and claim follow exponential distribution. As it was described in Albrecher et al. (2010), exponential distribution is of great importance in risk research since we can approximate any individual claims or individual premiums by a linear combination of exponential distribution. Let $F(x) = 1 - \mathrm{e}^{-ax}$, $G(y) = 1 - \mathrm{e}^{-by}$, $a > 0$, $b > 0$, $\lambda_1/a > \lambda_2/b$, and $\omega(x, y) \equiv 1$. Introduce $\xi_+(u) = 1 - \Phi_+(u)$ and $\xi_-(u) = 1 - \Phi_-(u)$, then Equation (3.1) can be reformulated as

$$\begin{aligned}
&-p\xi_+'(u) + (\lambda_1 + \lambda_2 + \alpha)\xi_+(u)\\
&= \alpha + \lambda_1 \int_0^\infty \xi_+(u + x)a\mathrm{e}^{-ax}\mathrm{d}x\\
&\quad + \lambda_2\Big[\int_0^u \xi_+(u - y)b\mathrm{e}^{-by}\mathrm{d}y + \int_u^{u+c/\delta} \xi_-(u - y)b\mathrm{e}^{-by}\mathrm{d}y\Big].
\end{aligned} \tag{4.1}$$

By taking the derivative with respect to u on both sides of Equation (4.1), we have

$$\begin{aligned}
-p\xi_+''(u) = &-(\lambda_1 + \lambda_2 + \alpha)\xi_+'(u)\lambda_1\Big[-a\xi_+(u) + a\int_0^\infty \xi_+(u + x)a\mathrm{e}^{-ax}\mathrm{d}x\Big]\\
&+ \lambda_2\Big[b\xi_+(u) - b\int_0^u \xi_+(u - y)b\mathrm{e}^{-by}\mathrm{d}y - b\int_u^{u+c/\delta} \xi_-(u - y)b\mathrm{e}^{-by}\mathrm{d}y\Big].
\end{aligned}$$

Taking the derivative with respect to u again yields

$$\begin{aligned}
&-p\xi_+'''(u) + (\lambda_1 + \lambda_2 + \alpha)\xi_+''(u) + (\lambda_1 a - \lambda_2 b)\xi_+'(u) + (\lambda_1 a^2 + \lambda_2 b^2)\xi_+(u)\\
&= \lambda_1 a^2 \int_0^\infty \xi_+(u + x)a\mathrm{e}^{-ax}\mathrm{d}x + \lambda_2 b^2\Big[\int_0^u \xi_+(u - y)b\mathrm{e}^{-by}\mathrm{d}y + \int_u^{u+c/\delta} \xi_-(u - y)b\mathrm{e}^{-by}\mathrm{d}y\Big].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&-p(a - b)\xi_+''(u) + [-pab + (a - b)(\lambda_1 + \lambda_2 + \alpha)]\xi_+'(u)\\
&+ (ab\alpha + \lambda_1 a^2 + \lambda_2 b^2)\xi_+(u) - ab\alpha\\
&= \lambda_1 a^2 \int_0^\infty \xi_+(u + x)a\mathrm{e}^{-ax}\mathrm{d}x\\
&\quad + \lambda_2 b^2\Big[\int_0^u \xi_+(u - y)b\mathrm{e}^{-by}\mathrm{d}y + \int_u^{u+c/\delta} \xi_-(u - y)b\mathrm{e}^{-by}\mathrm{d}y\Big],
\end{aligned} \tag{4.2}$$

together with Equation (4.2), it is followed with

$$p\xi_+'''(u) - (pa - pb + \lambda_1 + \lambda_2 + \alpha)\xi_+''(u) - (pab - \lambda_2 a - a\alpha + \lambda_1 b + b\alpha)\xi_+'(u) + ab\alpha\xi_+(u) = ab\alpha \tag{4.3}$$

and

$$\begin{aligned}
&-(u\delta+p)\xi_-'(u)+(\lambda_1+\lambda_2+\alpha)\xi_-(u)\\
=&\ \alpha+\lambda_1\Big[\int_0^{-u}\xi_-(u+x)a\mathrm{e}^{-ax}\mathrm{d}x+\int_{-u}^{\infty}\xi_+(u+x)a\mathrm{e}^{-ax}\mathrm{d}x\Big]\\
&+\lambda_2\int_0^{u+c/\delta}\xi_-(u-y)b\mathrm{e}^{-by}\mathrm{d}y. \qquad (4.4)
\end{aligned}$$

