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综述报告

Spectral Problems of Ergodic Schrödinger Operators

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Abstract

This article provides the readers with an introduction to the spectral theory of ergodic one dimensional Schrödinger operators. The theorems developed by the author are mainly discussed and their proofs are given in detail.

Keywords: Ergodic Schrödinger operator, reflectionless potential, integrated density of s-tates, Lyapounov exponent.

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Introduction

Schrödinger operators with stationary, ergodic potentials (to be explained later) are physical models to describe the movement of electrons in matters containing impurities, which was proposed by physicist P.W. Anderson in 1958. In this lecture, we consider dynamical properties of one electron under a stationary ergodic potential $V_\omega(x) = V(T_x\omega)$ governed by a Schrödinger equation in $\mathbb{R}^d$:

$$i\frac{\partial u}{\partial t} = -\Delta u + V_\omega u \qquad \Big(\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}\Big).$$

Most of its dynamical properties can be obtained from the spectral properties of a self-adjoint operator

$$H_\omega = -\Delta + V_\omega. \tag{1}$$

Before proceeding to ergodic Schrödinger equations let us recall a basic fact from quantum mechanics. Let H be a Schrödinger operator with a deterministic potential V

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and consider a Schrödinger equation

$$i\frac{\partial u}{\partial t} = Hu = -\Delta u + Vu, \qquad u(0,\cdot) = f(\cdot) \in L^2(\mathbb{R}^d).$$

Then

$$u(t,\cdot) = \big(\mathrm{e}^{-itH} f\big)(\cdot)$$

holds and the self-adjointness of H implies the unitarity of e^{-itH} for each fixed $t \in \mathbb{R}$, which shows

$$\int_{\mathbb{R}^d} |u(t,x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x \equiv \|f\|^2.$$

Hence, if the L^2-norm $\|f\|$ is normalized as $\|f\| = 1$, then $|u(t,x)|^2\mathrm{d}x$ turns to be a probability measure on $\mathbb{R}^d$. Quantum mechanics tells us that an electron moving under the potential V can be found in a domain $D \subset \mathbb{R}^d$ with probability

$$\int_D |u(t,x)|^2 \mathrm{d}x$$

at time t. Therefore the mean of the square of the distance from the origin is given by

$$\int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 \mathrm{d}x \equiv D(t),$$

which measures diffusion speed of the electron. Physically the increase of $D(t)$ implies the high conductivity of electricity. We compute the asymptotic behavior of $D(t)$ as $t \to \infty$ in two extreme cases. In the simplest case $V = 0$ one has $D(t)$:

$$\begin{aligned}(2\pi)^d D(t) &= \int_{\mathbb{R}^d} \big|\nabla_\xi \widehat{u}(t,\xi)\big|^2 \mathrm{d}\xi = \int_{\mathbb{R}^d} \big|\nabla_\xi \big(\mathrm{e}^{-it|\xi|^2} \widehat{f}(\xi)\big)\big|^2 \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} \big| -2it\xi \widehat{f}(\xi) + \nabla_\xi \widehat{f}(\xi)\big|^2 \mathrm{d}\xi,\end{aligned}$$

where $\widehat{f}$ denotes the Fourier transform of f. Therefore, if f has compact support and is smooth, then

$$D(t) \sim \frac{4t^2}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^2 \big|\widehat{f}(\xi)\big|^2 \mathrm{d}\xi = 4t^2 \int_{\mathbb{R}^d} |\nabla f(x)|^2 \mathrm{d}x \qquad \text{as } t \to \infty,$$

which shows that free electron moves linearly with respect to t like a classical free particle. The other extreme case is the one when

$$V(x) \to \infty \qquad \text{as } |x| \to \infty.$$

In this case, it is known that the operator H has countably many discrete eigenvalues $\{\lambda_j\}$. The associated normalized eigenfunctions $\{e_j\}$ forms a complete orthonormal basis of $L^2(\mathbb{R}^d)$. Therefore, if f is a finite linear combination of $\{e_j\}$, then

$$u(t,x) = \big(\mathrm{e}^{-itH} f\big)(x) = \sum_{j \in \text{a finite set}} f_j \mathrm{e}^{-it\lambda_j} e_j(x) \qquad \Big(f_j = \int_{\mathbb{R}^d} f(x)\overline{e_j(x)}\mathrm{d}x\Big),$$

and

$$D(t)=\int_{\mathbb{R}^d}|x|^2\Big|\sum_{j\in\text{ a finite set}} f_j \mathrm{e}^{-it\lambda_j}e_j(x)\Big|^2\mathrm{d}x\le\Big(\sum_j|f_j|^2\Big)\int_{\mathbb{R}^d}|x|^2\sum_{j\in\text{ a finite set}}|e_j(x)|^2\mathrm{d}x.$$

Since $e_j(x)$ decays exponentially fast as $|x|\to\infty$, the above right-hand side integral is finite, which shows $D(t)$ remains bounded as $t\to\infty$. This is consistent with a classical picture describing a particle moving under the potential V according to the Newton's law, because the particle can not exceed the potential barrier caused by V. If $D(t)$ remains bounded as $t\to\infty$ in a spectral region of H, then it is called that the (dynamical) **Anderson localization** occurs in that spectral region.

Although $D(t)$ describes the diffusion behavior of electrons well, it is necessary to study the time dependent Schrödinger equations to obtain properties for $D(t)$. Generally, it is much easier to investigate the spectral properties of the Schrödinger operator H. Like Hermitian matrices on a vector space with inner product any self-adjoint operator H on a Hilbert space $\mathbb{H}$ has a spectral representation

$$H=\int_{\mathbb{R}}\lambda E(\mathrm{d}\lambda),$$

where $\{E(\mathrm{d}\lambda)\}$ is a projection operators-valued measure on $\mathbb{R}$, that is

(R1) $E(A)$ is an orthogonal projection on $\mathbb{H}$ for each $A\in\mathscr{B}(\mathbb{R})$.

(R2) It holds $E(A)E(B)=E(A\cap B)$ for any $A,B\in\mathscr{B}(\mathbb{R})$, and $E(\phi)=0$, $E(\mathbb{R})=I$ (the identity operator).

Since $\sigma_f(\mathrm{d}\lambda)=(E(\mathrm{d}\lambda)f,f)$ is a ordinary measure on $\mathbb{R}$ with total mass $\|f\|^2$, it is not difficult to see that this $\{E(\mathrm{d}\lambda)\}$ has a Lebesgue decomposition

$$E(\mathrm{d}\lambda)=E_{\mathrm{ac}}(\mathrm{d}\lambda)\oplus E_{\mathrm{sc}}(\mathrm{d}\lambda)\oplus E_{\mathrm{p}}(\mathrm{d}\lambda)\qquad\text{(orthogonal sum)},\tag{2}$$

where $\{E_j(\mathrm{d}\lambda)\}_{j=\mathrm{ac,sc,p}}$ are again families of orthogonal projections on H satisfying the above (R1) and (R2) except the property $E(\mathbb{R})=I$. $\{E(\mathrm{d}\lambda)\}$ is called the **resolution of the identity** for H and the decomposition

$$\sigma_f(\mathrm{d}\lambda)=(E_{\mathrm{ac}}(\mathrm{d}\lambda)f,f)+(E_{\mathrm{sc}}(\mathrm{d}\lambda)f,f)+(E_{\mathrm{p}}(\mathrm{d}\lambda)f,f)$$

yields the ordinary Lebesgue decomposition of the measure σ_f. On the other hand e^{-itH} can be expressed as

$$\mathrm{e}^{-itH}=\int_{\mathbb{R}}\mathrm{e}^{-it\lambda}E(\mathrm{d}\lambda),$$

and

$$u(t,x)=\int_{\mathbb{R}}\mathrm{e}^{-it\lambda}E_{\mathrm{ac}}(\mathrm{d}\lambda)f+\int_{\mathbb{R}}\mathrm{e}^{-it\lambda}E_{\mathrm{sc}}(\mathrm{d}\lambda)f+\int_{\mathbb{R}}\mathrm{e}^{-it\lambda}E_{\mathrm{p}}(\mathrm{d}\lambda)f$$
$$=u_{\mathrm{ac}}(t,x)+u_{\mathrm{sc}}(t,x)+u_{\mathrm{p}}(t,x).$$

Therefore, Riemann-Lebesgue theorem for Fourier transform implies

$$\lim_{t\to\infty}u_{\mathrm{ac}}(t,x)=0 \ \text{ if } x \text{ remains in a fixed bounded domain of } \mathbb{R}^d.$$

Recalling that $|u(t,x)|^2\mathrm{d}x$ is a probability measure we see that if $E_{\mathrm{sc}}(\mathrm{d}\lambda)f=E_{\mathrm{p}}(\mathrm{d}\lambda)f=0$, then for any $R>0$

$$\lim_{t\to\infty}\int_{|x|\geq R}|u(t,x)|^2\mathrm{d}x=1$$

holds, which implies the quantum particle escapes from any bounded domain as $t\to\infty$. On the other hand, if $E_{\mathrm{ac}}(\mathrm{d}\lambda)f=E_{\mathrm{sc}}(\mathrm{d}\lambda)f=0$, which is the case if $V=0$, then

$$u(t,x)=\sum_j\mathrm{e}^{-it\lambda_j}(f,e_j)e_j(x)$$

as before, and under some additional conditions, one sees that the localization occurs. In this way, the problem of localization or delocalization can be translated into the problem of existence of the singularity of the resolution of identity, namely the existence of the absolutely continuous part or the point measure part. Physically $D(t)$ has information of the electricity conductivity.

The pioneering paper of 1958 by P.W. Anderson [1] considered the electricity conductivity of matters containing impurities, which was nothing but to study the spectral properties of H_ω defined in (1). The **ergodic potential** V_ω is defined by an ergodic transform $\{T_x\}_{x\in\mathbb{R}^d}$ on a probability space $(\Omega,\mathscr{F},\mathsf{P})$. That is,

(E1) T_x is a measurable map from Ω to Ω for each $x\in\mathbb{R}^d$ satisfying

$$T_{x+y}=T_xT_y \qquad \text{for any } x,y\in\mathbb{R}^d.$$

(E2) The probability measure P is invariant under $\{T_x\}_{x\in\mathbb{R}^d}$, namely

$$\mathsf{P}(T_x^{-1}A)=\mathsf{P}(A) \ \text{ holds for any } A\in\mathscr{F} \text{ and } x\in\mathbb{R}^d.$$

(E3) $\{T_x\}_{x\in\mathbb{R}^d}$ is ergodic in the sense that

$$\mathsf{P}(T_x^{-1}A\ominus A)=0 \ \text{ for any } x\in\mathbb{R}^d \text{ implies } \mathsf{P}(A)=0 \text{ or } 1.$$

For a real valued measurable function $V(\omega)$ on Ω an ergodic potential V_ω is defined by

$$V_\omega(x) = V(T_x\omega).$$

We give three typical examples of ergodic potentials.

Example 1 (periodic potentials) $\Omega = \mathbb{R}^d/\mathbb{Z}^d$, P is the Lebesgue measure on Ω, and $T_x\omega = \omega + \pi(x)$, where π is the natural map $\mathbb{R}^d \to \Omega$. Then for any real valued measurable function V on Ω

$$V_\omega(x) = V(T_x\omega) = V(\omega + \pi(x))$$

defines a periodic function on $\mathbb{R}^d$.

Example 2 (quasi-periodic potentials) $\Omega = \mathbb{R}^{Nd}/\mathbb{Z}^{Nd}$, P is the Lebesgue measure on Ω, and $T_x\omega = \omega + \pi(Ax)$, where π is the natural map $\mathbb{R}^{Nd} \to \Omega$, and A is an $Nd \times d$ matrix such that

$$Ax \notin \mathbb{Z}^{Nd} \qquad \text{for any } x \in \mathbb{R}^d \backslash \{0\}.$$

Then, for any real valued measurable function V on Ω

$$V_\omega(x) = V(T_x\omega) = V(\omega + \pi(Ax))$$

defines a quasi-periodic function on $\mathbb{R}^d$.

Example 3 (random potentials) Let $\{x_j(\omega)\} \subset \mathbb{R}^d$ be a set of countably many random points distributed in $\mathbb{R}^d$ according to a Poisson law with parameter μ, namely for any $D \in \mathscr{B}(\mathbb{R}^d)$

$$\mathsf{P}(\#\{x_j(\omega) \in D\} = k) = \mathrm{e}^{-\mu|D|}\frac{(\mu|D|)^k}{k!} \qquad \text{for } k = 0, 1, 2, \ldots,$$

and for any disjoint $D_i \in \mathscr{B}(\mathbb{R}^d)$, $i = 1, 2, \ldots, n$, the random variables $\#\{x_j(\omega) \in D_i\}$ are independent. Then, for any measurable function f on $\mathbb{R}^d$

$$V_\omega(x) = \sum_j f(x - x_j(\omega))$$

defines a stationary ergodic potential.

Another example of random potentials is given by a Gaussian random field $\{X_x(\omega)\}_{x\in\mathbb{R}^d}$ with mean 0 and a correlation

$$\rho(x - y) = \mathsf{E}(X_x(\omega)X_y(\omega)).$$

If $\rho(x-y) \to 0$ as $|x-y| \to \infty$, then $\{X_x(\omega)\}_{x\in\mathbb{R}^d}$ is known to be ergodic. For any measurable function f on $\mathbb{R}^d$

$$V_\omega(x) = f(X_x(\omega))$$

defines a stationary ergodic potential.

Exercise 4 Construct suitable Ω, $\{T_x\}_{x\in\mathbb{R}^d}$, P for Example 3.

It should be pointed out that Anderson [1] was not treating continuous Schrödinger operators of (1) but discrete Schrödinger operators

$$(H_\omega u)(x) = \Delta u(x) + V_\omega(x) \qquad \text{(Anderson's tight binding model)}$$

on $\ell^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian defined by

$$\Delta u(x) = \sum_{y\in\mathbb{Z}^d:|x-y|=1} u(y). \tag{3}$$

Discrete Schrödinger operators sometimes avoid unessential difficulties and all definitions and examples which have appeared so far can be replaced by analogous discrete ones.

There are two ways to classify this field. One way is to focus on the space dimension where Schrödinger operators are defined. From the point of view of mathematical tools, in one dimension there are many tools, whereas in higher dimension there is essentially one tool called multiscale analysis which is effective to show the existence of point spectrum. And phenomenologically it is supposed to exist difference between low dimension and high dimension, although it has not been settled mathematically. The other way is to look at the phenomena depending on the degree of randomness. For last 15 years the study of ergodic Schrödinger operators in one dimension has been separated into several fields, namely quasi-periodic (or more generally almost periodic) potentials, number theoretic potentials, and random potentials. The purpose of this lecture note is to give a basic knowledge of this field in one dimension to people who are not familiar with spectral theory for ergodic Schrödinger operators.

To people who want information of recent development Jitomirskaya [16] is recommendable. A general physical view of this field can be obtained by [28], and mathematical view by [7], [32]. To understand Anderson localization in general dimension [37], [18] are suitable to read in random case and [3] in quasi-periodic case.

The contents overlaps with Damanik [10], and will be as follows:

In the sequel the following notations are used:

$$\mathbb{R}_+ = \{x \in \mathbb{R};\, x > 0\}, \qquad \mathbb{R}_- = \{x \in \mathbb{R};\, x < 0\},$$
$$\mathbb{C}_+ = \{z \in \mathbb{C};\, \operatorname{Im} z > 0\}, \qquad \mathbb{C}_- = \{z \in \mathbb{C};\, \operatorname{Im} z < 0\}.$$

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§1. General Spectral Properties

Firstly we discuss general spectral properties for ergodic Schrödinger operators in general dimension, which state that each component of the spectrum is independent of individual samples. These results were obtained by L.A. Pastur [31]. Historically, the integrated density of states (IDS) was investigated exclusively, and still now plays an important role. The IDS is usually defined as a thermodynamic limit of the distribution of eigenvalues for ergodic Schrödinger operators considered in finite domains, although in this note we take a different approach for the definition.

