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Log-Operator Scaling Random Fields *

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Abstract

In this paper, we present a logarithm representation of operator scaling stable random fields which in particular contains a class of Log-fractional stable motion $\{\Delta_{\log}(x), x \geq 0\}$, and investigate the related sample paths regularity.

Keywords: Log-operator scaling, pseudo norm, Hölder regularity.

AMS Subject Classification: 60G50, 60F17.

§1. Introduction

Self-similar processes, first studied rigorously by Lamperti (1962) under the name "semi-stable", are processes that are invariant under suitable transformations of time and scale. There has been an extensive literature on self-similar processes. We refer the reader to Samorodnitsky and Taqqu (1994) for studies on Gaussian and stable self-similar processes and random fields, and to Embrechts and Maejima (2002) for an overview of self-similar processes in the one-dimensional case $d = 1$.

Unfortunately, the classical notion of self-similarity, defined for a field $\{X(x)\}_{x\in\mathbb{R}^d}$ on $\mathbb{R}^d$ by

$$\{X(ax)\}_{x\in\mathbb{R}^d} \stackrel{\mathscr{L}}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d}$$

for some $H \in \mathbb{R}$, is isotropic, that is variant under rotation of the underlying parameter space. In many applications, for example the modeling of fractured rock, however, random fields should have an anisotropic nature in the sense that they have different geometric characteristic in different directions. For this reason, an increasing interest has been paid in defining a suitable concept for anisotropic self-similarity. Many authors have developed techniques to handle anisotropy in the scaling.

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A model of anisotropic self-similar random field is the class of operator scaling random fields introduced by Biermé et al. (2007). These fields satisfy the following scaling property

$$\{X(a^E x)\}_{x\in\mathbb{R}^d} \stackrel{\mathscr{L}}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d} \tag{1.1}$$

for some matrix E whose eigenvalues have a positive real part. With this definition, note that if $E=I$, the identity matrix, then (1.1) is just the well-known self-similarity property. The authors present a moving average and a harmonizable representation of stable operator scaling random fields which are defined, respectively, as follows:

$$X_\varphi(x) = \int_{\mathbb{R}^d} (\varphi(x-y)^{H-q/\alpha} - \varphi(-y)^{H-q/\alpha}) Z_\alpha(\mathrm{d}y), \quad x\in\mathbb{R}^d,\ 0<\alpha\le 2,\ 0<H<\beta \tag{1.2}$$

and

$$X_\psi(x) = \mathrm{Re}\int_{R^d} (\mathrm{e}^{i\langle x,y\rangle}-1)\psi(y)^{-H-q/\alpha} W_\alpha(\mathrm{d}y), \quad x\in\mathbb{R}^d,\ 0<\alpha\le 2, \tag{1.3}$$

where $Z_\alpha(\mathrm{d}y)$ denotes an independently scattered symmetric α-stable ($S\alpha S$) random measure on $\mathbb{R}^d$ with Lebesgue control measure λ^d, and $W_\alpha(\mathrm{d}y)$ denotes a complex isotropic $S\alpha S$ random measure with Lebesgue control measure (Samorodnitsky and Taqqu, 1994; p.281). We refer the reader to Biermé et al. (2007; Theorem 3.1, Theorem 4.1) for the explicit definitions of (1.2) and (1.3). Biermé and Lacaux (2009) subsequently investigate the sample paths regularity of operator scaling α-stable random fields.

However, it is obvious that the representation (1.2) is not well defined when we are taking into account the case $H=q/\alpha$. Thus in this paper we discuss this case and analyze related sample paths regularity properties.

The paper is organized as follows. In Section 2, we recall and fix some notations and notions about operator scaling random fields. In Section 3, we prove the existence of logarithm representation of operator scaling random fields which was called the Log-operator scaling random fields. In Section 4, we analyze the related sample paths properties.

Throughout this paper, we adopt the following notations.

$\stackrel{\mathscr{L}}{=}$ denotes the equality of all finite dimensional marginal distributions.

We denote $M_d(\mathbb{R})$ the matrix with order $d\times d$.

$\varepsilon^+ = \{E\in M_d(\mathbb{R})$: the eigenvalues of E have positive real part$\}$.

Let $q=\mathrm{trace}(E)$, where $E\in M_d(\mathbb{R})$.

For any $E\in M_d(\mathbb{R})$, let us define

$$\rho_{\min}(E) = \min_{\lambda\in Sp(E)}(|\mathrm{Re}(\lambda)|), \qquad \rho_{\max}(E) = \max_{\lambda\in Sp(E)}(|\mathrm{Re}(\lambda)|),$$

where Re represents real part of a complex number.