Taking derivative with respect to u on both sides of (4.4) yields

$$\begin{aligned}
&-(u\delta+p)\xi_-''(u)+(\lambda_1+\lambda_2+\alpha-\delta)\xi_-'(u)+(\lambda_1a-\lambda_2b)\xi_-(u)\\
=&\ \lambda_1a\Big[\int_0^{-u}\xi_-(u+x)a\mathrm{e}^{-ax}\mathrm{d}x+\int_{-u}^{\infty}\xi_+(u+x)a\mathrm{e}^{-ax}\mathrm{d}x\Big]\\
&-\lambda_2b\int_0^{u+c/\delta}\xi_-(u-y)b\mathrm{e}^{-by}\mathrm{d}y. \qquad (4.5)
\end{aligned}$$

Repeating the previous step, we have

$$\begin{aligned}
&-(u\delta+p)\xi_-'''(u)+(\lambda_1+\lambda_2+\alpha-2\delta)\xi_-''(u)+(\lambda_1a-\lambda_2b)\xi_-'(u)+(\lambda_1a^2+\lambda_2b^2)\xi_-(u)\\
=&\ \lambda_1a^2\Big[\int_0^{-u}\xi_-(u+x)a\mathrm{e}^{-ax}\mathrm{d}x+\int_{-u}^{\infty}\xi_+(u+x)a\mathrm{e}^{-ax}\mathrm{d}x\Big]\\
&+\lambda_2b^2\int_0^{u+c/\delta}\xi_-(u-y)b\mathrm{e}^{-by}\mathrm{d}y. \qquad (4.6)
\end{aligned}$$

Let $(4.4)\times ab+(4.5)\times(a-b)$, it follows that

$$\begin{aligned}
&-(u\delta+p)(a-b)\xi_-''(u)+[-(u\delta+p)ab+(a-b)(\lambda_1+\lambda_2+\alpha-\delta)]\xi_-'(u)\\
&+(\lambda_1a^2+\lambda_2b^2+ab\alpha)\xi_-(u)-ab\alpha\\
=&\ \lambda_1a^2\Big[\int_0^{-u}\xi_-(u+x)a\mathrm{e}^{-ax}\mathrm{d}x+\int_{-u}^{\infty}\xi_+(u+x)a\mathrm{e}^{-ax}\mathrm{d}x\Big]\\
&+\lambda_2b^2\int_0^{u+c/\delta}\xi_-(u-y)b\mathrm{e}^{-by}\mathrm{d}y, \qquad (4.7)
\end{aligned}$$

together with (4.6), it follows that

$$\begin{aligned}
&(u\delta+p)\xi_-'''(u)-[(u\delta+p)(a-b)+\lambda_1+\lambda_2+\alpha-2\delta]\xi_-''(u)\\
&+[-(u\delta+p)ab+\lambda_2a+a\alpha-a\delta-b\lambda_1-b\alpha+b\delta]\xi_-'(u)+ab\alpha\xi_-(u)\\
=&\ ab\alpha. \qquad (4.8)
\end{aligned}$$

In the rest of this section, we focuses on the asymptotic estimation when initial surplus approaches to infinity, say $u\to\infty$, thus we just need to concentrate on Equation (4.3). Note that Equation (4.3) takes the form of the following third-order constant linear ordinary differential equation

$$y'''_{xxx} + a_2 y''_{xx} + a_1 y'_x + a_0 y = \frac{ab\alpha}{p}, \tag{4.9}$$

whose corresponding homogeneous equation is

$$y'''_{xxx} + a_2 y''_{xx} + a_1 y'_x + a_0 y = 0. \tag{4.10}$$

Denote by

$$\zeta(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0. \tag{4.11}$$

Then there are two probably solutions to the homogeneous equation.

(1) If the characteristic polynomial $\zeta(\lambda)$ is factorizable and

$$\zeta(\lambda) = (\lambda - \lambda_{01})(\lambda - \lambda_{02})(\lambda - \lambda_{03}), \tag{4.12}$$

where $\lambda_{01}, \lambda_{02}, \lambda_{03}$ are real numbers. Solution:

$$y = \begin{cases} C_1 \mathrm{e}^{\lambda_{01}x} + C_2^{\lambda_{02}x} + C_3^{\lambda_{03}x} & \text{if all roots are different;} \\ (C_1 + C_2 x)\mathrm{e}^{\lambda_{01}x} + C_3^{\lambda_{03}x} & \text{if } \lambda_{01} = \lambda_{02} \neq \lambda_{03}; \\ (C_1 + C_2 x + C_3 x^2)\mathrm{e}^{\lambda_{01}x} & \text{if } \lambda_{01} = \lambda_{02} \neq \lambda_{03}. \end{cases} \tag{4.13}$$

(2) If

$$\zeta(\lambda) = (\lambda - \lambda_{01})(\lambda^2 - k_1\lambda + k_0) = 0, \tag{4.14}$$

then the solution is of the form

$$y = C_1 \mathrm{e}^{\lambda_{01}x} + \mathrm{e}^{-k_1 x}(\cos\theta x + C_3 \sin\theta x), \tag{4.15}$$

where $\theta = \sqrt{b_0 - b_1^2}$.