For simplicity discrete Schrödinger operators are treated in the proof of several statements in this section, suitable modifications make it possible to obtain analogous results for continuous Schrödinger operators. Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space on which a multi-parameter ergodic dynamical system $\{T_x\}_{x\in\mathbb{Z}^d}$ sits. For a real valued bounded measurable function V on Ω define a Schrödinger operator

$$H_\omega = \Delta + V_\omega,$$

where Δ is the discrete Laplacian defined in (3) and $V_\omega(x) = V(T_x\omega)$. We do not put the "$-$" sign in front of Δ. Since Δ is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^d)$, H_ω can be realized as a self-adjoint operator for any $\omega \in \Omega$. Let $\{\theta_x\}$ be the shift operator on $\ell^2(\mathbb{Z}^d)$ defined by

$$(\theta_x f)(y) = f(y + x).$$

Then, a basic observation for H_ω is

$$H_{T_x\omega} = \theta_x H_\omega \theta_x^{-1} \qquad \text{for any } x \in \mathbb{Z}^d,\ \omega \in \Omega. \tag{4}$$

Let $E_\omega(\mathrm{d}\lambda)$ be the resolution of identity for H_ω. Since θ_x is a unitary operator on $\ell^2(\mathbb{Z}^d)$, without difficulty we have

$$E_{T_x\omega}(A) = \theta_x E_\omega(A)\theta_x^{-1}. \tag{5}$$

Set

$$\mathbb{H} = \ell^2(\mathbb{Z}^d).$$

Lemma 5 Suppose random orthogonal projection P_ω satisfies

$$P_{T_x\omega} = \theta_x P_\omega \theta_x^{-1} \tag{6}$$

for any $x \in \mathbb{Z}^d$, $\omega \in \Omega$. Then it holds that

$$\begin{cases} \dim P_\omega(\mathbb{H}) = 0 \ \text{ a.s. if } \ \mathsf{E}(P_\omega\delta_0, \delta_0) = 0 \\ \text{or} \\ \dim P_\omega(\mathbb{H}) = \infty \ \text{ a.s. if } \ \mathsf{E}(P_\omega\delta_0, \delta_0) > 0. \end{cases}$$

Proof For $j \in \mathbb{Z}^d$ define a complete orthonormal basis of $\mathbb{H}$ by

$$\delta_j(x) = \begin{cases} 1 & \text{for } x = j; \\ 0 & \text{for } x \neq j. \end{cases} \tag{7}$$

Then, noting

$$d(\omega) \equiv \dim P_\omega(\mathbb{H}) = \sum_{j\in\mathbb{Z}^d} (P_\omega\delta_j, \delta_j),$$

we have from (6)

$$d(T_x\omega) = \sum_{j\in\mathbb{Z}^d} (\theta_x P_\omega \theta_x^{-1}\delta_j, \delta_j) = \sum_{j\in\mathbb{Z}^d} (P_\omega \theta_x^{-1}\delta_j, \theta_x^{-1}\delta_j) = d(\omega),$$

since $\{\theta_x^{-1}\delta_j\}_j$ is also a complete orthonormal basis in $\mathbb{H}$. Then the ergodicity of $(\{T_x\}, \mathsf{P})$ implies $d(\omega)$ is equal to a constant a.s.. Let

$$\alpha = \mathsf{E}(P_\omega\delta_0, \delta_0).$$

Since

$$0 \le (P_\omega\delta_0, \delta_0) \le 1,$$

we have $0 \le \alpha \le 1$. Since

$$\mathsf{E}(P_\omega\delta_j, \delta_j) = \mathsf{E}(P_\omega\theta_{-j}\delta_0, \theta_{-j}\delta_0) \underset{(5)}{=} \mathsf{E}(P_{T_j\omega}\delta_0, \delta_0) \underset{\text{invariance of } T_j}{=} \alpha$$

holds for any $j \in \mathbb{Z}^d$, we have

$$d(\omega) = \mathsf{E}\, d(\cdot) = \begin{cases} 0 & \text{a.s. if } \alpha = 0; \\ \infty & \text{a.s. if } \alpha > 0, \end{cases}$$

which shows Lemma 5. □

In view of Lemma 5 we define a measure on $\mathbb{R}$ by

$$N(\mathrm{d}\lambda) = \mathsf{E}(E_\omega(\mathrm{d}\lambda)\delta_0, \delta_0). \tag{8}$$

The **spectrum** Σ_ω of H_ω is defined as

$$\Sigma_\omega = \operatorname{supp} E_\omega(\mathrm{d}\lambda) = \{\lambda \in \mathbb{R}; E_\omega((\lambda-\epsilon, \lambda+\epsilon)) \neq 0 \text{ for any } \epsilon > 0\}.$$

Then we have

Theorem 6 (Pastur) Σ_ω is independent of ω a.s. and coincides with $\operatorname{supp} N(\mathrm{d}\lambda)$.

Proof Since

$$E_\omega((\lambda-\epsilon, \lambda+\epsilon)) \neq 0 \Longleftrightarrow \dim E_\omega((\lambda-\epsilon, \lambda+\epsilon))(\mathbb{H}) > 0,$$

which is equivalent to

$$N((\lambda-\epsilon, \lambda+\epsilon)) = \mathsf{E}(E_\omega((\lambda-\epsilon, \lambda+\epsilon))\delta_0, \delta_0) > 0$$

due to Lemma 5, which shows the theorem. □

$N(\lambda) = N((-\infty, \lambda])$ is called the **integrated density of states** (IDS for short) for the discrete ergodic Schrödinger operator H^ω, which is a fundamental object in the study of ergodic Schrödinger operators.

Physically $N(\lambda)$ has another equivalent definition, namely for any rectangle Λ of $\mathbb{Z}^d$ let $\{\lambda^{\Lambda}_{\omega,k}\}_{k=1,2,\ldots,|\Lambda|}$ be the set of all eigenvalues of H_ω restricted to $\ell^2(\Lambda)$. Then one can show

$$N(\lambda) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \#\{k; \lambda^{\Lambda}_{\omega,k} \le \lambda\} \quad \text{a.s.} \qquad \text{for any } \lambda \in \mathbb{R}. \tag{9}$$

This identity is the origin of the name of $N(\lambda)$.

Remark 7 It is known that in any discrete ergodic Schrödinger operators

$$N(\{\lambda\}) = 0$$

for any $\lambda \in \mathbb{R}$, namely N is a continuous measure. This fact is believed to be valid also for continuous ergodic Schrödinger operators, however there has been no proof for this conjecture yet.

Since the Lebesgue decomposition (2) is unique for any resolution of the identity, $\{E_{\omega,j}(\mathrm{d}\lambda)\}_{j=\mathrm{ac,sc,p}}$ have the property (5) as well. Therefore denoting

$$\Sigma_{\omega,j} = \operatorname{supp} E_{\omega,j}(\mathrm{d}\lambda), \tag{10}$$

similarly we have

Theorem 8 (Pastur) $\Sigma_{\omega,\mathrm{ac}}$, $\Sigma_{\omega,\mathrm{sc}}$, $\Sigma_{\omega,\mathrm{p}}$ are independent of ω a.s..

$\Sigma_{\omega,\mathrm{ac}}$, $\Sigma_{\omega,\mathrm{sc}}$, $\Sigma_{\omega,\mathrm{p}}$ are called **absolutely continuous** (**ac in short**) **spectrum**, **singular continuous** (**sc in short**) **spectrum**, **point spectrum** (**p in short**) respectively.

Remark 9 Set

$$\widetilde{N}_j(\mathrm{d}\lambda) = \mathsf{E}((E_{\omega,j}(\mathrm{d}\lambda)\delta_0, \delta_0)).$$

Then obviously $\widetilde{N}_{\mathrm{ac}}(\mathrm{d}\lambda)$ is an absolutely continuous measure. However, it is not generally valid that $\widetilde{N}_{\mathrm{ac}}$ coincides with the absolutely continuous part of $N(\mathrm{d}\lambda)$ although we have generally

$$N_{\mathrm{ac}}(\mathrm{d}\lambda) \ge \widetilde{N}_{\mathrm{ac}}(\mathrm{d}\lambda).$$

This is because if $\{V_\omega(x)\}_{x\in\mathbb{Z}^d}$ are i.i.d. random variables whose distribution has a bounded, absolutely continuous density, then it is known that $N(\mathrm{d}\lambda)$ is always absolutely continuous. On the other hand, as we will see later, in one dimension we have only point spectrum with probability one. Needless to say, $N_j \ne \widetilde{N}_j$ for $j = \mathrm{sc}$, p in general.

§2. Ergodic Schrödinger Operators in One Dimension

The previous results can be applied also to one dimensional Schrödinger operators with ergodic potentials. However, among other things in one dimension one can show that the IDS $N(\lambda)$ can determine the absolutely continuous part $\Sigma_{\omega,\mathrm{ac}}$. The key word is the reflectionless property which characterize the absolutely continuous spectrum in general, and as a result it has been shown that ergodic Schrödinger operators have close relationship with a class of completely integrable systems.

We start with several basic facts of spectral properties of general deterministic 1D Schrödinger operators developed by Weyl-Stone-Titchmarsh-Kodaira. In one dimension the arguments are simpler and more transparent in continuous Schrödinger operators than in discrete operators, so in the proofs we consider mainly continuous models.

2.1 Weyl Function

In this subsection potentials V are deterministic functions. For simplicity we assume V is bounded. In one dimension the notion of Weyl function (sometimes called as Weyl-Titchmarsh function) is crucial. It has two important properties; one is its relationship with the spectrum and the other is its one-to-one correspondence with the potential.

Denote by $f_+(x,z)$ the unique solution u whose existence was guaranteed in Lemma 35, and define

$$m_+(z) = f_+'(0,z).$$

An identity $f_+(x,\overline{z}) = \overline{f_+(x,z)}$ due to $V(x)\in\mathbb{R}$ shows

$$m_+(\overline{z}) = \overline{m_+(z)}. \tag{11}$$

Similarly m_- is defined by using the solution $f_-(x,z)$ on the negative axis:

$$m_-(z) = -f_-'(0,z).$$

Lemma 10 For any $z,w\in\mathbb{C}\backslash\mathbb{R}$ we have

$$\frac{m_+(z)-m_+(w)}{z-w} = \int_0^\infty f_+(x,z)f_+(x,w)\mathrm{d}x.$$

Proof Integration by parts shows (see Exercise 36)

$$\begin{aligned}(z-w)\int_0^\infty f_+(x,z)f_+(x,w)\mathrm{d}x &= \int_0^\infty(-f_+''(x,z)+V(x)f_+(x,z))f_+(x,w)\mathrm{d}x\\ &\quad-\int_0^\infty(-f_+''(x,w)+V(x)f_+(x,w))f_+(x,z)\mathrm{d}x\\ &= f_+'(0,z)-f_+'(0,w) = m_+(z)-m_+(w). \qquad \square\end{aligned}$$

Setting $w = \bar{z}$ and noting (11) we have

$$\frac{\operatorname{Im} m_+(z)}{\operatorname{Im} z} = \int_0^\infty |f_+(x,z)|^2 \mathrm{d}x > 0. \tag{12}$$

m_+ is called **Weyl function** on $\mathbb{R}_+$ for the potential V. The Weyl function m_- on $\mathbb{R}_-$ can be defined similarly. Obviously $m_\pm$ are analytic on $\mathbb{C}\backslash\mathbb{R}$ and map the upper half plane into itself. Such a function m is called a **Herglotz function**. For necessary properties of Herglotz functions refer to Appendix B. If $V = 0$, then for $z \in \mathbb{C}\backslash\mathbb{R}$

$$f_+(x,z) = \mathrm{e}^{i\sqrt{z}x}, \qquad f_-(x,z) = \mathrm{e}^{-i\sqrt{z}x},$$

where $\sqrt{z}$ is chosen so that $\operatorname{Im}\sqrt{i} = \mathrm{e}^{\pi i/2}$. Hence, for $V = 0$

$$m_+(z) = i\sqrt{z}, \qquad m_-(z) = i\sqrt{z} \quad (\text{note here } m_-(z) = -f'_-(0,z)).$$

In (ii) of Lemma 37 we showed

$$|m_+(z) - i\sqrt{z}| \le \frac{C}{2\operatorname{Im}\sqrt{z}}\Big(1 + \frac{C}{\operatorname{Im} z}\Big),$$

where $C = \sup|V(x)|$. Hence $m_+(z)$ is close to $i\sqrt{z}$ as $\operatorname{Im} z \to \infty$. If V is smooth, then an analogous calculation shows

$$m_+(z) \sim i\sqrt{z} + V(0)(2i\sqrt{z})^{-1} - V'(0)(2i\sqrt{z})^{-2} + \cdots. \tag{13}$$

The **Green function** $g_z(x,y)$, which is the kernel of the resolvent operator $(-\Delta + V - z)^{-1}$ has an expression:

$$g_z(x,y) = g_z(y,x) = -\frac{f_+(x,z)f_-(y,z)}{m_+(z) + m_-(z)} \qquad \text{for } x \ge y. \tag{14}$$

Note

$$g_z(0,0) = -\frac{1}{m_+(z) + m_-(z)}. \tag{15}$$

Since $-(m_+ + m_-)^{-1}$ is again a Herglotz function, (ii) of Lemma 37 shows that there exists a measure σ such that

$$g_z(0,0) = \int_c^\infty \frac{1}{\lambda - z}\sigma(\mathrm{d}\lambda) \qquad \Big(\int_c^\infty \frac{1}{1+|\lambda|}\sigma(\mathrm{d}\lambda) < \infty\Big), \tag{16}$$

where $c = \inf V(x)$. The measure σ is called the **spectral measures** of $H = -\Delta + V$. It is known that the resolution of identity $E(\mathrm{d}\lambda)$ can be described by $m_\pm$, which is known as Weyl-Stone-Titchmarsh-Kodaira theorem. A brief introduction to the theorem can be found in Appendix A.

2.2 Homologous Relation under Shift

For a function V on $\mathbb{R}$ the shift operation θ_x is defined by

$$(\theta_x V)(\cdot) = V(\cdot + x).$$

Let f, g be functions of potentials V. We say that f, g are **homologous** with respect to $\{\theta_x\}$ if they satisfy

$$f(\theta_x V) - g(\theta_x V) = \frac{\mathrm{d}}{\mathrm{d}x} h(\theta_x V)$$

for an h. This relation will be used to show several identities between different random variables related to Weyl functions, which was first employed by Johnson-Moser [15].

We denote every quantity depending on a potential V by designating V explicitly if it is necessary. For instance $f_+(x, z)$ is defined through a potential V, so we denote it by $f_+(x, z, V)$. The fundamental observation is identities

$$f_\pm(x + y, z, V) = f_\pm(x, z, V) f_\pm(y, z, \theta_x V), \tag{17}$$

or its differential form:

$$f'_\pm(x, z, V) = \pm m_\pm(z, \theta_x V) f_\pm(x, z, V), \tag{18}$$

which obeys

$$f_\pm(x, z, V) = \exp\Big(\pm \int_0^x m_\pm(z, \theta_y V)\mathrm{d}y\Big). \tag{19}$$

The identity (17) is easily verified from Lemma 35 as follows. Since $f_+(x, z, V) \neq 0$, one can define

$$g(y) \equiv \frac{f_+(x + y, z, V)}{f_+(x, z, V)} \in L^2(\mathbb{R}_+),$$

and see

$$-g'' + (\theta_x V)g = zg, \qquad g(0) = 1.$$

Then Lemma 35 shows $g(y) = f_+(y, z, \theta_x V)$. (18) can be derived from (17) by taking the derivative.

From (18)

$$m_+(z, \theta_x V) = \frac{f'_+(x, z, V)}{f_+(x, z, V)}$$

and taking derivative, we see m_+ satisfies a Riccati equation:

$$\frac{\mathrm{d}}{\mathrm{d}x} m_+(z, \theta_x V) = V(x) - z - m_+(z, \theta_x V)^2.$$

We have a similar equation for m_-, and consequently

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x} m_+(z,\theta_x V) = V(x) - z - m_+(z,\theta_x V)^2, \\ \dfrac{\mathrm{d}}{\mathrm{d}x} m_-(z,\theta_x V) = -V(x) + z + m_-(z,\theta_x V)^2. \end{cases} \tag{20}$$

(20) leads us to the homologous relations between quantities related to $m_\pm$. In the lemma below log is defined on the upper half plane $\mathbb{C}_+$ so that it satisfies

$$\log i = \frac{\pi}{2} i.$$

Lemma 11 The followings are valid. Suppose $\operatorname{Im} z > 0$.

(i) $m_-(z,\theta_x V) - m_+(z,\theta_x V) = \dfrac{\mathrm{d}}{\mathrm{d}x} \log g_z(0,0,\theta_x V)$;

(ii) $\operatorname{Re} m_\pm(z,\theta_x V) + \dfrac{1}{2}\dfrac{\operatorname{Im} z}{\operatorname{Im} m_\pm(z,\theta_x V)} = \mp \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}x} \log \operatorname{Im} m_\pm(z,\theta_x V)$;

(iii) $g_z(0,0,\theta_x V) + \dfrac{\mathrm{d}}{\mathrm{d}z}(2g_z(0,0,\theta_x V))^{-1} = \dfrac{\mathrm{d}}{\mathrm{d}x} H(z,\theta_x V)$, where

$$H(z,V) = \frac{1}{2} \frac{\dfrac{\mathrm{d}}{\mathrm{d}z}(m_+(z,V) - m_-(z,V))}{m_+(z,V) + m_-(z,V)}.$$

Proof The identities (i), (ii) are direct from (20), if we note (15). To shorten the notations we use

$$m_\pm = m_\pm(z,\theta_x V).$$

To show (iii) take the derivative of (20) with respect to z and obtain

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\mathrm{d}m_+}{\mathrm{d}z} = -1 - 2m_+ \dfrac{\mathrm{d}m_+}{\mathrm{d}z}, \\ \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\mathrm{d}m_-}{\mathrm{d}z} = 1 + 2m_- \dfrac{\mathrm{d}m_-}{\mathrm{d}z}. \end{cases} \tag{21}$$

Then

$$\begin{aligned} & 2\frac{\mathrm{d}}{\mathrm{d}x} H(z,\theta_x V) \\ = & \frac{1}{m_+ + m_-} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{d}(m_+ - m_-)}{\mathrm{d}z} - \frac{1}{(m_+ + m_-)^2} \frac{\mathrm{d}(m_+ + m_-)}{\mathrm{d}x} \frac{\mathrm{d}(m_+ - m_-)}{\mathrm{d}z}. \end{aligned}$$

Using (21), (20) in the first term and the second term respectively, we have

$$\begin{aligned} &= \frac{1}{m_+ + m_-} \Big\{ (m_+ - m_-) \frac{\mathrm{d}(m_+ - m_-)}{\mathrm{d}z} - 2\Big(1 + m_+ \frac{\mathrm{d}m_+}{\mathrm{d}z} + m_- \frac{\mathrm{d}m_-}{\mathrm{d}z}\Big) \Big\} \\ &= -\frac{\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}z}(m_+ + m_-)^2 + 2}{m_+ + m_-} = \frac{\mathrm{d}}{\mathrm{d}z} g_z(0,0,\theta_x V)^{-1} + 2g_z(0,0,\theta_x V), \end{aligned}$$

which yields (iii). $\square$

2.3 Floquet Exponent

Originally Floquet exponent was introduced to describe the exponents of $f_\pm$ for periodic Schrödinger operators, and Johnson-Moser [15] generalized it to almost periodic Schrödinger operators. Without any difficulty one can extend it to general ergodic Schrödinger operators. The properties of Floquet exponent shown in this subsection will play a central role in the next subsection.

Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space and $\{T_x\}_{x\in\mathbb{R}}$ be an ergodic dynamical system. For a real valued bounded measurable function V on Ω define a Schrödinger operator

$$H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(T_x\omega).$$

For each $\omega \in \Omega$ let $m_\pm(z,\omega)$ be the Weyl functions for H_ω. The **Floquet exponents** $w_\pm(z)$ are defined by

$$w_\pm(z) = \mathsf{E}\, m_\pm(z,\omega). \tag{22}$$

The identities (19) and the ergodic theorem imply

$$\lim_{x\to\pm\infty} \frac{1}{|x|} \log f_\pm(x,z,\omega) = w_\pm(z) \qquad \text{a.s.}. \tag{23}$$

This is a meaning of the Floquet exponents for individual potential. Although for completely rigorous arguments it is necessary to ensure the finiteness of expectations, we omit this procedure from now on for simplicity.

Apparently $w_\pm$ are Herglotz functions. Applying Lemma 11, we have

Lemma 12 The following identities are valid.

(i) $w_+(z) = w_-(z)$;

(ii) $\mathrm{Re}\, w_\pm(z) = -\dfrac{1}{2}\mathsf{E}\Big(\dfrac{\mathrm{Im}\, z}{\mathrm{Im}\, m_\pm(z,\omega)}\Big)$;

(iii) $w'_\pm(z) = \mathsf{E}\, g_z(0,0,\omega)$.

We omit the proof, since from the homologous relations the identities immediately follow due to the invariance of P under $\{T_x\}$. The identities (i), (iii) were obtained by Johnson-Moser [15] and (ii) by Kotani [19].

Exercise 13 Show the following identities.

(i) $\mathsf{E}\dfrac{m_+(z,\omega)}{m_+(z,\omega)+m_-(z,\omega)} = \mathsf{E}\dfrac{m_-(z,\omega)}{m_+(z,\omega)+m_-(z,\omega)} = \dfrac{1}{2}$;

(ii) $\mathsf{E}\dfrac{m_+(z,\omega)m_-(z,\omega)}{m_+(z,\omega)+m_-(z,\omega)} = \mathsf{E}\dfrac{V(\omega)-z}{m_+(z,\omega)+m_-(z,\omega)}$.

Remark 14 $m_\pm(z,\omega)$ can be regarded as the random variables expressing the equilibrium states of a certain stochastic process. To simplify the situation we assume the potential is Gaussian white noise, namely we consider

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\mathrm{d}B_t}{\mathrm{d}t}, \qquad \text{where } \{B_t\}_{t\in\mathbb{R}} \text{ is a 1D Brownian motion.}$$

Then, (20) turns to be an SDE

$$\mathrm{d}m_-(z,T_t\omega) = (z + m_-(z,T_t\omega)^2)\mathrm{d}t - \mathrm{d}B_t(\omega), \tag{24}$$

and the distribution of $m_-(z,\omega)$ is nothing but the invariant measure of a diffusion process $Z_t \in \mathbb{C}_+$ determined by

$$\mathrm{d}Z_t = (z + Z_t^2)\mathrm{d}t - \mathrm{d}B_t.$$

A traditional approach to study the equilibrium state for $\{Z_t\}$ is to analyze the corresponding differential equations. However, the present analysis is a direct calculation using the SDE (24).

Since we have (i) of Lemma 12, we denote $w_\pm(z)$ by a single $w(z)$. Let $\sigma_\omega(\mathrm{d}\lambda)$ be the spectral measure for H_ω introduced in (16). Then the above identity (iii) of Lemma 12 shows

$$w'(z) = \int_c^\infty \frac{1}{\lambda - z}\mathsf{E}\,\sigma_\omega(\mathrm{d}\lambda). \tag{25}$$

On the other hand, from

$$g_z(0,0,\omega) = \big((-\Delta + V_\omega - z)^{-1}\delta_0, \delta_0\big) = \int_c^\infty \frac{1}{\lambda - z}(E_\omega(\mathrm{d}\lambda)\delta_0, \delta_0)$$

(δ_0 denotes the delta function in the continuous case) it follows that

$$\sigma_\omega(\mathrm{d}\lambda) = (E_\omega(\mathrm{d}\lambda)\delta_0, \delta_0).$$

Therefore, we have

$$\mathsf{E}\,\sigma_\omega(\mathrm{d}\lambda) = \mathsf{E}(E_\omega(\mathrm{d}\lambda)\delta_0, \delta_0) = \mathrm{d}N(\lambda), \tag{26}$$

where $N(\lambda)$ is the IDS for H_ω. Combining this with (25) yields

$$w'(z) = \int_c^\infty \frac{1}{\lambda - z}\mathrm{d}N(\lambda). \tag{27}$$

This identity was first noticed by Johnson-Moser [15]. For $z \in \mathbb{C}_+$ (ii) of Lemma 12 implies $\mathrm{Re}\, w(z) < 0$, which is consistent with $f_+ \in L^2(\mathbb{R}_+)$ in (23). The quantity

$$\gamma(z) = -\mathrm{Re}\, w(z) > 0 \tag{28}$$

is called as the **Lyapounov exponent**. In conclusion, we have

$$w,\ \ w',\ \ -iw \ \text{ are of Herglotz.} \tag{29}$$

The integrated version of (27) is

$$w(z) = w(i) + \int_c^{\infty} \{\log(\lambda - z)^{-1} - \log(\lambda - i)^{-1}\} \mathrm{d}N(\lambda), \tag{30}$$

and the property $\mathrm{Re}\, w(z) < 0$ implies that for any finite $a < b$ there exists a constant c such that

$$\int_a^b \log \frac{1}{|\lambda - z|} \mathrm{d}N(\lambda) \leq c \qquad \text{for any } z \text{ satisfying } 0 < \mathrm{Im}\, z < 1,$$

which shows the log-Hölder continuity of N, namely

$$|N(\lambda_1) - N(\lambda_2)| \leq -c \log |\lambda_1 - \lambda_2| \ \text{ for } \ \lambda_1, \lambda_2 \in [a, b] \ \text{ s.t. } |\lambda_1 - \lambda_2| \leq 1. \tag{31}$$

If a analytic function w on $\mathbb{C}_+$ satisfies (29), then $\mathrm{Im}\, w'(z) > 0$ implies the monotone increasing property of $\mathrm{Re}\, w(\lambda + iy)$ with respect to $y > 0$, which shows that w has a finite limit

$$w(\lambda + i0) = \lim_{\epsilon \downarrow 0} w(\lambda + i\epsilon) \qquad \text{for any } \lambda \in \mathbb{R}.$$

In the discrete case we have similar identities to those of Lemma 2 (see [34]), and w has the same properties. Especially (30) takes a simpler form

$$w(z) = \int_{-\infty}^{\infty} \log(\lambda - z)^{-1} \mathrm{d}N(\lambda)$$

if we assume the boundedness of V. Taking the real part, we have

$$\gamma(z) = -\mathrm{Re}\, w(z) = \int_{-\infty}^{\infty} \log |\lambda - z| \mathrm{d}N(\lambda) \qquad \text{for any } z \in \mathbb{C}, \tag{32}$$

which is called the **Thouless formula**.

Historically the Lyapounov exponent was introduced to study the exponential growth of solutions to a system of a first order linear ODE:

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V_\omega - z & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{33}$$

which is equivalent to $-u'' + V_\omega u = zu$. Let $U_\omega(x, z)$ be the fundamental matrix for (33), that is, $U_\omega \in SL(2, \mathbb{C})$ is the solution to

$$\frac{\mathrm{d}}{\mathrm{d}x} U(x) = \begin{pmatrix} 0 & 1 \\ V(T_x\omega) - z & 0 \end{pmatrix} U(x), \qquad U(0) = I, \tag{34}$$

where I is the 2×2 identity matrix. Then the Lyapounov exponent is defined by

$$\widetilde{\gamma}(z)=\lim_{x\to\infty}\frac{1}{x}\log\|U_\omega(x,z)\| \qquad \text{a.s..}$$

It should be noted that the other Lyapounov exponent for $\{U_\omega(x,z)\}$ is $-\widetilde{\gamma}(z)$ since $\det U_\omega(x,z)=1$ (see Appendix D). The existence is known by the subadditive ergodic theorem, and the limit is independent of ω because of the ergodicity. Since $f_\pm(x,z,\omega)$ are linearly independent solutions to $-u''+V_\omega u=zu$ for $z\in\mathbb{C}\backslash\mathbb{R}$, $U_\omega(x,z)$ can be expressible by $f_\pm$:

$$\begin{aligned}U_\omega(x,z)&=\begin{pmatrix} f_+(x,z,\omega) & f_-(x,z,\omega)\\ f_+'(x,z,\omega) & f_-'(x,z,\omega)\end{pmatrix}\begin{pmatrix} f_+(0,z,\omega) & f_-(0,z,\omega)\\ f_+'(0,z,\omega) & f_-'(0,z,\omega)\end{pmatrix}^{-1}\\ &=\begin{pmatrix} f_+(x,z,\omega) & f_-(x,z,\omega)\\ f_+'(x,z,\omega) & f_-'(x,z,\omega)\end{pmatrix}\begin{pmatrix} 1 & 1\\ m_+(z,\omega) & -m_-(z,\omega)\end{pmatrix}^{-1}\end{aligned}$$

from which the identity

$$\widetilde{\gamma}(z)=\lim_{x\to\infty}\frac{1}{x}\log\|U_\omega(x,z)\|=-\mathrm{Re}\,w(z)=\gamma(z) \tag{35}$$

follows. Although this identity is valid for $z\in\mathbb{C}\backslash\mathbb{R}$, both sides have finite values for all $z\in\mathbb{C}$. Craig-Simon [8] showed the identity holds for all $z\in\mathbb{C}$ by using subharmonicity. Note that $\gamma(\lambda)$ may vanish for real λ. Oseledec theorem [30] implies that if $\gamma(\lambda)>0$, then there exists a non-trivial solution f which decays exponentially fast with exponent $-\gamma(\lambda)$ as $x\to\infty$ (see Theorem 47 in Appendix). This property is closely related to the existence of point spectrum, or equivalently eigenvalues whose eigenfunctions decay exponentially fast.

2.4 AC Spectrum and Reflectionless Property

In this subsection, the relation between absolutely continuous spectrum and Floquet exponent is investigated. The key word is reflectionless property, which was known for decaying or periodic potentials historically. Most of the results in this subsection were obtained by the author [19].

The key observation for the main results is an identity

$$-\frac{\mathrm{Re}\,w(z)}{\mathrm{Im}\,z}-\mathrm{Im}\,w'(z)=\mathsf{E}\Big\{\Big(\frac{1}{\mathrm{Im}\,m_+}+\frac{1}{\mathrm{Im}\,m_-}\Big)\Big|\frac{m_++\overline{m}_-}{m_++m_-}\Big|^2\Big\} \tag{36}$$

due to Lemma 12, where $m_\pm=m_\pm(z,\omega)$. To study the limit of the left hand side of (36) as $\mathrm{Im}\,z\downarrow 0$, we need two lemmas.

For the IDS N and the Lyapounov exponent γ set

$$\begin{cases} A = \Big\{\lambda \in \mathbb{R} : \lim\limits_{\epsilon\downarrow 0} \dfrac{N(\lambda+\epsilon)-N(\lambda-\epsilon)}{2\epsilon} \text{ exists finitely}\Big\}, \\ \mathscr{Z} = \{\lambda \in \mathbb{R} : \gamma(\lambda) = 0\}, \end{cases}$$

and for $\lambda \in A$

$$N'(\lambda) \equiv \lim_{\epsilon\downarrow 0} \frac{N(\lambda+\epsilon)-N(\lambda-\epsilon)}{2\epsilon}.$$

Lemma 15 For $\lambda \in \mathscr{Z} \cap A$ we have

$$-\lim_{\epsilon\downarrow 0} \frac{\operatorname{Re} w(\lambda+i\epsilon)}{\epsilon} = \lim_{\epsilon\downarrow 0} \operatorname{Im} w'(\lambda+i\epsilon) = \pi N'(\lambda).$$

Proof Set

$$\gamma(x,y) = -\operatorname{Re} w(x+iy).$$

Then, Cauchy-Riemann equation implies

$$w'(z) = -\frac{\partial\gamma}{\partial x} + i\frac{\partial\gamma}{\partial y}.$$

Therefore, from (27)

$$\frac{\partial\gamma}{\partial y}(\lambda,\epsilon) = \int_{-\infty}^{\infty} \frac{\epsilon}{(x-\lambda)^2+\epsilon^2} \mathrm{d}N(x),$$

and (78) in Appendix shows for $\lambda \in A$

$$\lim_{\epsilon\downarrow 0} \frac{\partial\gamma}{\partial y}(x,\epsilon) = \pi N'(\lambda) < \infty.$$

Consequently, we have, for $\lambda \in A \cap \mathscr{Z}$

$$-\lim_{\epsilon\downarrow 0} \frac{\operatorname{Re} w(\lambda+i\epsilon)}{\epsilon} = \lim_{\epsilon\downarrow 0} \frac{\gamma(\lambda,\epsilon)-\gamma(\lambda,0)}{\epsilon} = \pi N'(\lambda). \qquad \square$$

Set

$$\begin{cases} \mathfrak{A}_{\pm} = \{(\lambda,\omega) \in \mathbb{R}\times\Omega : \exists\, m_{\pm}(\lambda+i0,\omega) \text{ finitely in } \mathbb{C}_{+}\}, \\ \mathfrak{A}_{\pm}^{\omega} = \{\lambda \in \mathbb{R} : (\lambda,\omega) \in \mathfrak{A}_{\pm}\}. \end{cases} \tag{37}$$

Then they are measurable sets of $\mathscr{B}(\mathbb{R}) \times \mathscr{F}$. To state the lemma below we need the notion of essential closure. The **essential closure** of $A \in \mathscr{B}(\mathbb{R})$ w.r.t. Lebesgue measure is defined by

$$\overline{A}^{\text{ess}} \equiv \{\lambda \in \mathbb{R} : \text{ for any } \epsilon > 0,\ |A \cap (\lambda-\epsilon,\lambda+\epsilon)| > 0\},$$

where $|A|$ denotes the Lebesgue measure of $A \in \mathscr{B}(\mathbb{R})$.

Lemma 16 There exist measurable sets $\mathscr{A}_\pm$ of $\mathscr{B}(\mathbb{R})$ such that with probability one

$$\mathfrak{A}_\pm^\omega = \mathscr{A}_\pm \quad \text{a.e.}, \qquad \text{namely } |\mathfrak{A}_\pm^\omega \ominus \mathscr{A}_\pm| = 0 \tag{38}$$

hold. Moreover, the ac spectrum $\Sigma_{\omega,\mathrm{ac}}$ of H_ω is given by

$$\Sigma_{\omega,\mathrm{ac}} = \overline{\mathscr{A}_+ \cup \mathscr{A}_-}^{\,\mathrm{ess}} \qquad \text{a.s.}. \tag{39}$$

Proof Note for the fundamental matrix U_ω of (34)

$$m_+(z, T_x\omega)^{-1} = U_\omega(x,z)\cdot m_+(z,\omega)^{-1}$$

holds, where for $m \in \mathbb{C}$

$$U\cdot m = \frac{am+b}{cm+d} \qquad \text{if } U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $U_\omega(x,z)$ are entire w.r.t. z and define elements of $SL(2,\mathbb{R})$ if $z\in\mathbb{R}$, we see without difficulty

$$\mathfrak{A}_+^{T_x\omega} = \mathfrak{A}_+^\omega \qquad \text{for any } x\in\mathbb{R},\ \omega\in\Omega.$$

Hence $f(\lambda,\omega) = I_{\mathfrak{A}_+}(\lambda,\omega)$ obeys $f(\lambda, T_x\omega) = f(\lambda,\omega)$, which implies that for each $\lambda\in\mathbb{R}$

$$f(\lambda,\omega) = 1 \quad \text{a.s.} \qquad \text{or} \qquad f(\lambda,\omega) = 0 \quad \text{a.s.}$$

holds. Then the function

$$f(\lambda) \equiv \mathsf{E}\, f(\lambda,\omega)$$

takes 1 or 0 for each $\lambda\in\mathbb{R}$. Due to Fubini theorem the set

$$\mathscr{A}_+ \equiv \{\lambda\in\mathbb{R} : f(\lambda) = 1\} \in \mathscr{B}(\mathbb{R})$$

satisfies

$$\mathfrak{A}_+^\omega = \mathscr{A}_+ \qquad \text{a.e.}$$

with probability one. For $\mathfrak{A}_-^\omega$ one can define $\mathscr{A}_-$ similarly.