For any real $a > 0$, a^E denotes the matrix

$$a^E = \exp(E \log a) = \sum_{k=0}^{\infty} \frac{E^k (\log a)^k}{k!}.$$

The constant C with or without indexes will denote different constants (depending on the indexes) whose values are not important.

§2. Preliminaries

Now let us recall and fix some notations and notions about operator scaling random fields.

Definition 2.1 Let $E \in \varepsilon^+$. A function ρ defined on $\mathbb{R}^d$ is a $(\mathbb{R}^d, E)$-pseudo norm, if it satisfies the three following properties:

(a) ρ is continuous on $\mathbb{R}^d$;

(b) ρ is strictly positive on $\mathbb{R}^d \backslash \{0\}$;

(c) ρ is E-homogeneous, i.e. $\rho(a^E x) = a\rho(x)$, $\forall\, x \in \mathbb{R}^d$, $\forall\, a > 0$.

Remark 1 The existence of $(\mathbb{R}^d, E)$-pseudo norm for any matrix $E \in \varepsilon^+$ is proved in Biermé et al. (2007) or Lemarié-Rieusset (1994).

Let us recall some classical properties of the pseudo norm. First, let us introduce the anisotropic sphere $S_0^E(\rho)$ for the $(\mathbb{R}^d, E)$-pseudo norm ρ defined by

$$S_0^E(\rho) = \{x \in \mathbb{R}^d; \rho(x) = 1\}.$$

The following result can be found in Meerschaert and Scheffler (2001).

Proposition 2.1 For all $x \in \mathbb{R}^d \backslash \{0\}$, there exists a unique couple $(r, \theta) \in \mathbb{R}_+^* \times S_0^E(\rho)$ such that $x = r^E \theta$. Moreover $S_0^E(\rho)$ is a compact of $\mathbb{R}^d$ and the map

$$(r, \theta) \to x = r^E \theta$$

is a homeomorphism from $\mathbb{R}_+^* \times S_0^E(\rho)$ to $\mathbb{R}^d \backslash \{0\}$.

Remark 2 Since $\rho(x) = \rho(r^E \theta) = r\rho(\theta)$ and $\rho(\theta) = 1$, we can get $r = \rho(x)$. Then it follows from Meerschaert and Scheffler (2001) that for a given $(\mathbb{R}^d, E)$-pseudo norm ρ, we can write any $x \in \mathbb{R}^d \backslash \{0\}$ uniquely as $x = \rho(x)^E \ell(x)$, where $\ell(x) \in S_0^E(\rho)$. Furthermore, we have $\rho(x) = \rho(-x)$ and $\ell(x) = -\ell(-x)$.

The term of pseudo norm is justified by the fact that these functions satisfy the triangular inequality which can be get from the following proposition.

Proposition 2.2 (Clausel and Vedel, 2011) Let ρ be $(\mathbb{R}^d, E)$-pseudo norm. Then

(a) there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\rho(x+y) \le C(\rho(x) + \rho(y));$$

(b) there exists a constant $C' > 0$ such that for all $x \in \mathbb{R}^d$,

$$\frac{1}{C'}\rho_1(x) \le \rho_2(x) \le C'\rho_1(x).$$

The following result gives bounds on the growth rate of any $(\mathbb{R}^d, E)$-pseudo norm in terms of the euclidian norm $\|\cdot\|$.

Proposition 2.3 (Biermé and Lacaux, 2009) Let ρ be $(\mathbb{R}^d, E)$-pseudo norm. There exist strictly positive constant $C_1(\rho), C_2(\rho), C_3(\rho), C_4(\rho)$ depending only on the pseudo norm ρ such that

(a) for all $x \in \mathbb{R}^d$ with $\|x\| \le 1$ or $\|x\|_E < 1$, one has

$$\begin{aligned} &C_1(\rho)\|x\|^{1/\rho_{\min}(E)}(1 + |\log(\|x\|)|)^{-d/\rho_{\min}(E)} \\ &\le \rho(x) \le C_2(\rho)\|x\|^{1/\rho_{\max}(E)}(1 + |\log(\|x\|)|)^{d/\rho_{\max}(E)}; \end{aligned}$$

(b) for all $x \in \mathbb{R}^d$ with $\|x\| \ge 1$ or $\|x\|_E > 1$, one has

$$\begin{aligned} &C_3(\rho)\|x\|^{1/\rho_{\max}(E)}(1 + |\log(\|x\|)|)^{-d/\rho_{\max}(E)} \\ &\le \rho(x) \le C_4(\rho)\|x\|^{1/\rho_{\min}(E)}(1 + |\log(\|x\|)|)^{d/\rho_{\min}(E)}. \end{aligned}$$

The following proposition provides an integration in polar coordinates formula which play an important role in the proof of existence of random integral.