Obviously, since the ruin time $T_\delta \to \infty$ when $u \to \infty$, it follows that $\Psi_+(u) \to 0$ and $\xi_+(u)$ is positive and $\xi_+(u) \to 1$. Note that $\xi_+(u) \equiv 1$ is a special solution to Equation (4.3), together with preceding analysis and Equations (4.13)-(4.15), we claim that asymptotic estimation of the Gerber-Shiu penalty function is

$$\Psi_+(u) \sim \widetilde{C}\mathrm{e}^{\widetilde{\lambda}_{01}u}, \qquad u \to \infty, \tag{4.16}$$

where the coefficients $\widetilde{C}$ is the largest real number among C_1, C_2, C_3 subject to the constraints $\{\lambda_k > 0\}$, which is to be determined according to the solution of characteristic polynomial $\zeta(\lambda) = 0$ and comparing coefficients method.

Particularly, by putting $\omega \equiv 1$ and $\alpha = 0$, $\Psi_+(u)$ is reduced to ruin probability. Denoted by $\Psi_+(u)$ the ruin probability and $\overline{\Psi}_+(u) = 1 - \Psi_+(u)$, then we have

$$p\overline{\Psi}_+'''(u) - (p(a-b) + \lambda_1 + \lambda_2)\overline{\Psi}_+''(u) - (pab - \lambda_2 a - \lambda_1 b)\overline{\Psi}_+'(u) = 0. \tag{4.17}$$

Denote by $g(u)$ the $\overline{\Psi}_+'(u)$, then preceding equations is rewritten as

$$pg''(u) - (p(a-b) + \lambda_1 + \lambda_2)g'(u) - (pab - \lambda_2 a - \lambda_1 b)g(u) = 0. \tag{4.18}$$

The characteristic equation of Equation (4.18) is

$$p\lambda^2 - (p(a-b) + \lambda_1 + \lambda_2)\lambda - (pab - \lambda_2 a - \lambda_1 b) = 0. \tag{4.19}$$

With standard calculations on constant linear second-order ODE, we have

$$g(u) = C_1 e^{\lambda_{01} u} + C_2 e^{\lambda_{02}} u. \tag{4.20}$$

Thus $\overline{\Psi}(u) \sim 1 - \widetilde{C} e^{\widetilde{\lambda} u}$ and

$$\Psi(u) \sim \widetilde{C} e^{\widetilde{\lambda} u}, \tag{4.21}$$

where $\widetilde{C}(>0)$ and $\widetilde{\lambda}(<0)$ is determined by comparison coefficients method.

Remark 2 Due to the fact that most distribution function can be approximated by the linear combination of exponential distribution. If the distribution of individual claims and individual random premiums are both finite combination of exponential distribution, we can easily come to an an exponential asymptotic estimation of ruin probability, with an analogue form of Equation (4.21). Motivated by such a result, it is easy to find that when the individual random premium income and individual claim are both "light tailed" distributed, the decay of absolute ruin probability does not differ to the one for the model without extra random premium income. As to the case that both individual random premium and claim are both heavy-tailed, Wei et al. (2008) shows that under the decay of ruin probability is mainly dominated by the decay of the distribution function of individual claim. It is a remaining topic for the absolute ruin probability under the model with random premium income.

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随机保费模型下绝对破产概率的可微性以及渐近性

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本文研究了具有随机保费收入的风险模型的Gerber-Shiu罚金函数的可微性以及渐近性质, 随机保费收入通过一个复合泊松过程刻画. 本文得到了Gerber-Shiu函数所满足的积分微分方程, 给出了Gerber-Shiu罚金函数二次可微与三次可微的充分条件. 当所讨论的罚金函数是三次可微的时候, 前述积分微分方程可以转化为一般的常微分方程. 利用常微分方程的标准方法, 当个体随机保费和随机理赔都是指数分布的时候, 得到了绝对破产概率在初始盈余趋向于无穷大时的渐近性质.

关键词: 绝对破产时间, Gerber-Shiu罚金函数, 随机保费, 可微性, 渐近性质.
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