For a general Schrödinger operator H_V the ac spectrum is the sum of the supports of the ac parts of the spectral measures for the two Herglotz functions

$$m_1 = -(m_+ + m_-)^{-1}, \qquad m_2 = m_+ m_-(m_+ + m_-)^{-1}.$$

Define

$$\mathfrak{A}_i = \{\lambda\in\mathbb{R} : \exists\, m_i(\lambda + i0) \text{ finitely in } \mathbb{C}_+\}$$

for $i=1,2$. Then, from Lemma 42 in Appendix B without difficulty we see

$$\mathfrak{A}_1 = \mathfrak{A}_2 \qquad \text{a.e.},$$

and the ac spectrum of H_V is given by $\overline{\mathfrak{A}_1}^{\mathrm{ess}}$ $(=\overline{\mathfrak{A}_2}^{\mathrm{ess}})$. Since, for each $\omega \in \Omega$

$$\mathfrak{A}_1^{\omega} = \mathfrak{A}_+^{\omega} \cup \mathfrak{A}_-^{\omega} \qquad \text{a.e.}$$

holds, the ac spectrum of H_V is equal to $\overline{\mathfrak{A}_+^{\omega} \cup \mathfrak{A}_-^{\omega}}^{\mathrm{ess}} = \overline{\mathscr{A}_+ \cup \mathscr{A}_-}^{\mathrm{ess}}$, which yields (39). $\square$

Now one can show the main theorem. Set

$$\begin{cases} \Sigma = \text{ spectrum of } H_\omega = \operatorname{supp} \mathrm{d}N, \\ \Sigma_{\mathrm{ac}} = \text{ absolutely continuous spectrum of } H_\omega. \end{cases}$$

From Theorems 6, 8 we know that Σ, Σ_{ac} are independent of ω a.s..

Theorem 17 The identity

$$\Sigma_{\mathrm{ac}} = \overline{\mathscr{Z}}^{\mathrm{ess}} \tag{40}$$

holds. Moreover, with probability one we have

$$m_+(\lambda + i0, \omega) = -\overline{m_-(\lambda + i0, \omega)} \qquad \text{for a.e. } \lambda \in \mathscr{Z}. \tag{41}$$

Proof For simplicity we assume $\mathscr{A}_\pm$ are bounded. First we show $\Sigma_{\mathrm{ac}} \subset \overline{\mathscr{Z}}^{\mathrm{ess}}$. The identity

$$\frac{\epsilon}{\operatorname{Im} m_+(\lambda + i\epsilon, \omega)} = \Big(\int_c^{\infty} \frac{1}{(\lambda - x)^2 + \epsilon^2} \sigma_+(\mathrm{d}x, \omega)\Big)^{-1}$$

due to (2) of Lemma 12 implies that the left side is monotone increasing with respect to ϵ. Therefore

$$\int_{\mathscr{A}_+} \frac{\epsilon}{\operatorname{Im} m_+(\lambda + i\epsilon, \omega)} \mathrm{d}\lambda$$

converges to 0 as $\epsilon \searrow 0$ decreasingly, and

$$2\int_{\mathscr{A}_+} \gamma(\lambda)\mathrm{d}\lambda = \lim_{\epsilon \downarrow 0} \int_{\mathscr{A}_+} 2\gamma(\lambda + i\epsilon)\mathrm{d}\lambda = \lim_{\epsilon \downarrow 0} \mathsf{E} \int_{\mathscr{A}_+} \Big(\frac{\epsilon}{\operatorname{Im} m_+(\lambda + i\epsilon, \omega)}\Big)\mathrm{d}\lambda = 0.$$

Similarly we have

$$\int_{\mathscr{A}_-} \gamma(\lambda)\mathrm{d}\lambda = 0,$$

which shows

$$\gamma(\lambda) = 0 \qquad \text{a.e. on } \mathscr{A}_+ \cup \mathscr{A}_-.$$

Therefore, Lemma 16 implies $\Sigma_{\rm ac} \subset \overline{\mathscr{Z}}^{\rm ess}$.

Secondly we prove $\Sigma_{\rm ac} \supset \overline{\mathscr{Z}}^{\rm ess}$ and (41) simultaneously. For simplicity we assume $\mathscr{Z}$ is bounded. Since A in Lemma 16 satisfies $A = \mathbb{R}$ a.e. due to Lebesgue theorem, applying Lemma 16 to (36), we see

$$\begin{aligned} 0 &= \lim_{\epsilon\downarrow 0}\int_{A\cap\mathscr{Z}}\Big(\frac{\gamma(\lambda+i\epsilon)}{\epsilon} - \operatorname{Im} w'(\lambda+i\epsilon)\Big)\mathrm{d}\lambda \\ &= \lim_{\epsilon\downarrow 0}\int_{\mathscr{Z}}\Big(\frac{\gamma(\lambda+i\epsilon)}{\epsilon} - \operatorname{Im} w'(\lambda+i\epsilon)\Big)\mathrm{d}\lambda \\ &= \lim_{\epsilon\downarrow 0}\int_{\mathscr{Z}}\mathsf{E}\Big\{\Big(\frac{1}{\operatorname{Im} m_+} + \frac{1}{\operatorname{Im} m_-}\Big)\Big|\frac{m_+ + \overline{m}_-}{m_+ + m_-}\Big|^2(\lambda+i\epsilon)\Big\}\mathrm{d}\lambda \\ &\geq \mathsf{E}\Big\{\int_{\mathscr{Z}}\Big(\frac{1}{\operatorname{Im} m_+} + \frac{1}{\operatorname{Im} m_-}\Big)\Big|\frac{m_+ + \overline{m}_-}{m_+ + m_-}\Big|^2(\lambda+i0)\mathrm{d}\lambda\Big\} \qquad \text{(Fatou)}, \end{aligned}$$

hence with probability one

$$\Big(\frac{1}{\operatorname{Im} m_+} + \frac{1}{\operatorname{Im} m_-}\Big)\Big|\frac{m_+ + \overline{m}_-}{m_+ + m_-}\Big|^2(\lambda+i0) = 0 \qquad \text{for a.e. } \lambda \in \mathscr{Z} \tag{42}$$

holds. Since $\operatorname{Im} m_\pm(\lambda+i0) < \infty$, $0 < |(m_+ + m_-)(\lambda+i0)| < \infty$ for a.e. $\lambda \in \mathscr{Z}$, (42) is valid if and only if (41) holds. It is clear that (41) implies $\mathscr{Z} \subset \mathscr{A}_+ \cup \mathscr{A}_-$ a.e.. And the proof is complete. $\square$

In the theorem, the inclusion $\Sigma_{\rm ac} \subset \overline{\mathscr{Z}}^{\rm ess}$ was shown by Ishii [14] and Pastur [31] by using Lemma 38 and Fubini's theorem. As for this fact Deift-Simon [11] also gave a proof by using $m_\pm$. The inclusion $\Sigma_{\rm ac} \supset \overline{\mathscr{Z}}^{\rm ess}$ and the identity (41) were proved by Kotani [19].

Corollary 18 $\mathscr{A}_+ = \mathscr{A}_- = \mathscr{Z}$ a.e. and $\Sigma_{\rm ac}^+ = \Sigma_{\rm ac}^- = \Sigma_{\rm ac} = \overline{\mathscr{Z}}^{\rm ess}$ hold.

The property (41) is very strong. For instance, suppose $\mathscr{Z}$ contains an interval I. Then (41) implies $\operatorname{Re} g_{\lambda+i0}(0,0,\omega) = 0$ on I, which enables us extend $g_z(0,0,\omega)$ analytically down to $\mathbb{C}_-$ through I. Therefore, if $\mathscr{Z}$ is a sufficiently regular set and $\Sigma = \mathscr{Z}$, then $g_z(0,0,\omega)$ can be determined as a analytic function on a Riemannian surface, which makes it possible to describe the potential V. This was a main theme in 1970s and 1980s relating to completely integrable systems. If two Herglotz functions $m_\pm$ satisfy

$$m_+(\lambda+i0) = -\overline{m_-(\lambda+i0)} \qquad \text{for a.e. } \lambda \in F \tag{43}$$

for a measurable set F with $|F| > 0$, then we call $\{m_\pm\}$ is **reflectionless** on F. Historically this property appeared for decaying potentials when the reflection coefficients vanishes, and in the study of a certain class of completely integrable systems like KdV equation, non-linear Schrödinger equation etc this property has appeared implicitly. We remark that all periodic potentials are reflectionless on their spectrum in this sense, since the spectra are purely absolutely continuous.

Theorem 17 asserts that the ac spectrum is supported exactly on $\mathscr{Z}$. However it does not exclude the possibility of the existence of the singular spectrum in $\mathscr{Z}$. The theorem below is related to this problem. The spectral measure σ_ω was defined as the measure satisfying

$$g_z(0,0,\omega)=\int_c^\infty \frac{1}{\lambda-z}\sigma(\mathrm{d}\lambda,\omega),$$

where $g_z(x,y,\omega)$ is the Green function of $H_\omega=-\Delta+V_\omega$. Let

$$\sigma(\mathrm{d}\lambda,\omega)=\sigma_{\mathrm{ac}}(\mathrm{d}\lambda,\omega)+\sigma_{\mathrm{s}}(\mathrm{d}\lambda,\omega)$$

be the Lebesgue decomposition. Then

$$N(\mathrm{d}\lambda)=\mathsf{E}(\sigma(\mathrm{d}\lambda,\omega))=\mathsf{E}(\sigma_{\mathrm{ac}}(\mathrm{d}\lambda,\omega))+\mathsf{E}(\sigma_{\mathrm{s}}(\mathrm{d}\lambda,\omega)) \tag{44}$$

holds, and clearly $\mathsf{E}(\sigma_{\mathrm{ac}}(\mathrm{d}\lambda,\omega))$ is absolutely continuous. However, $\mathsf{E}(\sigma_{\mathrm{s}}(\mathrm{d}\lambda,\omega))$ may have non-trivial absolutely continuous part, hence (44) does not necessarily yield the Lebesgue decomposition of $N(\mathrm{d}\lambda)$. The result obtained in Kotani [26] is clarified as follows. Decompose $\mathscr{Z}$ into three parts by the IDS N as

$$\mathscr{Z}=\mathscr{Z}_{\mathrm{ac}}\cup\mathscr{Z}_{\mathrm{s}}\cup\mathscr{Z}_{\mathrm{ex}},$$

where

$$\begin{cases}\mathscr{Z}_{\mathrm{ac}}=\mathscr{Z}\cap A=\Big\{\lambda\in\mathscr{Z}:\lim\limits_{\epsilon\downarrow 0}\dfrac{N(\lambda+\epsilon)-N(\lambda-\epsilon)}{2\epsilon}<\infty\Big\},\\ \mathscr{Z}_{\mathrm{s}}=\Big\{\lambda\in\mathscr{Z}:\lim\limits_{\epsilon\downarrow 0}\dfrac{N(\lambda+\epsilon)-N(\lambda-\epsilon)}{2\epsilon}=\infty\Big\},\\ \mathscr{Z}_{\mathrm{ex}}=\mathscr{Z}\backslash(\mathscr{Z}_{\mathrm{ac}}\cup\mathscr{Z}_{\mathrm{s}}).\end{cases}$$

Each subset can be redefined also through w':

$$\begin{cases}\mathscr{Z}_{\mathrm{ac}}=\Big\{\lambda\in\mathscr{Z}:\lim\limits_{\epsilon\downarrow 0}\mathrm{Im}\,w'(\lambda+i\epsilon)<\infty\Big\},\\ \mathscr{Z}_{\mathrm{s}}=\Big\{\lambda\in\mathscr{Z}:\lim\limits_{\epsilon\downarrow 0}\mathrm{Im}\,w'(\lambda+i\epsilon)=\infty\Big\},\\ \mathscr{Z}_{\mathrm{ex}}=\Big\{\lambda\in\mathscr{Z}:\lim\limits_{\epsilon\downarrow 0}\mathrm{Im}\,w'(\lambda+i\epsilon)\text{ does not exist}\Big\}.\end{cases}$$

It should be noted that the IDS N is purely absolutely continuous on $\mathscr{Z}_{\mathrm{ac}}$ if $|\mathscr{Z}|>0$.

Theorem 19 It holds that

$$|\mathscr{Z}|=|\mathscr{Z}_{\mathrm{ac}}|,\qquad |\mathscr{Z}_{\mathrm{s}}|=0,\qquad N(\mathscr{Z}_{\mathrm{ex}})=0, \tag{45}$$

and

$$\begin{cases} \mathsf{E}(\sigma_{\rm ac}(\mathrm{d}\lambda,\omega)) = N(\mathrm{d}\lambda\cap\mathscr{Z}_{\rm ac}) = N_{\rm ac}(\mathrm{d}\lambda\cap\mathscr{Z}_{\rm ac}), \\ \mathsf{E}(\sigma_{\rm s}(\mathrm{d}\lambda,\omega)) = N(\mathrm{d}\lambda\cap(\mathscr{Z}_{\rm s}\cup(\mathbb{R}\backslash\mathscr{Z}))) = N_{\rm s}(\mathrm{d}\lambda\cap\mathscr{Z}_{\rm s}) + N(\mathrm{d}\lambda\cap(\mathbb{R}\backslash\mathscr{Z})). \end{cases} \tag{46}$$

Therefore, if $|\mathscr{Z}|>0$, then the ac spectrum of H_ω is concentrated on $\mathscr{Z}_{\rm ac}$ and no other spectra on $\mathscr{Z}_{\rm ac}$.

Proof (45) follows from Lemma 42 in Appendix immediately. On the other hand, from Theorem 17

$$\mathsf{E}(\sigma_{\rm ac}(\mathrm{d}\lambda,\omega)) = \mathsf{E}(\sigma_{\rm ac}(\mathrm{d}\lambda\cap\mathscr{Z}_{\rm ac},\omega))$$

holds and clearly this is dominated by $N_{\rm ac}(\mathrm{d}\lambda\cap\mathscr{Z}_{\rm ac})$ from above. To show the converse we use

$$p_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2+x^2},$$

and for a bounded set K of $\mathscr{Z}$ define

$$\widetilde{p}_\epsilon(\lambda,x) = C_\epsilon(\lambda)^{-1}p_\epsilon(\lambda-x), \qquad \text{where } C_\epsilon(\lambda) = \int_K p_\epsilon(\lambda-x)\mathrm{d}x.$$

Schwarz inequality shows

$$\Big(\int_K f(x)^{-1}\widetilde{p}_\epsilon(\lambda,x)\mathrm{d}x\Big)^{-1} \le \int_K f(x)\widetilde{p}_\epsilon(\lambda,x)\mathrm{d}x$$

for any non-negative f. Applying this inequality to

$$f(x) = \frac{1}{\operatorname{Im} m_+(x+i0,\omega)},$$

we have

$$C_\epsilon(\lambda)^2\Big(\int_K \operatorname{Im} m_+(x+i0,\omega)p_\epsilon(\lambda-x)\mathrm{d}x\Big)^{-1} \le \int_K \frac{1}{\operatorname{Im} m_+(x+i0,\omega)}p_\epsilon(\lambda-x)\mathrm{d}x. \tag{47}$$

Note

$$\begin{aligned}\int_K \operatorname{Im} m_+(x+i0,\omega)p_\epsilon(\lambda-x)\mathrm{d}x &\le \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\epsilon}{(\lambda-x)^2+\epsilon^2}\operatorname{Im} m_+(x+i0,\omega)\mathrm{d}x \\ &\le \int_{-\infty}^{\infty}\frac{\epsilon}{(\lambda-x)^2+\epsilon^2}\sigma_+(\mathrm{d}x,\omega) = \operatorname{Im} m_+(\lambda+i\epsilon,\omega).\end{aligned}$$

Then (47) turns to

$$C_\epsilon(\lambda)^2(\operatorname{Im} m_+(\lambda+i\epsilon,\omega))^{-1} \le \int_K \frac{1}{\operatorname{Im} m_+(x+i0,\omega)}p_\epsilon(\lambda-x)\mathrm{d}x,$$

and taking the expectation leads us to

$$C_\epsilon(\lambda)^2\frac{2\gamma(\lambda+i\epsilon)}{\epsilon}\leq\int_K\Big(\mathsf{E}\frac{1}{\operatorname{Im}m_+(x+i0,\omega)}\Big)p_\epsilon(\lambda-x)\mathrm{d}x. \tag{48}$$

From Theorem 17

$$\begin{aligned}\operatorname{Im}g_{x+i0}(0,0,\omega)&=-\operatorname{Im}\frac{1}{m_+(x+i0,\omega)+m_-(x+i0,\omega)}\\&=-\operatorname{Im}\frac{1}{m_+(x+i0,\omega)-\overline{m_+(x+i0,\omega)}}\\&=\frac{1}{2}\frac{1}{\operatorname{Im}m_+(x+i0,\omega)}\end{aligned}$$

follows, and (48) yields

$$C_\epsilon(\lambda)^2\frac{\gamma(\lambda+i\epsilon)}{\epsilon}\leq\int_K\mathsf{E}(\operatorname{Im}g_{x+i0}(0,0,\omega))p_\epsilon(\lambda-x)\mathrm{d}x.$$

Since $C_\epsilon(\lambda)\to 1$ for a.e. $\lambda\in K$, Lemma 15 shows

$$\pi\int_K N'(\lambda)\mathrm{d}\lambda\leq\int_K\mathsf{E}(\operatorname{Im}g_{\lambda+i0}(0,0,\omega))\mathrm{d}\lambda=\pi\,\mathsf{E}(\sigma_{\mathrm{ac}}(K,\omega)),$$

which implies

$$\mathsf{E}(\sigma_{\mathrm{ac}}(\mathrm{d}\lambda,\omega))=N_{\mathrm{ac}}(\mathrm{d}\lambda\cap\mathscr{Z}_{\mathrm{ac}}).$$

Moreover, applying (iii) of Lemma 42 in Appendix to w' we have $N_{\mathrm{s}}(\mathbb{R}\backslash\mathscr{Z}_{\mathrm{s}})=0$, hence $N(\mathscr{Z}_{\mathrm{ac}})=N_{\mathrm{ac}}(\mathscr{Z}_{\mathrm{ac}})$. Consequently $\mathsf{E}(\sigma_{\mathrm{s}}(\mathrm{d}\lambda,\omega))=N(\mathrm{d}\lambda\cap(\mathscr{Z}_{\mathrm{s}}\cup(\mathbb{R}\backslash\mathscr{Z})))$ holds. □

Since the IDS N completely determines the Lyapounov exponent γ, Theorem 17 asserts that the IDS completely characterizes the ac spectrum and the singular spectrum for any one-dimensional ergodic Schrödinger operators.