Proposition 2.4 (Biermé et al., 2007) For a given $(\mathbb{R}^d, E)$-pseudo norm ρ, there exists a unique finite Random measure σ on $S_0^E(\rho)$ such that for all $f \in L^1(\mathbb{R}^d, \mathrm{d}x)$, we have

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}x = \int_0^\infty \int_{S_0^E(\rho)} f(r^E\theta)\sigma(\mathrm{d}\theta)r^{q-1}\mathrm{d}r.$$

Corollary 2.1 (Biermé et al., 2007) For a given $(\mathbb{R}^d, E)$-pseudo norm ρ, let $\beta \in \mathbb{R}$ and suppose that $f : \mathbb{R}^d \to \mathbb{C}$ is measurable such that $|f(x)| = O(\rho(x)^\beta)$, then

(a) if $\beta > -q$, then f is integrable near 0;

(b) if $\beta < -q$, then f is integrable near infinity.

Definition 2.2 A scalar valued random field is called operator scaling if there exists a matrix $E \in \varepsilon^+$ and $H > 0$ such that

$$\{X(a^E x)\}_{x\in\mathbb{R}^d} \overset{\mathscr{L}}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d}.$$

Remark 3 The matrix E and the real H are respectively called an exponent (of scaling) or an anisotropy and a Hurst index of the field. In general, the couple (H,E) of an operator scaling random field is not unique.

We shall need the following definition which is introduced in Biermé et al. (2007).

Definition 2.3 Let τ be a $(\mathbb{R}^d,E)$-pseudo norm and $\beta>0$. A function $\varphi(x)$ is called (τ,β,E)-admissible if it satisfies the following three properties:

(a) $\varphi(x)$ is continuous on $\mathbb{R}^d$;

(b) $\varphi(x)$ is strictly positive on $\mathbb{R}^d\backslash\{0\}$;

(c) For any $0<A<B$, there exists a positive constant $C>0$, such that, for $A\leq\|y\|\leq B$,

$$\tau(x)\leq 1\Rightarrow|\varphi(x+y)-\varphi(y)|\leq C\tau(x)^{\beta}.$$

Remark 4 As can be seen from Biermé et al. (2007), if a function $\varphi(x)$ is (τ,β,E)-admissible, then $\beta\leq\rho_{\min}(E)$.

§3. Log-Operator Scaling Random Fields

In this section we extend the moving average representation to the case $H=q/\alpha$ and derive its basic properties. We first give sufficient conditions such that the integral representation exists. Then we can define Log-operator scaling stable random fields.

Throughout this section we choose two fixed $(\mathbb{R}^d,E)$-pseudo norm φ and τ with $E\in\{E\in\varepsilon^+:\rho_{\min}(E)>1\}$.

Theorem 3.1 Let $\beta>1$ $(\leq\rho_{\min}(E))$. Let φ be a (τ,β,E)-admissible function. Then for any $0<\alpha\leq 2$ and $1<q/\alpha<\beta$, the random field

$$X_{\varphi}(x)=\int_{\mathbb{R}^d}(\log\varphi(x-y)-\log\varphi(-y))Z_{\alpha}(\mathrm{d}y)$$

exists and is stochastically continuous.

Before proceeding, we need some preparations.

Lemma 3.1 (Embrechts and Maejima, 2002) Let $0<\alpha\leq 2$, $A\in\mathbb{R}^d$. If

$$\int_A|f(x)|^{\alpha}\mathrm{d}x<\infty,$$

then a stable integral

$$I(f):=\int_A f(x)\mathrm{d}Z_{\alpha}(x)$$

can be defined in the sense of convergence in probability.

Lemma 3.2 (Samorodnitsky and Taqqu, 1994) Let f_j, $j=1,2,\ldots$, f be non-random functions. Let $X_j=\int_{\mathbb{R}^d} f_j(x)Z_\alpha(\mathrm{d}x)$, $j=1,2,\ldots$, and $X=\int_{\mathbb{R}^d} f(x)Z_\alpha(\mathrm{d}x)$. Then

$$\lim_{j\to\infty} X_j = X, \qquad \text{a.e.}$$

if and only if

$$\lim_{j\to\infty}\int_{\mathbb{R}^d} |f_j(x)-f(x)|^\alpha Z_\alpha(\mathrm{d}x)=0.$$

Proof of Theorem 3.1 First we prove the existence of random integral. Let us recall that $X_\varphi(x)$ exists if and only if

$$\Gamma_\varphi^\alpha(x)=\int_{\mathbb{R}^d} |\log\varphi(x-y)-\log\varphi(-y)|^\alpha \mathrm{d}y<\infty.$$

Noting that we can always choose a sufficiently little R_1, such that for all $\tau(y)\le R_1$,

$$|\log\tau(-y)|\le C_1\tau(-y)^{-1}, \tag{3.1}$$

hence by C_r inequality and (b) of Proposition 2.2, we have

$$\begin{aligned}|\log\varphi(x-y)-\log\varphi(-y)|^\alpha &\le C_2(|\log^\alpha\varphi(x-y)|+|\log^\alpha\varphi(-y)|)\\ &\le C_3(|\log^\alpha\tau(x-y)|+|\log^\alpha\tau(-y)|).\end{aligned} \tag{3.2}$$

For fixed $x\in\mathbb{R}^d\backslash\{0\}$, it is obvious that

$$\int_{\tau(y)\le R_1} |\log^\alpha\tau(x-y)|\mathrm{d}y<\infty.$$

By (3.1) and Corollary 2.1, since $\alpha<q$, we can conclude that

$$\int_{\tau(y)\le R_1} |\log^\alpha\tau(-y)|\mathrm{d}y<\int_{\tau(y)\le R_1}|\tau(-y)|^{-\alpha}\mathrm{d}y<\infty.$$

Therefore by (3.2), we have

$$\int_{\tau(y)\le R_1} |\log\varphi(x-y)-\log\varphi(-y)|^\alpha\mathrm{d}y<\infty.$$

In the case that y is sufficiently close to x, we can prove similarly that $|\log\varphi(x-y)-\log\varphi(-y)|^\alpha$ is integrable.

It remains to show that for some sufficiently large $R_2=R_2(x)>0$, we have

$$\int_{\tau(y)>R_2} |\log\varphi(x-y)-\log\varphi(-y)|^\alpha\mathrm{d}y<\infty.$$

Through a change of variable, we know that

$$\int_{\tau(y)>R_2} |\log \varphi(x-y) - \log \varphi(-y)|^\alpha \mathrm{d}y < \infty,$$

holds if and only if

$$\int_{\tau(y)>R_2} |\log \varphi(x+y) - \log \varphi(y)|^\alpha \mathrm{d}y < \infty.$$

Since $\tau(y) > R_2$, $\varphi(y) > 0$ and φ is $(\mathbb{R}^d, E)$-pseudo norm, we have

$$\varphi(x+y) = \varphi(\varphi(y)^E(\varphi(y)^{-E}x + \varphi(y)^{-E}y)) = \varphi(y)\varphi(\varphi(y)^{-E}x + \varphi(y)^{-E}y).$$

Again by $\varphi(\varphi(y)^{-E}y) = 1$ and φ is (τ, β, E)-admissible, we can find $C_4 > 0$ such that

$$|\varphi(\varphi(y)^{-E}x + \varphi(y)^{-E}y) - 1| \leq C_4\tau(\varphi(y)^{-E}x)^\beta = C_4\varphi(y)^{-\beta}\tau(x)^\beta.$$

Obvious that we can always choose sufficiently large R_2, such that for all $\tau(y) > R_2$,

$$C_4\varphi(y)^{-\beta}\tau(x)^\beta \leq \frac{1}{2},$$

thus we have

$$\varphi(\varphi(y)^{-E}x + \varphi(y)^{-E}y) \geq \frac{1}{2}. \tag{3.3}$$

Now by (3.3) and mean valued theorem we can get that for all $\tau(y) > R_2$,

$$\begin{aligned}
|\log \varphi(x+y) - \log \varphi(y)| &= |\log \varphi(\varphi(y)^{-E}x + \varphi(y)^{-E}y)| \\
&= |\log \varphi(\varphi(y)^{-E}x + \varphi(y)^{-E}y) - \log 1| \\
&\leq C_5|\varphi(\varphi(y)^{-E}x + \varphi(y)^{-E}y) - 1| \\
&\leq C_6\varphi(y)^{-\beta}\tau(x)^\beta.
\end{aligned}$$

By (b) of Proposition 2.2, we can get $\varphi(y)^{-\beta} \leq C\tau(y)^{-\beta}$, and hence we have

$$|\log \varphi(x+y) - \log \varphi(y)|^\alpha \leq C_7\tau(y)^{-\beta\alpha}\tau(x). \tag{3.4}$$

Therefore by Corollary 2.1, since $q/\alpha < \beta$, we have

$$\int_{\tau(y)>R_2} |\log \varphi(x-y) - \log \varphi(-y)|^\alpha \mathrm{d}y < \infty.$$

Consequently, we obtain that the random field

$$X_\varphi(x) = \int_{\mathbb{R}^d} (\log \varphi(x-y) - \log \varphi(-y)) Z_\alpha(\mathrm{d}y)$$

exists.