It should be noted that Theorem 19 implies

Corollary 20 H_ω has purely ac spectrum if and only if $N(\mathbb{R}\backslash\mathscr{Z}_{\mathrm{ac}})=0$ or equivalently

$$N(\mathrm{d}\lambda)\ \text{is absolutely continuous and}\ N(\mathbb{R}\backslash\mathscr{Z})=0$$

holds.

The main theorems 17, 19 are stated by the IDS N and the Lyapounov exponent γ. Its significance lies in the constructive definitions (9), (35) of N, γ, which sometimes makes it possible to compute N,γ in some way.

For discrete Schrödinger operators a theorem analogous to Theorem 17 was established by Simon [34]. Theorem 19 can be shown for discrete systems without any change of the proof. As for the extension of Theorem 17 refer Kotani-Simon [24].

§3. Applications

Theorem 17 has several applications because of the reflectionless property of the ac spectrum. In this section, we provide several theorems obeying from Theorem 17. There is a common belief that the increase of randomness creates the irregularity of the spectrum. The first two applications realize this belief.

3.1 Nondeterministic Potentials

Lemma 43 in Appendix B and Lemma 46 in Appendix C imply that the Weyl functions $m_{\pm}$ given on $A \subset \mathscr{B}(\mathbb{R})$ with positive Lebesgue measure recover the potential V on $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively. On the other hand, For a random process $\{X_x(\omega)\}_{x\in\mathbb{R}}$ are said to be **deterministic** if two σ-fields

$$\mathscr{F}_- = \sigma\text{-}\{X_x(\omega) : x < 0\}, \qquad \mathscr{F}_+ = \sigma\text{-}\{X_x(\omega) : x > 0\}$$

for $\{X_x(\omega)\}_{x\in\mathbb{R}}$ satisfy

$$\mathscr{F}_- = \mathscr{F}_+ \qquad \text{a.s..}$$

On the contrary, if $\mathscr{F}_- \neq \mathscr{F}_+$ a.s. holds, then the process is called **nondeterministic**. If the process is random intuitively, then it is supposed to be non-deterministic. Combining the two lemmas in Appendix with Theorem 17, this notion makes it possible to state

Theorem 21 (Kotani [19]) Assume $\{V(T_x\omega)\}_{x\in\mathbb{R}}$ is nondeterministic. Then, the Lyapounov exponent $\gamma(\lambda)$ is a.e. positive, and there is no absolutely continuous spectrum a.s..

In discrete case, if a potential $\{V_n(\omega)\}_{n\in\mathbb{Z}}$ is i.i.d., then it is certainly nondeterministic. Therefore, in this case the corresponding discrete Schrödinger operator has no ac spectrum. Historically, the positivity of the Lyapounov exponents for i.i.d. potentials was known by applying Furstenberg theorem.

3.2 Support Theorem

To compare the randomness of two ergodic potentials we use their induced probability measures on a set of potentials on $\mathbb{R}$.

Let C be a positive constant and set

$$\mathscr{V} = \{V : V \text{ is real valued measurable on } \mathbb{R} \text{ and } |V(x)| \leq C \text{ for all } x \in \mathbb{R}\}.$$

We impose a metric on $\mathscr{V}$, for instance, for $\{\varphi_n\}_{n=1,2,\dots} \subset C_0^\infty(\mathbb{R})$ which are dense in $C_0(\mathbb{R})$ define

$$\mathrm{d}(V_1, V_2) = \sum_{n=1}^{\infty} \frac{1}{2^n}\Big(\Big| \int_{\mathbb{R}} (V_1(x) - V_2(x))\varphi_n(x)\mathrm{d}x \Big| \wedge 1\Big).$$

On $\mathscr{V}$ the shift θ_x is defined by $(\theta_x V)(\cdot) = V(\cdot + x)$. Then one can consider a shift invariant probability measure on $\mathscr{V}$, and the notion of ergodicity with respect to $\{\theta_x\}_{x\in\mathbb{R}}$ can be defined as usual. Let $\mathscr{P}$ be the set of all ergodic probability measures on $\mathscr{V}$. An ergodic potential $\{V(T_x\omega)\}$ in the previous sense on a certain probability space $(\Omega, \mathscr{F}, \mathsf{P})$ can be regarded as an ergodic potential in the present sense by a map

$$\phi : \Omega \to \mathscr{V}, \qquad \phi(\omega)(x) = V(T_x\omega),$$

if we define $\mu \in \mathscr{P}$ by

$$\mu(A) = \mathsf{P}(\phi^{-1}(A)), \qquad A \in \mathscr{B}(\mathscr{V}) : \text{the Borel } \sigma\text{-field on } \mathscr{V}.$$

For $\mu \in \mathscr{P}$ the support is

$$\operatorname{supp}\mu = \{V \in \mathscr{V} : \mu(O) > 0 \text{ for any open } O \text{ such that } O \ni V\}.$$

One can think that the larger the support is, the more random the ergodic potential is. Since the spectrum and three spectral components depend only on $\mu \in \mathscr{P}$, we denote them by

$$\Sigma^\mu, \quad \Sigma^\mu_{\mathrm{ac}}, \quad \Sigma^\mu_{\mathrm{sc}}, \quad \Sigma^\mu_{\mathrm{p}}.$$

Theorem 22 (Kotani [22]) Suppose $\mu_1, \mu_2 \in \mathscr{P}$ satisfy $\operatorname{supp}\mu_1 \subset \operatorname{supp}\mu_2$. Then

(i) (this is valid in any dimension)

$$\Sigma^{\mu_1} \subset \Sigma^{\mu_2}.$$

(ii) (this is valid only in one dimension)

$$\Sigma^{\mu_1}_{\mathrm{ac}} \supset \Sigma^{\mu_2}_{\mathrm{ac}}.$$

Proof First note that the convergence of V_n to V in $\mathscr{V}$ implies the convergence of the associated $m_n^\pm$ (see Exercise 23), that is

$$m_n^\pm(z) \to m^\pm(z) \qquad \text{for any } z \in \mathbb{C}_+. \tag{49}$$

Let I be a bounded open interval such that $I \subset \mathbb{R}\backslash\Sigma^{\mu_2}$. Then, $\sigma_V(I) = 0$ holds for a.s. V with respect to μ_2. (49) and Lemma 45 imply $\sigma_V(I) = 0$ holds for any $V \in \operatorname{supp}\mu_2$, hence for any $V \in \operatorname{supp}\mu_1$, which shows $\mathbb{R}\backslash\Sigma^{\mu_2} \subset \mathbb{R}\backslash\Sigma^{\mu_1}$.

The second statement follows similarly. Suppose $m_V^\pm$ are reflectionless on A for a.e. V with respect to μ_2. Then, (49) and Lemma 45 imply $m_V^\pm$ are reflectionless on A for any $V \in \operatorname{supp}\mu_2$, hence for any $V \in \operatorname{supp}\mu_1$, which shows $\Sigma_{\mathrm{ac}}^{\mu_1} \supset \Sigma_{\mathrm{ac}}^{\mu_2}$. □

Exercise 23 Show that the convergence of V_n to V in $\mathscr{V}$ implies the convergence of the associated $m_n^\pm$.

Theorem 22 says that the increase of the support implies the less ac spectrum. We provide typical applications of the theorem in discrete system. Let

$$(H_V u)(x) = \sum_{y:\,|y-x|=1} u(y) + V(x)u(x).$$

Assume $\{V(x)\}_{x\in\mathbb{Z}^d}$ are i.i.d. with distribution $F(x)$. Then

Claim 24 Suppose $\operatorname{supp}\mathrm{d}F = [a_-, a_+]$. Then

$$\Sigma = \operatorname{spec} H_V = [-2d + a_-, 2d + a_+]. \tag{50}$$

This is shown as follows. Let μ be the probability measure on $\mathscr{V}$ induced by the i.i.d. potentials. The spectrum of Δ is $[-2d, 2d]$. Let $a \in \operatorname{supp}\mathrm{d}F$. Then, the potential $V_a(x) = a$ (constantly) is an element of $\operatorname{supp}\mu$, hence (i) of the theorem implies $\Sigma \supset [-2d, 2d] + a$, hence

$$\Sigma \supset [-2d, 2d] + \operatorname{supp}\mathrm{d}F = [-2d + a_-, 2d + a_+].$$

Conversely, since $a_- \le V_\omega(x) \le a_+$ a.s., we have

$$(-2d + a_-)\|u\|^2 \le (H_V u, u) \le (-2d + a_+)\|u\|^2 \qquad \text{for any } \ u \in \ell^2(\mathbb{Z}),$$

it is clear that $\Sigma \subset [-2d + a_-, 2d + a_+]$.

As an example to apply (ii) of the theorem we consider a one dimensional discrete system with potential

$$V_\omega(x) = f(X_x(\omega)),$$

where $\{X_x(\omega)\}_{x\in\mathbb{Z}}$ is a stationary Gaussian process with mean 0 and covariance

$$\rho(x-y) = \mathsf{E}(X_x(\omega)X_y(\omega)) = \int_0^{2\pi} \mathrm{e}^{i\lambda(x-y)}\nu(\mathrm{d}\lambda),$$

where ν is a finite measure on $[0, 2\pi)$ with no atoms. Then, $\{X_x(\omega)\}_{x\in\mathbb{Z}}$ becomes ergodic. Suppose f is a bounded continuous function on $\mathbb{R}$. Then $f(X_x(\omega))$ induces an ergodic probability measure μ on $\mathscr{V}$. Since, for any $x_1 < x_2 < \cdots < x_n \in \mathbb{Z}$ the matrix

$$(\rho(x_i - x_j))_{1\le i,j\le n}$$

is strictly positive definite, the support of the distribution of $\{X_{x_i}(\omega)\}_{1\le i\le n}$ is $\mathbb{R}^n$. Therefore, in view of the locality of the distance of $\mathscr{V}$ it follows that any periodic potential taking its value in $[\inf f, \sup f]$ is contained in $\operatorname{supp}\mu$. Conversely we easily see that any $V \in \operatorname{supp}\mu$ can be approximated by periodic functions, hence we have

$$\operatorname{supp}\mu = \overline{\{\text{periodic potential taking its value in } [\inf f, \sup f]\}}.$$

Consequently, an identity

$$\Sigma = [-2d + \inf f,\ 2d + \sup f]$$

follows from (i) of the theorem. Moreover, for any interval I of Σ one can create a periodic potential V with bounds $\inf f$ and $\sup f$ such that $(\operatorname{spec}, H_V) \cap I = \phi$. Then, applying (ii) of the theorem to μ and the probability measure μ_1 on V generated by $\{\theta_x V\}_{x\in\mathbb{Z}}$, we have

$$I \subset (\Sigma^{\mu_1})^c = (\Sigma^{\mu_1}_{\text{ac}})^c \subset (\Sigma^{\mu}_{\text{ac}})^c,$$

which implies $\Sigma^{\mu}_{\text{ac}} = \phi$. The boundedness of f can be removed.

Claim 25 Let $\{X_x(\omega)\}_{x\in\mathbb{Z}}$ be an ergodic stationary Gaussian process and f be a nonconstant continuous function. Then, the discrete Schrödinger operator with an ergodic potential $V_\omega(x) = f(X_x(\omega))$ has no ac spectrum a.s..

In this way Theorem 22 is useful to show the absence of the ac spectrum for some deterministic potentials. A further application of the theorem can be found in Kirsch-Kotani-Simon [17].

3.3 Potentials Taking Finitely Many Values

Under a certain situation, Theorem 21 has a stronger statement, especially for the discrete system when ergodic potentials take only finitely many values. A typical example is the **Sturmian potential** (see Damanik [9]): for $\kappa > 0$ and $\theta \in (0,1)$ irrational

$$V_\omega(x) = \kappa\chi_{[1-\theta,1)}(x\theta + \omega), \qquad x \in \mathbb{Z},\ \omega \in \Omega \equiv [0,1).$$

In this section we give a result which is valid for general such potentials.

Before we state the result we prepare a lemma. For a finite subset S of $\mathbb{R}$ set

$$\Omega = S^{\mathbb{Z}}, \quad \Omega_- = S^{\mathbb{Z}_-}, \qquad \text{where } \mathbb{Z}_- = \{x \in \mathbb{Z} : x \le -1\}.$$

Define a projection $\pi_- : \Omega \to \Omega_-$ by

$$\pi_-(\omega)(x) = \omega(x) \qquad \text{for } x \in \mathbb{Z}_-.$$

Ω turns to be compact by a metric d:

$$d(\omega_1,\omega_2)=\sum_{x\in\mathbb{Z}}2^{-|x|}|\omega_1(x)-\omega_2(x)|.$$

One can define a similar metric on Ω_-, and abuse the same notation. Define a shift operation on Ω by

$$(\theta_x\omega)(\cdot)=\omega(\cdot+x).$$

Lemma 26 Let Ω_1 be a shift invariant closed subset of Ω satisfying

$$\pi_-:\Omega_1\to\Omega_- \text{ is injective.} \tag{51}$$

Then Ω_1 becomes a finite set, and any element of Ω_1 is periodic.

Proof The condition (51) implies that there exists a bijection ϕ between $\pi_-(\Omega_1)$ and Ω_1. Since π_- is continuous and Ω_1 is compact, ϕ turns to be continuous. Therefore, for any $\epsilon>0$ there exists $\delta>0$ such that

$$\sum_{x\le -1}2^x|\omega_1(x)-\omega_2(x)|<\delta\Longrightarrow|\omega_1(0)-\omega_2(0)|<\epsilon.$$

Since S is a finite set, we have

$$\delta_0=\min_{a,b\in S,\,a\neq b}|a-b|>0,\qquad \delta_1=\max_{a,b\in S}|a-b|<\infty.$$

Choose $\epsilon<\delta_0$. Then, for $n>\log_2(\delta_1/\delta)$

$$\sum_{x\le -n-1}2^x|\omega_1(x)-\omega_2(x)|\le\delta_1\sum_{x\le -n-1}2^x=\delta_1 2^{-n}<\delta$$

holds. Therefore, we have

$$\begin{aligned}&\omega_1(x)=\omega_2(x)\ \text{ for any }\ -n\le x\le -1\\ \Longrightarrow&\sum_{x\le -1}2^x|\omega_1(x)-\omega_2(x)|<\delta\Longrightarrow|\omega_1(0)-\omega_2(0)|<\epsilon\Longrightarrow\omega_1(0)=\omega_2(0).\end{aligned}$$

Since Ω_1 is shift invariant, this property of Ω_1 implies that for any $\omega\in\Omega_1$ the value $\omega(x)$ is determined from $\{\omega(-1),\omega(-2),\ldots,\omega(-n)\}$. Consequently we see $\#\Omega_1\le(\#S)^n<\infty$. The rest of the proof is easy. $\square$

Theorem 27 (Kotani [25]) Suppose $V(\omega)$ takes finitely many values. Then, the associated ergodic Schrödinger operators have no ac spectrum a.s., unless they are periodic.

Proof Let S be all possible values taken by V and Ω_1 be the closure of the set of all potentials $V(T_x\omega)$. Then Ω_1 becomes a shift invariant closed set of $\Omega \equiv S^{\mathbb{Z}}$. Assume the corresponding Schrödinger operators have ac spectrum. Then, the Weyl functions satisfy

$$m_+(\lambda + i0, \omega) = -\overline{m_-(\lambda + i0, \omega)} \qquad \text{for a.e. } \lambda \in \mathscr{L}, \tag{52}$$

and, Lemma 45 in Appendix implies that for any $\omega \in \Omega_1$ (52) holds. Hence, from (ii) of Lemma 43 we know that Ω_1 satisfies the condition (51). Applying Lemma 26, we have the conclusion. □

3.4 Point Spectrum

Generally the IDS can not distinguish singular continuous spectrum and point spectrum. There are examples indicating this fact. They are discrete quasi-periodic Schrödinger operators defined by

$$H_\omega = \Delta + \kappa \tan(2\pi\alpha n + \omega) \text{ on } l^2(\mathbb{Z}) \text{ for irrational } \alpha.$$

This operator has the IDS

$$N_\kappa(\lambda) = \frac{1}{\pi}\mathrm{Im}\int_{-\infty}^{\infty} \frac{1}{x - \lambda - i\kappa}\mathrm{d}N_0(x),$$

where N_0 is the IDS for the free Laplacian Δ. The IDS $N_\kappa(\lambda)$ is independent of α. For α a quantity

$$L(\alpha) = \limsup_{n\to\infty}(-n^{-1}\log|\sin\pi\alpha n|)$$

measures a distance between α and rational numbers, since there exist integers p_n, q_n such that

$$\left|\alpha - \frac{q_n}{p_n}\right| \sim \mathrm{e}^{-L(\alpha)q_n}.$$

Simon [35] proved that

$$\begin{cases} L(\alpha) = 0 \Longrightarrow \text{the spectrum is pure point}, \\ L(\alpha) = \infty \Longrightarrow \text{the spectrum is purely singular continuous}, \end{cases}$$

which implies that the IDS can not determine the components of the singular spectrum. It should be noted, however, that these potentials are not quasi-periodic in an ordinary sense, since $\tan x$ is unbounded.

There is a method to show the existence of the point spectrum, which was discovered by Carmona [4] and developed by Kotani [23], [26]. This method is useful only in one dimension when potentials are nondeterministic.

Define the future and the past by

$$\mathscr{F}_+ = \sigma\text{-}\{V(T_x\omega); x \geq 0\}, \qquad \mathscr{F}_- = \sigma\text{-}\{V(T_x\omega); x \leq 0\}.$$

Let $\sigma(\mathrm{d}\lambda, \omega)$ be the spectral measure of the Herglotz function $g_z(0,0,\omega)$. The conditional expectation

$$\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_\pm)$$

can be realized as measures on $\mathbb{R}$ for fixed ω by using the regular conditional probability given $\mathscr{F}_\pm$.

Theorem 28 (Lemma 2 in Kotani [23]) Assume an ergodic potential $\{V(T_x\omega)\}$ is nondeterministic. Let μ be the Lebesgue measure on $\mathbb{R}$. For a Borel subset A of $\mathbb{R}$ with $\mu(A) > 0$ assume with probability one

$$\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_\pm) \text{ are absolutely continuous on } A \text{ w.r.t. } \mu. \tag{53}$$

Then, the corresponding schrödinger operators H_ω have purely point spectrum on A and all the eigenfunctions decay like $\mathrm{e}^{-\gamma(\lambda)|x|}$.

Proof Set

$$S_\pm(\omega) = \Big\{\lambda \in A; \lim_{x\to\pm\infty} \frac{1}{|x|} \log \|U_\omega(x,\lambda)\| = \gamma(\lambda) > 0\Big\}.$$

Then, the assumption and the subadditive ergodic theorem imply that

$$\mu(A \backslash S_\pm(\omega)) = 0 \qquad \text{a.s.} \tag{54}$$

holds. Define

$$f_\pm(\lambda, \omega) = \chi_{A\backslash S_\pm(\omega)}(\lambda).$$

Since f_+ is measurable with respect to $\mathscr{B}(\mathbb{R}) \times \mathscr{F}_+$, we have

$$\mathsf{E}\Big(\int_A f_+(\lambda,\omega)\sigma(\mathrm{d}\lambda,\omega)\Big) = \mathsf{E}\Big(\int_A f_+(\lambda,\omega)\mathsf{E}(\sigma(\mathrm{d}\lambda,\omega)|\mathscr{F}_+)\Big),$$

which is equal to 0 due to (53), (54). Hence

$$\sigma(A\backslash S_+(\omega), \omega) = \int_A f_+(\lambda,\omega)\sigma(\mathrm{d}\lambda,\omega) = 0 \qquad \text{a.s.},$$

and

$$\sigma(A\backslash S_+(\omega), \omega) = 0 \qquad \text{a.s..}$$

Similarly we have

$$\sigma(A\backslash S_-(\omega), \omega) = 0 \qquad \text{a.s..}$$

Therefore, we see that for a.e. $\lambda \in A$ with respect to $\sigma(\mathrm{d}\lambda, \omega)$ there exist solutions $u_\pm$ to $H_\omega u_\pm = \lambda u_\pm$ satisfying

$$u_\pm(x),\ u'_\pm(x) \sim \mathrm{e}^{-\gamma(\lambda)(\pm x)} \qquad \text{as } x \to \pm\infty,$$

and any other solution u to $H_\omega u = \lambda u$ which is independent of $u_\pm$ satisfies

$$u(x)^2 + u'(x)^2 \sim \mathrm{e}^{\mp x\gamma(\lambda)} \qquad \text{as } x \to \pm\infty \quad (\text{see Theorem 47}).$$

On the other hand, it is known that for a.e. λ with respect to $\sigma(\mathrm{d}\lambda, \omega)$ there exists a nontrivial solution f to

$$H_\omega f = \lambda f \text{ on } \mathbb{R} \text{ satisfying } f(x)^2 + f'(x)^2 = O(|x|^\rho) \text{ as } |x| \to \infty,$$

for some finite $\rho > 0$ (see Lemma 40). Therefore, this f should be

$$f(x) = \begin{cases} \text{const.}u_+(x) & \text{for } x > 0; \\ \text{const.}u_-(x) & \text{for } x < 0, \end{cases}$$

which means f is an element of $L^2(\mathbb{R})$, and $\sigma(\mathrm{d}\lambda, \omega)$ has only point part on A. □

If $\{V(T_x\omega)\}_{x\in\mathbb{R}}$ is deterministic, namely $\mathscr{F}_- = \mathscr{F}_+$ holds, then $\mathscr{F}_- = \mathscr{F}_+ =$ the whole σ-field $\mathscr{F}$, hence

$$\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_-) = \sigma(\mathrm{d}\lambda, \omega)$$

is valid, hence the condition (53) is useless. Assume $\{V(T_x\omega)\}_{x\in\mathbb{R}}$ is nondeterministic on the contrary, and set

$$\mathscr{F}_n = \sigma\text{-}\{V(T_x\omega); x \le n\}, \qquad \mathscr{F}_{-\infty} = \bigcap_{n\le -1} \mathscr{F}_n.$$

Then, it is easily seen that $f_-(\lambda, \omega)$ is measurable w.r.t. $\mathscr{B}(\mathbb{R}) \times \mathscr{F}_n$ for any $n \le -1$, hence

$$\mathsf{E}\Big(\int_A f_-(\lambda, \omega)\sigma(\mathrm{d}\lambda, \omega)\Big) = \mathsf{E}\Big\{\mathsf{E}\Big(\int_A f_-(\lambda, \omega)\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_n)\Big)\Big\}.$$

One might be tempted to take the limit $n \to -\infty$ and conclude

$$\begin{aligned} \mathsf{E}\Big(\int_A f_-(\lambda, \omega)\sigma(\mathrm{d}\lambda, \omega)\Big) &= \mathsf{E}\Big\{\lim_{n\to-\infty} \mathsf{E}\Big(\int_A f_-(\lambda, \omega)\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_n)\Big)\Big\} \\ &= \mathsf{E}\Big\{\mathsf{E}\Big(\int_A f_-(\lambda, \omega)\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_{-\infty})\Big)\Big\}. \end{aligned} \tag{55}$$

This identity is valid if $f_-(\lambda, \omega)$ is measurable w.r.t. $\mathscr{B}(\mathbb{R}) \times \mathscr{F}_{-\infty}$, but generally

$$\mathscr{B}(\mathbb{R}) \times \mathscr{F}_{-\infty} \subsetneq \bigcap_{n\le -1} (\mathscr{B}(\mathbb{R}) \times \mathscr{F}_n),$$

and one can not hope (55).

The first application of Theorem 28 is to discrete Schrödinger operators with i.i.d. potentials.

Claim 29 Assume that the potential $\{V(x,\omega)\}$ are i.i.d., and the common distribution has a bounded density $\tau(x)$. Then, the corresponding H_ω has purely point spectrum and every its eigenfunction decays exponentially fast with probability one.

Proof We use the identity

$$g_z(0,0,\omega) = -\frac{1}{m_+(z,\omega)+z-V(1,\omega)+m_-(z,\omega)},$$

where $m_\pm(z,\omega)$ are the Weyl functions on $x \geq 2$ and $x \leq 0$ respectively. Set

$$w = m_+(z,\omega)+z+m_-(z,\omega) \in \mathbb{C}_+.$$

Noting the independence of w and $V(1,\omega)$, we see

$$\begin{aligned}\mathsf{E}(\operatorname{Im} g_z(0,0,\omega)|\mathscr{F}_-) &= \mathsf{E}(\mathsf{E}(\operatorname{Im} g_z(0,0,\omega)|\{V(x,\omega); x\neq 1\})|\mathscr{F}_-)\\ &= \int_{-\infty}^{\infty} \operatorname{Im}\frac{1}{x-w}\tau(x)\mathrm{d}x \leq c\int_{-\infty}^{\infty}\operatorname{Im}\frac{1}{x-w}\mathrm{d}x = c\pi,\end{aligned}$$

independently of w. Therefore, we have

$$\mathsf{E}(\sigma(\mathrm{d}\lambda,\omega)|\mathscr{F}_-) \leq c\,\mathrm{d}\lambda. \tag{56}$$

Since a similar argument shows (56) for $\mathscr{F}_+$, one can apply Theorem 28. □

The pure point property of the spectrum of discrete Schrödinger operators with i.i.d. random potentials was first established by Goldsheid-Molchanov-Pastur [13] under some regularity condition on the distribution of $V(x,\omega)$. Carmona-Klein-Martinelli [5] removed the regularity condition by applying the multiscale analysis method and proved the Anderson localization (pure point property of the spectrum) under a quite general condition on the distribution of $V(x,\omega)$ including Bernoulli distribution.

Another application of Theorem 28 is to Gaussian potentials of Claim 25. Let $\{X_x(\omega)\}_{x\in\mathbb{Z}}$ be a stationary Gaussian process with mean 0 and covariance $\rho(x)$. Bochner's theorem asserts that $\rho(x)$ is expressed by a finite measure ν on $[0,2\pi)$ as

$$\rho(x) = \int_0^{2\pi} \mathrm{e}^{ix\lambda}\nu(\mathrm{d}\lambda).$$

Lemma 30 Assume that the absolutely continuous part of ν has a density ν' satisfying

$$\int_0^{2\pi} \frac{1}{\nu'(\lambda)} \mathrm{d}\lambda < \infty. \tag{57}$$

Then, for any function f on $\mathbb{R}$ such that for positive α, β

$$|f'(x)| \geq \alpha \mathrm{e}^{-\beta|x|} \tag{58}$$

holds on $\mathbb{R}$, the potential $f(X_x(\omega))$ satisfies the condition (53).

Proof We give a sketch of the proof. It can be shown that the condition (57) is equivalent to

$$\mathscr{F}_1 \equiv \sigma\text{-}\{X_x(\omega); x \neq 1\} \subsetneqq \sigma\text{-}\{X_x(\omega); x \in \mathbb{Z}\}. \tag{59}$$

Therefore, the random variable $X_0(\omega)$ has a Gaussian distribution with variance $v(\omega) > 0$ a.s. under the regular conditional probability given $\mathscr{F}_1$. Then, denoting the mean and the variance of $X_0(\omega)$ by $m(\omega)$, $v(\omega)$ respectively, we have

$$\begin{aligned}
\mathsf{E}(g_z(0,0,\omega)|\mathscr{F}_0) &= \mathsf{E}\Big(\frac{1}{f(X_0(\omega)) - w}\Big|\mathscr{F}_0\Big) \\
&= \frac{1}{\sqrt{2\pi v(\omega)}} \int_{-\infty}^{\infty} \frac{1}{f(x) - w} \mathrm{e}^{-(x-m(\omega))^2/2v(\omega)} \mathrm{d}x \\
&= \int_{\inf f}^{\sup f} \frac{1}{y - w} \tau_\omega(y) \mathrm{d}y,
\end{aligned}$$

where

$$\tau_\omega(y) = \mathrm{e}^{-(f^{-1}(y) - m(\omega))^2/2v(\omega)} \frac{1}{f'(f^{-1}(y))}.$$

The condition (58) implies the boundedness of $\tau_\omega(y)$, hence

$$\mathsf{E}(\operatorname{Im} g_z(0,0,\omega)|\mathscr{F}_0) \leq c(\omega) \int_{-\infty}^{\infty} \operatorname{Im} \frac{1}{y - w} \mathrm{d}y = \pi c(\omega).$$

Therefore, the measure $\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_0)$ is absolutely continuous for each fixed ω with respect to Lebesgue measure, hence so is $\mathsf{E}(\sigma(\mathrm{d}\lambda, \omega)|\mathscr{F}_-)$. $\square$

Since (59) implies that $\{f(X_x(\omega))\}$ is nondeterministic, we have

Claim 31 Under the conditions of Lemma 30, the Schrödinger operator H_ω has pure point spectrum and every its eigenfunction decays exponentially fast.

A typical example of purely nondeterministic Gaussian process satisfying the condition (57) is a stationary Gaussian process with covariance

$$\rho(x) = c(1 + |x|)^{-\alpha}$$

with $c, \alpha > 0$.

Exercise 32 Show that the spectral density

$$\nu'(\lambda) \equiv \frac{c}{2\pi} \sum_{x\in\mathbb{Z}} \mathrm{e}^{-ix\lambda}(1+|x|)^{-\alpha}$$

is positive and continuous by using the fact:

$$d(x) = \rho(x) - \rho(x+1) \text{ is positive and decreasing for } x \geq 0.$$

Consequently, one can say that whatever the speed of decay of correlation of $\{X_x(\omega)\}_{x\in\mathbb{Z}}$ is, we have always Anderson localization, namely, pure point property of the spectrum.

This method of averaging the spectral measures has been further developed as a rank one perturbation theorem by Simon-Wolff [36], which is sometimes effective also in higher dimension.

There is another method called **multiscale analysis** which was initiated by Fröhlich-Spencer (see [18], [37]). This method is effective also for deterministic potentials and in higher dimension.

§4. Some Deterministic Potentials

If ergodic potentials are nondeterministic, then we know that the corresponding Schrödinger operators have no ac spectrum, and in most of purely nondeterministic potentials the spectra consist only of point part. The next task is to investigate the spectrum for deterministic ergodic potentials. It should be remarked though that Theorems 22, 27 are applicable to some deterministic potentials. Since the definition of nondeterminism $\mathscr{F}_- = \mathscr{F}_+$ is too wide to obtain some significant results, we restrict ourselves to a narrower class including almost periodic potentials. Namely, we assume Ω is a compact metric space and $\{T_x\}$ is a continuous flow on it. In this section we give one theorem on ac spectrum which holds **for every** $\omega \in \Omega$ (not for a.e. ω) under the condition that the underlying flow is uniquely ergodic. A flow on a compact metric space is called **uniquely ergodic** if it posses a unique invariant probability measure μ such that $\operatorname{supp}\mu = \Omega$.

Theorem 33 (Avron-Simon [2]) The spectrum is independent of any $\omega \in \Omega$.

Proof We omit the proof. □

Theorem 34 (Theorem 7.4 of Kotani [26]) For a continuous function $V(\omega)$ on Ω the corresponding Schrödinger operator H_ω with ergodic potential $V(T_x\omega)$ has the ac spectrum coinciding with $\overline{\mathscr{Z}}^{\mathrm{ess}}$ for every $\omega \in \Omega$. Moreover, $m_\pm(z,\omega)$ are reflectionless on $\mathscr{Z}$ for every $\omega \in \Omega$.