Let us now show that $X_\varphi(x)$ is stochastic continuous. Since $X_\varphi(x)$ is a $S\alpha S$-random field, it follows from Lemma 3.2 that $X_\varphi(x)$ is stochastically continuous if and only if, for all $x_0\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}|\log\varphi(x_0+x-y)-\log\varphi(x_0-y)|^\alpha \mathrm{d}y\to 0,\qquad \text{as}\ \ x\to 0.$$

Through a change of variables, this holds if and only if

$$\Gamma_\varphi^\alpha(x)=\int_{\mathbb{R}^d}|\log\varphi(x-y)-\log\varphi(-y)|^\alpha \mathrm{d}y\to 0,\qquad \text{as}\ \ x\to 0.$$

Since φ is $(\mathbb{R}^d,E)$-pseudo norm, φ is continuous on $\mathbb{R}^d$, so we have that for all $y\in\mathbb{R}^d$,

$$|\log\varphi(x-y)-\log\varphi(-y)|^\alpha\to 0,\qquad \text{as}\ \ x\to 0.$$

We can split $|\log\varphi(x-y)-\log\varphi(-y)|^\alpha$ into three parts:

$$\begin{aligned}|\log\varphi(x-y)-\log\varphi(-y)|^\alpha&=|\log\varphi(x-y)-\log\varphi(-y)|^\alpha 1_{\tau(y)\le R_1}\\&\quad+|\log\varphi(x-y)-\log\varphi(-y)|^\alpha|1_{R_1<\tau(y)\le R_2}\\&\quad+\log\varphi(x-y)-\log\varphi(-y)|^\alpha 1_{\tau(y)>R_2}\\&:=I_1+I_2+I_3.\end{aligned}$$

The first term I_1 is dealt with as follows. By (a) of Proposition 2.2, we have

$$\{y:\tau(x-y)\le R_1\}\subset\{y:\tau(y)\le C(R_1+\tau(x))\},$$

and hence for $\alpha'>\alpha$ which is sufficiently close to α and $\tau(x)\le 1$, we have

$$\begin{aligned}\sup_{\tau(x)\le 1}\int_{\tau(y)\le R_1}|\log\varphi(x-y)|^{\alpha'}\mathrm{d}y&=\sup_{\tau(x)\le 1}\int_{\tau(x-y)\le R_1}|\log\varphi(y)|^{\alpha'}\mathrm{d}y\\&\le\sup_{\tau(x)\le 1}\int_{\tau(y)\le C(R_1+\tau(x))}|\log\varphi(y)|^{\alpha'}\mathrm{d}y\\&\le\sup_{\tau(x)\le 1}\int_{\tau(y)\le C(R_1+1)}|\log\varphi(y)|^{\alpha'}\mathrm{d}y.\end{aligned}$$

Noting that we can always choose a sufficiently little R, such that for all $\tau(y)\le R$,

$$|\log\tau(-y)|\le C\tau(-y)^{-1},$$

then by (b) of Proposition 2.2 and Corollary 2.1, since $\alpha'<q$, we have

$$\begin{aligned}&\int_{\tau(y)\le C(R_1+1)}|\log\varphi(y)|^{\alpha'}\mathrm{d}y\\&\le\int_{\tau(y)\le R}|\log\varphi(y)|^{\alpha'}\mathrm{d}y+\int_{R<\tau(y)\le C(R_1+1)}|\log\varphi(y)|^{\alpha'}\mathrm{d}y\\&\le C\int_{\tau(y)\le R}|\tau(y)|^{-\alpha'}\mathrm{d}y+\int_{R<\tau(y)\le C(R_1+1)}|\log\varphi(y)|^{\alpha'}\mathrm{d}y<\infty.\end{aligned}$$

This implies that

$$\sup_{\tau(x)\le 1}\int_{\tau(y)\le R_1} |\log\varphi(x-y)|^{\alpha'}\mathrm{d}y<\infty,$$

and hence we obtain that $\{|\log\varphi(x-y)|^{\alpha}\}_{\tau(x)\le 1}$ is uniformly integrable. Consequently, applying Vitali dominated convergence theorem, we have

$$\int_{\tau(y)\le R_1} |\log\varphi(x-y)-\log\varphi(-y)|^{\alpha}\mathrm{d}y\to 0,\qquad \text{as}\ \ x\to 0. \tag{3.5}$$

For the term I_2, it is obvious that

$$\int_{R_1<\tau(y)\le R_2} |\log\varphi(x-y)-\log\varphi(-y)|^{\alpha}\mathrm{d}y\to 0,\qquad \text{as}\ \ x\to 0. \tag{3.6}$$

Now we deal with the term I_3. By (3.4), we have

$$|\log\varphi(x-y)-\log\varphi(-y)|^{\alpha}1_{\tau(y)>R_2}\le C_4\tau(-y)^{-\beta\alpha}1_{\tau(y)>R_2}.$$

Therefore by $\int_{\tau(y)>R_2}\tau(-y)^{-\beta\alpha}\mathrm{d}y<\infty$ and Lebesgue dominated convergence theorem, we have

$$\int_{\tau(y)>R_2} |\log\varphi(x-y)-\log\varphi(-y)|^{\alpha}\mathrm{d}y\to 0,\qquad \text{as}\ \ x\to 0. \tag{3.7}$$

Then from (3.5), (3.6) and (3.7), we deduce that $X_\varphi(x)$ is stochastic continuous. □

According to this theorem, we can define Log-operator scaling random fields.

Definition 3.1 Let $\beta>1$ $(\le\rho_{\min}(E))$, φ be $(\mathbb{R}^d,E)$-pseudo norm, and (τ,β,E)-admissible. Then for $0<\alpha\le 2$, $1<q/\alpha<\beta$, we define Log-operator random fields $X_\varphi(x)$ as follows:

$$X_\varphi(x)=\int_{\mathbb{R}^d}(\log\varphi(x-y)-\log\varphi(-y))Z_\alpha(\mathrm{d}y).$$

Remark 5 It is worth mentioning that Log-operator scaling random fields extends the definition of Log-fractional stable motion $\{\Delta_{\log}(x),x\ge 0\}$ (cf. Samorodnitsky and Taqqu, 1994; Section 7.6) which was defined as follows:

$$\Delta_{\log}(x)=\int_{-\infty}^{\infty}\log\left\|\frac{x-y}{y}\right\|Z_\alpha(\mathrm{d}y),\qquad 1<\alpha\le 2.$$

Log-operator scaling random fields have stationary increment and operator scaling.

Theorem 3.2 Under the condition of Theorem 3.1, the random field $X_\varphi(x)$ has the following properties:

(a) Operator scaling, that is, for any $c>0$, the random field $X_\varphi(x)$ satisfies the following property

$$\{X_\varphi(c^E x)\}_{x\in\mathbb{R}^d}\overset{\mathscr{L}}{=}\{c^{q/\alpha}X_\varphi(x)\}_{x\in\mathbb{R}^d};$$

(b) Stationary increments, that is, for any $h \in \mathbb{R}^d$, the random field $X_\varphi(x)$ satisfies the following property

$$\{X_\varphi(x+h) - X_\varphi(h)\}_{x\in\mathbb{R}^d} \stackrel{\mathscr{L}}{=} \{X_\varphi(x)\}_{x\in\mathbb{R}^d}.$$

Proof (a) For any fixed $x_1, x_2, \ldots, x_p \in \mathbb{R}^d$, and $t_1, t_2, \ldots, t_p \in \mathbb{R}$, $c > 0$, if we can show that

$$\sum_{j=1}^p t_j X_\varphi(c^E x_j) \stackrel{\mathscr{L}}{=} \sum_1^p t_j c^{q/\alpha} X_\varphi(x_j),$$

then we can get the conclusion. By a change of variable together with $\varphi(c^E x) = c\varphi(x)$ and $Z_\alpha(c^E \mathrm{d}z) \stackrel{\mathscr{L}}{=} c^{q/\alpha} Z_\alpha(\mathrm{d}z)$, we can get that

$$\begin{aligned}
\sum_{j=1}^p t_j X_\varphi(c^E x_j) &= \int_{\mathbb{R}^d} \sum_{j=1}^p t_j (\log\varphi(c^E x - y) - \log\varphi(-y)) Z_\alpha(\mathrm{d}y) \\
&= \int_{\mathbb{R}^d} \sum_{j=1}^p t_j (\log\varphi(c^E x - c^E z) - \log\varphi(-c^E z)) Z_\alpha(\mathrm{d}y) \\
&\stackrel{d}{=} c^{q/\alpha} \int_{\mathbb{R}^d} \sum_{j=1}^p t_j (\log\varphi(x - z) - \log\varphi(-z)) Z_\alpha(\mathrm{d}z) \\
&= c^{q/\alpha} \sum_{j=1}^p t_j X_\varphi(x_j),
\end{aligned}$$

and the proof is completed.