Proof Theorem 17 obeys for a.e. ω

$$\Sigma_{\omega,\mathrm{ac}} = \overline{\mathscr{Z}}^{\mathrm{ess}},$$

and the reflectionless property on $\mathscr{Z}$, namely

$$m_+(\lambda + i0, \omega) = -\overline{m_-(\lambda + i0, \omega)} \qquad \text{for a.e. } \lambda \in \mathscr{Z}.$$

Therefore, Lemma 45 in Appendix implies for every $\omega \in \Omega$

$$\Sigma_{\omega,\mathrm{ac}} \supset \overline{\mathscr{Z}}^{\mathrm{ess}}$$

holds and $m_\pm(z,\omega)$ are reflectionless on $\mathscr{Z}$. To show the converse inclusion we use Theorem 52. Let $H_\pm(\omega)$ be the Schrödinger operators on $L^2(\mathbb{R}_\pm)$ respectively with Dirichlet boundary condition at 0. Then, $H(\omega)$ is a rank one perturbation of $H_+(\omega) \oplus H_-(\omega)$, hence their ac spectrum coincides. Therefore, for a.e. λ w.r.t. the ac part of the spectral measure of $H(\omega)$, one of the equations

$$-u'' + V(T_x\omega)u = \lambda u \text{ on } \mathbb{R}_\pm \text{ with } u(0) = 0$$

has a nontrivial polynomially bounded solution (see Lemma 38). Suppose the equation on $\mathbb{R}_+$ has such a solution ψ_λ. Then, one has (assuming $V \geq 0$, hence $\lambda > 0$)

$$\lim_{x\to\infty} \frac{1}{x} \log \sqrt{\psi'(x,\lambda)^2 + \lambda\psi(x,\lambda)^2} = 0.$$

Then, applying Theorem 52 $\gamma(\lambda) = 0$ follows, which implies $\Sigma_{\omega,\mathrm{ac}} \subset \overline{\mathscr{Z}}^{\mathrm{ess}}$ for each $\omega \in \Omega$.
□

A similar result was shown by Last-Simon [27] when the underlying flow is **minimal**, that is, the orbit $\{T_x\omega; x \in \mathbb{R}\}$ is dense in Ω for each $\omega \in \Omega$, which is milder than the unique ergodicity.

The **almost Mathieu operator** is a typical quasi-periodic discrete Schrödinger operator in one dimension:

$$(H_{\omega,\alpha,\kappa}u)(x) = u(x+1) + u(x-1) + \kappa\cos(2\pi(\alpha x + \omega))u(x),$$

where $\kappa > 0$ and α is irrational. This operator arises in connection with quantum Hall effect in condensed matter physics. The basic facts for this operator are

$$\begin{cases} \text{the Lebesgue measure of } \Sigma_{\alpha,\kappa} = 2|2-\kappa| \\ \gamma_{\alpha,\kappa}(\lambda) = \max\{0, \log(\kappa/2)\} \text{ for } \lambda \in \Sigma_{\alpha,\kappa} \end{cases}$$

hold for any ω and any irrational α, where $\Sigma_{\alpha,\kappa}$ and $\gamma_{\alpha,\kappa}$ denote the spectrum of $H_{\omega,\alpha,\kappa}$ and the Lyapounov exponent. To establish these results many people have contributed, especially for the second one by Bourgain-Jitomirskaya. Applying Theorem 34, we have for **every** irrational α and ω

$$\text{the ac spectrum of } H_{\omega,\alpha,\kappa} = \overline{\mathscr{Z}}^{\text{ess}} \text{ if } 0 < \kappa < 2.$$

However, Theorem 34 does not say anything about the pure ac property of the spectrum, although Corollary 20 implies that $H_{\omega,\alpha,\kappa}$ has purely ac spectrum for a.e. ω, since the absolute continuity of the IDS is known by Avila-Damanik. And Avila finally has proved that $H_{\omega,\alpha,\kappa}$ has purely ac spectrum for every ω and any irrational α.

The Cantor nature of the spectrum was called "Ten martini problem", and solved completely by Avila-Jitomirskaya. One of the reasons for the Fields prize awarded to Avila in 2014 was his great contribution to the spectral theory for the almost Mathieu operator. Many efforts by many people now have completed the study of the almost Mathieu operator except for several very delicate problems. In all these problems, a serious point to overcome was to extend a.e. α or ω statements to every α or ω statements.

§5. Open Problems

Open problems in the field of ergodic Schrödinger operators are divided into two parts:

[a] Higher dimensional problems

[b] One dimensional problems.

The most important open problem in [a] is "delocalization problem" or "Metal-Insulator transition problem", which states that in dimension three Schrödinger operators with random enough potentials have a critical energy E_c such that

$$\begin{cases} \text{the spectrum on } [E_0, E_c] \text{ is purely point spectrum,} \\ \text{the spectrum on } [E_c, \infty] \text{ is purely ac spectrum,} \end{cases}$$

where E_0 is the bottom of the spectrum. Only progress for this problem is the result that there exists an energy E_1 strictly greater than E_0 such that the spectrum on $[E_0, E_1]$ is of pure point, which was a great achievement of multiscale analysis exploited by Frölich-Spencer. However, for these 30 years essentially no progress has been made. One of the reasons of the difficulty in higher dimension is a lack of suitable quantity measuring localization. In one dimension, to prove the localization (existence of point spectrum) the Lyapounov exponents played a big role because they are computable. Among several possibilities to challenge this problem there is one way which might be computable. The underlying space $\mathbb{Z}^{d+1}$ can be decomposed to $\mathbb{Z}\times\mathbb{Z}^d$, so, instead of $\mathbb{Z}^{d+1}$ we consider the problem in $\mathbb{Z}\times[-L,L]^d$. Then, the operator will be

$$(Hu)(x,y)=u(x+1,y)+u(x-1,y)+(Au(x,\cdot))(y)+V(x,y)u(x,y),$$

where $y\in[-L,L]^d$ and for $v(y)$ the operator A is defined by

$$(Av)(y)=\sum_{z\in[-L,L]^d;\,\lceil z-y\rceil=1} v(z). \tag{60}$$

A is a $(2L)^d\times(2L)^d$ symmetric matrix. This is a quasi one dimensional model, which approximate the original higher dimensional model by letting $L\to\infty$. A merit of this model is to be able to define Lyapounov exponents, since eigenvalue equation $Hu=\lambda u$ for H turns to

$$\begin{pmatrix}\boldsymbol{u}(x+1)\\ \boldsymbol{u}(x)\end{pmatrix}=\begin{pmatrix}\lambda I-\boldsymbol{V}(x)-A\boldsymbol{u}(x) & -I\\ I & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{u}(x)\\ \boldsymbol{u}(x-1)\end{pmatrix}, \tag{61}$$

where $\boldsymbol{u}(x)=(u(x,y))_{y\in[-L,L]^d}\in\mathbb{R}^{(2L)^d}$, I is the $(2L)^d\times(2L)^d$ identity matrix and $\boldsymbol{V}(x)$ is a $(2L)^d\times(2L)^d$ diagonal matrix for fixed x with elements $V(x,y)$. Define

$$T(x)=\begin{pmatrix}\lambda I-\boldsymbol{V}(x)-A\boldsymbol{u}(x) & -I\\ I & 0\end{pmatrix}.$$

Then

$$\begin{pmatrix}\boldsymbol{u}(x+1)\\ \boldsymbol{u}(x)\end{pmatrix}=T(x)T(x-1)\cdots T(1)\begin{pmatrix}\boldsymbol{u}(1)\\ \boldsymbol{u}(0)\end{pmatrix},$$

which makes it possible to study asymptotic behavior of $\boldsymbol{u}(x)$ as $x\to\infty$ by using the Lyapounov exponents for this product of matrices. A general theory as an extension of one dimensional ergodic operators was obtained by Kotani-Simon [24]. Due to the symplectic property of $T(x)$ the Lyapounov exponents turn out to be symmetric w.r.t. 0, namely

$$-\gamma_d(\lambda)\le-\gamma_{d-1}(\lambda)\le\cdots\le-\gamma_1(\lambda)\le 0\le\gamma_1(\lambda)\le\gamma_2(\lambda)\le\cdots\le\gamma_d(\lambda).$$

The magnitude of localization can be measured by the smallest Lyapounov exponent $\gamma_1(\lambda)$. Generally H has no ac spectrum if $\gamma_1(\lambda)>0$. And if the potential $V(x,y)$ are i.i.d., it is known that $\gamma_1(\lambda) > 0$ for every λ and the spectrum is of pure point a.s.. This is valid for any d and L. Denote the Lyapounov exponent by $\gamma_1^L(\lambda)$. An interesting problem here is to show, if $d \geq 2$

$$\lim_{L\to\infty} \gamma_1^L(\lambda) = 0 \tag{62}$$

for λ near the middle of the spectrum of H on $\mathbb{Z}^{d+1}$. This is already a very hard problem and nobody has succeeded to give any answer, although computer simulations indicate its validity. Our experiences suggest that sometimes continuous versions are computable, so we consider the problem in $\mathbb{R} \times [-L, L]^d$ ($[-L, L]$ denotes here an interval consisting of integers), and Gaussian white noise potential in place of i.i.d. random potential. Then, (61) turns to

$$\mathrm{d}\begin{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{u}'(t) \end{pmatrix} = \begin{pmatrix} 0 & I\mathrm{d}t \\ -A\mathrm{d}t - \lambda I\mathrm{d}t + \mathrm{d}\boldsymbol{B}_t & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}(t) \\ \boldsymbol{u}'(t) \end{pmatrix},$$

where $\boldsymbol{B}_t$ is a d-dimensional Brownian motion. Then one can obtain the positive minimum Lyapounov exponent $\gamma_1^L(\lambda)$, and the problem is to prove or disprove (62). This proposes a very important and interesting problem to stochastic analysis. If $d = 0$ (in the original space $d + 1 = 1$), the Lyapounov exponent is explicitly computable.

As for [b] one basic problem in which the author is interested is to extract properties of ergodic potentials with a given IDS N or equivalently Lyapounov exponent γ. One may call it as the inverse spectral problem for ergodic potentials. Let us formulate the problem: For a fixed real number c set

$$\Omega = \Omega_c = \{V; V \text{ is real valued, bounded on } \mathbb{R} \text{ and } \inf \operatorname{spec} H_V \geq c\}.$$

One can introduce a metric on Ω as in Section 3.2 and a shift operation θ_x. Let $\mathscr{P}$ be the set of all shift invariant ergodic probability measures on Ω and $w = w_\mu$ be the Floquet exponent for a $\mu \in \mathscr{P}$. Then, as necessary conditions for w we have

(i) w is analytic on $\mathbb{C}\backslash[c, \infty)$, real valued on $(-\infty, c)$ and

$$w(z) = i\sqrt{z} + \frac{c_1}{i\sqrt{z}} + o\big(\sqrt{z}^{-1}\big) \qquad \text{as } \sqrt{z} \to \infty;$$

(ii) w, w', $-iw$ are Herglotz functions.

Conversely, for w satisfying (i), (ii) set

$$\mathscr{P}_w = \{\mu \in \mathscr{P}; w_\mu = w\}.$$

Then the fundamental problem is

$$\text{to investigate the structure of } \mathscr{P}_w. \tag{63}$$

First of all we have to show $\mathscr{P}_w \neq \phi$, which is nontrivial, although we believe its validity. It is easy to show the existence of a shift-invariant probability measure μ on Ω satisfying $w_\mu = w$, however, to choose ergodic one among them is nontrivial. The only result we have is $\mathscr{P}_w \neq \phi$ if

$$\int_c^\infty \gamma(\lambda)\mathrm{d}N(\lambda) = 0 \tag{64}$$

(see Kotani [20], [21], and Carmona-Kotani [6]).

Once $\mathscr{P}_w \neq \phi$ is established, then the next step is to investigate the structure of $\mathscr{P}_w$. As we have seen in Section 3.4, if $\gamma(\lambda) > 0$ for all λ, then there is a possibility that a quasi-periodic potential and a quite random potential may have the same w, which implies $\mathscr{P}_w$ is too large to give a reasonable description of its structure. On the other hand, if γ vanishes on $\operatorname{supp} \mathrm{d}N$ (= the spectrum) like (64) and $\operatorname{supp} \mathrm{d}N$ is a disjoint sum of finite numbers of intervals. Then we have a complete description of $m_\pm$ from the reflectionless property of $m_\pm$ on $\operatorname{supp} \mathrm{d}N$, which results in

$$V(x) = c - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\Theta(x\boldsymbol{a} + \boldsymbol{b}) \qquad \text{(Its-Matveev)},$$

where $c \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, and Θ is the θ-function on a compact Riemannian surface relating to $\operatorname{supp} \mathrm{d}N$. Especially we know that Θ is a smooth function on $\mathbb{R}^n/\mathbb{Z}^n$, hence $V(x)$ is quasi-periodic. Therefore, in this case our problem can be reduced to a finite dimensional problem. This argument leads us to the following working hypothesis: Suppose w satisfies

$$\begin{cases} (1)\ \ N(\lambda)\ \left(= \dfrac{1}{\pi}\operatorname{Im} w(\lambda + i0)\right) \text{ is absolutely continuous,} \\ (2)\ \ \gamma(\lambda)\ (= -\operatorname{Re} w(\lambda + i0)) = 0 \text{ a.e. w.r.t. } \mathrm{d}N. \end{cases} \tag{65}$$

The condition (65) is equivalent to the pure ac property of the spectrum of the corresponding ergodic operators (Corollary 20).

$$\boxed{\begin{array}{l}\text{Suppose the Floquet exponent } w \text{ for an ergodic Schrödinger operator} \\ \text{satisfies (65). Then, the corresponding potentials are almost periodic?}\end{array}} \tag{66}$$

Sodin-Yuditski solved (66) affirmatively under some homogeneity condition on $\operatorname{supp} \mathrm{d}N$. However, recently counter examples to (66) were discovered, so the problem determining the structure of $\mathscr{P}_w$ for w satisfying (65) remains open.

There was another remarkable evolution concerning the reflectionless property. Remling [33] found a deterministic version of our main Theorem 17. Namely, let V be a potential defined on $\mathbb{R}_+$ and H_V^+ be the corresponding Schrödinger operator on $\mathbb{R}_+$ with Dirichlet boundary condition at 0. Assume H_V^+ has ac spectrum on A. Then, he proved that for any limit point $\widetilde{V}$ (under some weak topology on potentials) of $\{\theta_x V\}_{x>0}$ as $x \to \infty$ the corresponding $H_{\widetilde{V}}$ has the reflectionless property on A.

Finally the author would like to remark the other possibility which should be considered as an extension of the present framework. In our argument, the property on the shift operation θ_x:

$$H_{T_x\omega} = \theta_x H_\omega \theta_x^{-1} \tag{67}$$

played a central role. In the KdV hierarchy, the shift operation is considered to be the first equation and KdV equation is the second one. The hierarchy is a collection of infinitely many nonlinear differential equations (only the first equation concerning the shift). P. Lax found relations similar to (67) with some unitary transforms in place of θ_x for every equation belonging to the KdV hierarchy. Recently the author could prove invariance of the Floquet exponent under the time evolution by one of the KdV hierarchy, which may have some relationship with the above inverse spectral problem. The difficult open problem here is to construct solutions to KdV equation starting from general nondecaying initial functions.

Appendix

A Generalized Eigenfunctions Expansion

This section is devoted to a brief introduction to Weyl-Stone-Titchmarsh-Kodaira theorem. This theorem can be considered as a generalized eigenfunctions expansion for Schrödinger operators (more generally Sturm-Liouville operators), which is a continuous analogue of the correspondence between Jacobi matrices and moment problems or orthogonal polynomials.

A.1 Existence and Uniqueness of L^2-Solution

The following lemma is fundamental to introduce Weyl function. The proof is performed by using functional analysis. Let

$$\mathscr{D} = \{u \in L^2(\mathbb{R}_+); u'' \in L^2(\mathbb{R}_+) \text{ and } u(0) = 0\},$$

and define self-adjoint operators on $L^2(\mathbb{R}_+)$ with domain $\mathscr{D}$ by

$$H_0^+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad H^+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V. \tag{68}$$

Lemma 35 (Unique existence of L^2-solution) For any fixed $z \in \mathbb{C}\backslash\mathbb{R}$ there exists uniquely a solution $u \in L^2(\mathbb{R}_+)$ to

$$-u'' + Vu = zu, \qquad u(0) = 1.$$

Proof The self-adjoitness of H^+ implies that for $z \in \mathbb{C}$ and $u \in \mathscr{D}$

$$\begin{aligned}\|H^+u - zu\|^2 &= ((H^+ - \operatorname{Re} z)u - (i\operatorname{Im} z)u, (H^+ - \operatorname{Re} z)u - (i\operatorname{Im} z)u)\\ &= \|(H^+ - \operatorname{Re} z)u\|^2 + |\operatorname{Im} z|^2\|u\|^2\end{aligned} \tag{69}$$

is valid. Hence, letting

$$\mathscr{D}_0 = \{u \in \mathscr{D}; u'(0) = 0\},$$

we see that $(H^+ - z)\mathscr{D}_0$ is closed in $L^2(\mathbb{R}_+)$ if $\operatorname{Im} z \neq 0$. Since $(H^+ - z)^{-1}$ exists as a bounded operator and $(H^+ - z)^{-1}L^2(\mathbb{R}_+) = \mathscr{D}$, we have

$$(H^+ - z)\mathscr{D}_0 \subsetneq L^2(\mathbb{R}_+).$$

Therefore, there exists a $u \in L^2(\mathbb{R}_+)$ such that

$$0 \neq u \in ((H^+ - z)\mathscr{D}_0)^{\perp}.$$

This u satisfies

$$-u'' + Vu - \overline{z}u = 0.$$

Suppose $u(0) = 0$. Then (69) shows $u = 0$, which is a contradiction again, and we see $u(0) \neq 0$. One can assume $u(0) = 1$. The uniqueness can be shown by contradiction. Suppose $v \in L^2(\mathbb{R}_+)$ satisfies

$$-v'' + Vv - \overline{z}v = 0, \qquad v(0) = 1.$$

Then, $w = u - v \in L^2(\mathbb{R}_+)$ satisfies $w(0) = 0$, hence $w \in \mathscr{D}$. Applying (69) to w, we have $w = 0$, which concludes the lemma. □

Exercise 36 By showing that u, $u'' \in L^2(\mathbb{R}_+)$ implies

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty} u'(x) = 0$$

prove the closedness of $(H^+ - z)\mathscr{D}_0$ in $L^2(\mathbb{R}_+)$ if $\operatorname{Im} z \neq 0$.