(b) For any fixed $x_1, x_2, \ldots, x_p \in \mathbb{R}^d$, and $t_1, t_2, \ldots, t_p \in \mathbb{R}$, $h \in \mathbb{R}^d$, if we can show that

$$\sum_{j=1}^p t_j (X_\varphi(x_j + h) - X_\varphi(h) \stackrel{\mathscr{L}}{=} \sum_1^p t_j X_\varphi(x_j),$$

then we can get the conclusion. By a change of variable, we get

$$\begin{aligned}
\sum_{j=1}^p t_j X_\varphi((x_j + h) - X_\varphi(h)) &= \sum_{j=1}^p t_j \Big(\int_{\mathbb{R}^d} (\log\varphi(x_j + h - y) - \log\varphi(h - y)) Z_\alpha(\mathrm{d}y) \Big) \\
&= \int_{\mathbb{R}^d} \sum_{j=1}^p t_j (\log\varphi(x_j + h - y) - \log\varphi(h - y)) Z_\alpha(\mathrm{d}y) \\
&\stackrel{d}{=} \int_{\mathbb{R}^d} \sum_{j=1}^p t_j (\log\varphi(x_j - z) - \log\varphi(-z)) Z_\alpha(\mathrm{d}z) \\
&= \sum_1^p t_j X_\varphi(x_j),
\end{aligned}$$

and the proof is completed. □

§4. Hölder Critical Exponent of Log-Gaussian Operator Scaling Random Fields

In this section, we are interested in the smoothness of the sample paths of Log-Gaussian operator scaling random fields. Through this section, we fix $E \in \varepsilon^+$, with $1 < a_1 < a_2 < \cdots < a_p$ denoting the real parts of the eigenvalues of E. Following Meerschaert and Scheffler (2001; Section 2.1), let $V_1, V_2, \ldots, V_p$ be spectral decomposition of $\mathbb{R}^d$ with respect to E. For $i = 1, 2, \ldots, p$, let us define $W_i = V_1 \oplus V_2 \oplus V_i$ and $W_0 = 0$. Observe that $E|_{W_i}$ has $1 < a_1 < a_2 < \cdots < a_i$ as real parts of the eigenvalues.

Definition 4.1 (Biermé et al., 2007) Let $\gamma \in (0, 1)$. A random field $\{X(x)\}_{x\in\mathbb{R}^d}$ is said to have Hölder critical exponent γ whenever it satisfies the following two properties:

(a) For any $s \in (0, \gamma)$, the sample paths of random fields $\{X(x)\}_{x\in\mathbb{R}^d}$ satisfy almost surely a uniform Hölder condition of order s on any compact set, that is, for any compact set $K \subseteq \mathbb{R}^d$, there exists a positive random variable A such that for all $x, y \in K$,

$$|X(x) - X(y)| \le A\|x - y\|^s.$$

(b) For any $s \in (\gamma, 1)$, almost surely the sample paths of random field $\{X(x)\}_{x\in\mathbb{R}^d}$ fail to satisfy any uniform Hölder condition of order s.

Proposition 4.1 (Biermé et al., 2007) Let $\{X(x)\}_{x\in\mathbb{R}^d}$ be a Gaussian random field with stationary increments.

(a) Let $\gamma \in (0, 1)$, and assume that

$$\gamma = \sup\{s > 0; \mathsf{E}((X(x) - X(0))^2) = o_{\|x\|\to 0}(\|x\|^{2s})\},$$

then, for any $s \in (0, \gamma)$, any continuous version of random field $\{X(x)\}_{x\in\mathbb{R}^d}$ satisfy almost surely a uniform Hölder condition of order s on any compact set.

(b) If moreover

$$\gamma = \inf\{s > 0; \|x\|^{2s} = o_{\|x\|\to 0}(\mathsf{E}(X(x) - X(y))^2)\},$$

then any continuous version of random field $\{X(x)\}_{x\in\mathbb{R}^d}$ admit γ as the Hölder critical exponent.

Definition 4.2 (Biermé et al., 2007) Let S^{d-1} be the Euclidean unit sphere. A real-valued random field $\{X(x)\}_{x\in\mathbb{R}^d}$ admits $\gamma(u)$ as directional regularity in $u \in S^{d-1}$, if the process $\{X(tu)\}_{t\in\mathbb{R}}$ admits $\gamma(u)$ Hölder critical exponent.

Let us investigate sample paths properties for $(E, q/2)$-Log-Gaussian operator scaling random field.

Theorem 4.1 Let $\beta > 1$ $(\le a_1)$, $1 < q/2 < \beta$. Let X_φ be Log-Gaussian operator scaling random field given by Definition 3.1. Then any continuous version X_φ admits $q/(2a_p)$ as Hölder critical exponent. Moreover, for any $i = 1, 2, \ldots, p$, if $u \in (W_i \setminus W_{i-1}) \cap S^{d-1}$, the random field X_φ admits $q/(2a_i)$ as directional regularity in the direction u.

Proof By Remark 2, we can write any $x \in \mathbb{R}^d \setminus \{0\}$ uniquely as $x = \tau(x)^E \ell(x)$ for a given $(\mathbb{R}^d, E)$-pseudo norm $\tau(x)$, where $\ell \in S_0^E(\tau)$. Observing that $X_\varphi(0) = 0$, hence we define

$$\Gamma_\varphi^2(x) := \mathsf{E}[(X_\varphi(x) - X_\varphi(0))^2] = \int_{R^d} |\log \varphi(x-y) - \log \varphi(-y)|^2 \mathrm{d}y.$$

Thus by the $(E, q/2)$-operator scaling of X_φ, it is straightforward to see that

$$\Gamma_\varphi^2(x) = \tau(x)^q \Gamma_\varphi^2(\ell(x)).$$

As in the proof of Theorem of 3.1, since $S_0^E(\tau)$ is a compact of $\mathbb{R}^d$, we have that $\Gamma_\varphi^2(x)$ is continuous and positive on the $x \in S_0^E(\tau)$. Thus for all $\theta \in S_0^E(\tau)$, $0 < m \le \Gamma_\varphi^2(\theta) \le M$.

Let $u \in (W_i \setminus W_{i-1}) \cap S^{d-1}$, with $1 \le i \le p$. According to Proposition 2.3, for $|t|$ small enough,

$$C_1(\tau)|t|^{1/a_i}(1 + |\log|t||)^{-d/a_i} \le \tau(tu) \le C_2(\tau)|t|^{1/a_i}(1 + |\log|t||)^{d/a_i}.$$

Thus we have

$$mC_1^q(\tau)|t|^{q/a_i}(1 + |\log|t||)^{-dq/a_i} \le \Gamma_\varphi^2(tu) \le MC_2^q(\tau)|t|^{q/a_i}(1 + |\log|t||)^{dq/a_i}.$$

Therefore by Proposition 4.1, X_φ admits $q/(2a_i)$ as directional regularity in the directional $u \in (W_i \setminus W_{i-1}) \cap S^{d-1}$.

Since $q/(2a_p)$ is Hölder critical exponent of X_φ in any $u \in (W_p \setminus W_{p-1}) \cap S^{d-1}$, it follows from this that for any $s \in (q/(2a_p), 1)$ almost surely the sample paths of X_φ fail to satisfy any uniform Hölder condition of order s. Again by Proposition 2.3, we know that for $|x|$ small enough,

$$\tau(x) \le C_2(\tau)\|x\|^{1/a_p}(1 + |\log(\|x\|)|)^{d/a_p}.$$

Then for $|x|$ small enough, we have

$$\Gamma_\varphi^2(x) \le MC_2^q(\tau)\|x\|^{q/a_p}(1 + |\log(\|x\|)|)^{dq/a_p}.$$

Therefore according to Proposition 4.1, it follows that any continuous version of X_φ satisfies almost surely a uniform Hölder condition of order $s < q/(2a_p)$ on any compact set. Then X_φ admits $q/(2a_p)$ as Hölder critical exponent. $\square$

The authors proved in Biermé and Lacaux (2009) that harmonizable operator scaling stable random fields share many properties with Gaussian operator scaling random fields. In particular, they have locally Hölder sample pathes and critical directional Hölder exponent depending on the directions. However, for stable laws ($\alpha \in (0,2)$), Log-operator scaling random fields do not have the same behavior with Gaussian operator scaling random fields as we see in this section. The following theorem turn out this fact.

Theorem 4.2 Let $\beta > 1$ $(\le a_1)$, $0 < \alpha < 2$, $1 < q/\alpha < \beta$. Let $\{X(x)\}_{x\in\mathbb{R}^d}$ be Log-operator scaling random fields given by Definition 3.1. Then, any modification of the random field $\{X(x)\}_{x\in\mathbb{R}^d}$ are almost surely unbounded on every open ball.

Proof For any open ball U, let $U^* = U \cap \mathbb{Q}^d$ be a dense sequence in U. Noting that φ is a $(\mathbb{R}^d, E)$-pseudo norm, then for any $y \in U$, we have

$$f^*(U^*, y) \triangleq \sup_{x\in U^*} |\log\varphi(x-y) - \log\varphi(-y)| = +\infty,$$

and hence $\int_{\mathbb{R}^d} f^*(U^*, y)^\alpha dy = +\infty$. Again by necessary condition for sample boundedness (cf. Samorodnitsky and Taqqu, 1994; Theorem 10.2.3), we obtain that any modification of the random field $\{X(x)\}_{x\in\mathbb{R}^d}$ are almost surely unbounded on every open ball and hence conclude the proof. $\square$

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Log算子标度随机场

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在这篇文章中, 我们给出了算子标度稳定随机场的一个对数表示, 特别的, 它包含了一类Log-分数稳定运动$\{\Delta_{\log}(x), x \geq 0\}$. 同时我们还考察了它的相关的样本轨道的正则性.

关键词: Log算子标度, 伪范数, Hölder正则性.

学科分类号: O211.4, O211.6.