To show the existence of L^2-solutions there is a traditional method by which the boundedness of V can be relaxed to a milder condition $V \in L^1_{\text{loc}}(\mathbb{R})$, namely the limit circle and limit point method which was first used by H. Weyl in 1910. Suppose $\operatorname{Im} z > 0$. For $a > 0$ define

$$D_a = \Big\{w \in \mathbb{C};\ \int_0^a |\varphi(x) + w\psi(x)|^2 \mathrm{d}x \le \frac{\operatorname{Im} w}{\operatorname{Im} z}\Big\},$$

where φ, ψ are linearly independent solutions to

$$-u'' + Vu - zu = 0, \qquad \varphi(0) = \psi'(0) = 1, \qquad \varphi'(0) = \psi(0) = 0. \tag{70}$$

D_a becomes a non-empty (actually $\psi(a) \ne 0$ and $-\varphi(a)/\psi(a) \in D_a$) closed disc in $\mathbb{C}_+$, and decreases as a increases. Then

$$D_\infty = \bigcap_{a>0} D_a$$

becomes a non-empty closed disc or one point. For $w \in D_\infty$ we have easily

$$\int_0^\infty |\varphi(x) + w\psi(x)|^2 \mathrm{d}x \le \frac{\operatorname{Im} w}{\operatorname{Im} z} \Longrightarrow \varphi + w\psi \in L^2(\mathbb{R}_+).$$

The boundary ∞ is called as **limit circle type** if D_∞ is a disc, and is called as **limit point type** if D_∞ is one point. Lemma 35 shows that if V is bounded, then ∞ is of limit point type.

A.2 Estimate of Green Functions for Large z

In this subsection, we estimate the Green functions of H^+, H and the Weyl functions $m_\pm(z)$. For $z \in \mathbb{C}\backslash\mathbb{R}$ denote

$$R(z) = (H^+ - z)^{-1}, \qquad R_0(z) = (H_0^+ - z)^{-1}.$$

A key is the following identity:

$$R(z) = R_0(z) - R_0(z)VR_0(z) + R_0(z)VR(z)VR_0(z). \tag{71}$$

Rewriting (71) by the Green functions $g_+(x,y)$ of H^+ and $g_+^0(x,y)$ of H_0^+ yields

$$\begin{aligned} g_+(x,y) = g_+^0(x,y) &- \int_0^\infty g_+^0(x,s)V(s)g_+^0(s,y)\mathrm{d}s \\ &+ \int_0^\infty g_+^0(x,s)V(s)\mathrm{d}s \int_0^\infty g_+(s,t)V(t)g_+^0(t,y)\mathrm{d}t, \end{aligned} \tag{72}$$

where the spectral parameter $z\in\mathbb{C}\backslash\mathbb{R}$ is fixed. On the other hand, it is known that $g_+(x,y)$, $g_+^0(x,y)$ are given by

$$\begin{cases} g_+(x,y)=g_+(y,x)=\psi(x,z)f_+(y,z)\\ g_+^0(x,y)=g_+^0(y,x)=\dfrac{1}{\sqrt{z}}\sin(\sqrt{z}x)\mathrm{e}^{i\sqrt{z}y}\end{cases}\tag{73}$$

for $0\le x\le y$, where $\psi(x,z)$ is the solution ψ of (70). Set

$$C\equiv\sup|V(x)|,\qquad \delta(z)=\mathrm{dist}(z,\mathrm{spec}\,H^+)\quad(\ge|\mathrm{Im}\,z|).$$

Lemma 37 The followings hold.

(i) $|g_+(x,x)-g_+^0(x,x)|\le\dfrac{\mathrm{Im}\,g_+^0(x,x)}{\mathrm{Im}\,z}C(1+C\delta(z)^{-1})$;

(ii) $|m_+(z)-i\sqrt{z}|\le\dfrac{C}{2\mathrm{Im}\,\sqrt{z}}(1+C\delta(z)^{-1})$.

Proof Since $\|R(z)\|\le\delta(z)^{-1}$, we have

$$\begin{aligned}\int_0^\infty\mathrm{d}s\Big|\int_0^\infty g_+(s,t)V(t)g_+^0(t,x)\mathrm{d}t\Big|^2&\le\delta(z)^{-2}\int_0^\infty|V(t)g_+^0(t,x)|^2\mathrm{d}t\\&\le C^2\delta(z)^{-2}\int_0^\infty|g_+^0(t,x)|^2\mathrm{d}t\\&=C^2\delta(z)^{-2}(\mathrm{Im}\,z)^{-1}\mathrm{Im}\,g_+^0(x,x).\end{aligned}$$

In the last step we have used the resolvent identity

$$\frac{g_+^0(x,y,z)-g_+^0(x,y,w)}{z-w}=\int_0^\infty g_+^0(x,s,z)g_+^0(s,y,w)\mathrm{d}s.$$

Then, Schwarz inequality shows

$$\begin{aligned}&\Big|\int_0^\infty g_+^0(x,s)V(s)\mathrm{d}s\int_0^\infty g_+(s,t)V(t)g_+^0(t,x)\mathrm{d}t\Big|\\&\le C\int_0^\infty|g_+^0(x,s)|\mathrm{d}s\,\Big|\int_0^\infty g_+(s,t)V(t)g_+^0(t,x)\mathrm{d}t\Big|\\&\le C^2\delta(z)^{-1}\sqrt{\int_0^\infty|g_+^0(x,s)|^2\mathrm{d}s}\,\sqrt{(\mathrm{Im}\,z)^{-1}\mathrm{Im}\,g_+^0(x,x)}\\&=C^2\delta(z)^{-1}(\mathrm{Im}\,z)^{-1}\mathrm{Im}\,g_+^0(x,x).\end{aligned}$$

Therefore, (i) can be obtained by (72).

To show (ii) first note

$$m_+(z)=\lim_{y\to0}\frac{\partial^2}{\partial x\partial y}g_+(x,y)\Big|_{0<x<y},$$

which follows from (73). This together with (72) yields

$$m_+(z) = i\sqrt{z} - \int_0^\infty \mathrm{e}^{2is\sqrt{z}} V(s)\mathrm{d}s + \int_0^\infty \mathrm{e}^{is\sqrt{z}} V(s)\mathrm{d}s \int_0^\infty g_+(s,t)V(t)\mathrm{e}^{it\sqrt{z}}\mathrm{d}t.$$

The rest of the proof is the same as that of (i). □

The estimate for the Green function of H can be obtained similarly.

A.3 Generalized Eigenfunctions

In this subsection we consider the eigenfunctions expansion for the self-adjoint operators H^+, H. Let $\varphi(x,z)$, $\psi(x,z)$ be the solutions φ, ψ to (70) respectively. Then

$$f_+(x,z) = \varphi(x,z) + m_+(z)\psi(x,z),$$

hence, for $0 \le x \le y$ the Green function of H^+ is

$$g_z^+(x,y) = \psi(x,z)f_+(y,z) = \psi(x,z)\varphi(y,z) + m_+(z)\psi(x,z)\psi(y,z).$$

Denote by σ_+ the representing measure of $m_+(z)$ (see (76)). Then $g_z^+(x,x)$ is

$$g_z^+(x,x) = G_x(z) + \psi(x,z)^2 \int_{-\infty}^\infty \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big)\sigma_+(\mathrm{d}\lambda),$$

where

$$G_x(z) = \psi(x,z)\varphi(x,z) + (\alpha + \beta z)\psi(x,z)^2.$$

Noting that $G_x(z)$ takes real values for $z \in \mathbb{R}$, we see

$$\lim_{\epsilon\downarrow 0} \frac{1}{\pi}\int_a^b \operatorname{Im} g_{\lambda+i\epsilon}^+(x,x)\mathrm{d}\lambda = \int_a^b \psi(x,\lambda)^2\sigma_+(\mathrm{d}\lambda)$$

for any $a < b$. Therefore, from (77) the representing measure of $g_z^+(x,x)$ turns to $\psi(x,\lambda)^2\sigma_+(\mathrm{d}\lambda)$. On the other hand, (i) of Lemma 37 implies

$$g_z^+(x,x) = O\big(z^{-1/2}\big)$$

as $z \to \infty$. Consequently, the α and β terms vanish and we have

$$g_z^+(x,x) = \int_c^\infty \frac{\psi(x,\lambda)^2}{\lambda - z}\sigma_+(\mathrm{d}\lambda).$$

Here we have used the fact $g_z^+(x,x)$ is analytic on $\mathbb{C}\backslash(c,\infty)$ and takes real values on $(-\infty,c)$, where $c = \inf V(x)$. Since

$$c_1\overline{c_1}g_z^+(x,x) + c_1\overline{c_2}g_z^+(x,y) + c_2\overline{c_1}g_z^+(y,x) + c_2\overline{c_2}g_z^+(y,y)$$

is a Herglotz function for any complex c_1, c_2, without difficulty we obtain a formula

$$g_z^+(x,y)=\int_c^\infty \frac{\psi(x,\lambda)\psi(y,\lambda)}{\lambda-z}\sigma_+(\mathrm{d}\lambda) \tag{74}$$

for any $x,y\geq 0$. $\psi(x,\lambda)$ are the **generalized eigenfunctions** of H^+ for a.e. λ w.r.t. σ_+. As a corollary we have

Lemma 38 For a.e. $\lambda\in\mathbb{R}$ with respect to σ_+ generalized eigenfunctions $\psi(x,\lambda)$ of H^+ satisfy

$$\psi(x,\lambda)=O(x^\rho) \qquad \text{as } x\to\infty$$

for any $\rho>2$.

Proof Choose $\lambda_0<0$ such that

$$\lambda_0<-C.$$

Then, $\lambda_0\notin \operatorname{spec} H^+$, hence the estimate (i) of Lemma 37 shows

$$\int_1^\infty g_{\lambda_0}^+(x,x)x^{-\beta}\mathrm{d}x<\infty$$

for any $\beta>1$. Therefore, (74) implies

$$\int_c^\infty \frac{1}{\lambda-\lambda_0}\Big\{\int_1^\infty \psi(x,\lambda)^2x^{-\beta}\mathrm{d}x\Big\}\sigma_+(\mathrm{d}\lambda)<\infty,$$

and consequently, for a.e. $\lambda\in\mathbb{R}$ with respect to σ_+

$$\int_1^\infty \psi(x,\lambda)^2x^{-\beta}\mathrm{d}x<\infty$$

holds. On the other hand, from the equation

$$\psi(x,\lambda)=x+\int_0^x(x-y)(V(y)-\lambda)\psi(y,\lambda)\mathrm{d}y,$$

it follows that

$$|\psi(x,\lambda)|\leq x+C_1\int_0^x y|\psi(y,\lambda)|\mathrm{d}y$$

with $C_1=\sup|V(y)-\lambda|<\infty$. Hence

$$|\psi(x,\lambda)|\leq x+C_1\int_0^1 y|\psi(y,\lambda)|\mathrm{d}y+C_1\sqrt{\int_1^x y^{2+\beta}\mathrm{d}y}\sqrt{\int_1^x \psi(y,\lambda)^2y^{-\beta}\mathrm{d}y},$$

which completes the proof. $\square$

Although the generalized eigenfunctions expansion of H on the whole R is unnecessary in this note, it may be better to briefly explain it. The Green function $g_z(x,y)$ of H is given by (14), namely

$$g_z(x,y) = g_z(y,x) = -\frac{f_+(x,z)f_-(y,z)}{m_+(z)+m_-(z)} \qquad \text{for } x \geq y.$$

$f_\pm$ can be expressed by $\varphi(x,z)$, $\psi(x,z)\psi$ as

$$f_+(x,z) = \varphi(x,z) + m_+(z)\psi(x,z), \qquad f_-(x,z) = \varphi(x,z) - m_-(z)\psi(x,z).$$

Therefore, introducing

$$\begin{cases} M(z) = \begin{pmatrix} -\dfrac{1}{m_+(z)+m_-(z)} & -\dfrac{m_+(z)}{m_+(z)+m_-(z)} + \dfrac{1}{2} \\ -\dfrac{m_+(z)}{m_+(z)+m_-(z)} + \dfrac{1}{2} & \dfrac{m_+(z)m_-(z)}{m_+(z)+m_-(z)} \end{pmatrix}, \\ \boldsymbol{\phi}(x,z) = (\varphi(x,z), \psi(x,z)), \end{cases}$$

we have for $x \geq y$

$$g_z(x,y) = \boldsymbol{\phi}(x,z)M(z)\boldsymbol{\phi}(y,z)^{\mathsf{T}} + \frac{1}{2}(\varphi(x,z)\psi(y,z) - \varphi(y,z)\psi(x,z)). \tag{75}$$

Exercise 39 For $m_\pm \in \mathbb{C}_+$ define a symmetric matrix M by

$$M = \begin{pmatrix} -\dfrac{1}{m_+ + m_-} & -\dfrac{m_+}{m_+ + m_-} + \dfrac{1}{2} \\ -\dfrac{m_+}{m_+ + m_-} + \dfrac{1}{2} & \dfrac{m_+ m_-}{m_+ + m_-} \end{pmatrix}.$$

Then, show $\operatorname{Im} M = (M - M^*)/(2i)$ is positive definite.

This exercise implies that for any $\boldsymbol{u} \in \mathbb{C}^2$

$$\operatorname{Im}(M(z)\boldsymbol{u}, \boldsymbol{u}) \geq 0,$$

which means that $(M(z)\boldsymbol{u}, \boldsymbol{u})$ is a Herglotz function. Then one can show that there exist a self-adjoint matrix A, a nonnegative definite matrix B and a matrix valued nonnegative definite measure $\Sigma(\mathrm{d}\lambda)$ such that

$$M(z) = A + Bz + \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big)\Sigma(\mathrm{d}\lambda).$$

Then (75) shows

$$g_z(x,y) = \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big)\boldsymbol{\phi}(x,z)\Sigma(\mathrm{d}\lambda)\boldsymbol{\phi}(y,z)^{\mathsf{T}} + G_{x,y}(z),$$

where

$$G_{x,y}(z)=\boldsymbol{\phi}(x,z)(A+Bz)\boldsymbol{\phi}(y,z)^{\mathsf{T}}+\frac{1}{2}(\varphi(x,z)\psi(y,z)-\varphi(y,z)\psi(x,z)).$$

$G_{x,y}(z)$ is an entire function for each fixed x,y, and takes real values if $z\in\mathbb{R}$ and $x=y$. Then, the rest of the argument is almost similar to that of H^{+} and we see

$$\int_{c}^{\infty}\frac{1}{1+|\lambda|}\boldsymbol{\phi}(x,\lambda)\Sigma(\mathrm{d}\lambda)\boldsymbol{\phi}(x,\lambda)^{\mathsf{T}}<\infty$$

and

$$g_z(x,y)=\int_{c}^{\infty}\frac{1}{\lambda-z}\boldsymbol{\phi}(x,\lambda)\Sigma(\mathrm{d}\lambda)\boldsymbol{\phi}(y,\lambda)^{\mathsf{T}}.$$

From this formula a generalized Fourier transform is obtained as follows. For $f\in L^2(\mathbb{R})$ define a transform by

$$\widehat{f}(\lambda)=\int_{-\infty}^{\infty}f(x)\boldsymbol{\phi}(x,\lambda)\mathrm{d}x.$$

Then, we see that an inversion formula

$$f(x)=\lim_{z\to\infty}\{-zG_zf(x)\}=\int_{c}^{\infty}\boldsymbol{\phi}(x,\lambda)\Sigma(\mathrm{d}\lambda)\widehat{f}(\lambda)^{\mathsf{T}},$$

and Parseval's identity

$$\int_{-\infty}^{\infty}|f(x)|^2\mathrm{d}x=\lim_{z\to\infty}\{-z(G_zf,f)\}=\int_{c}^{\infty}\overline{\widehat{f}(\lambda)}\Sigma(\mathrm{d}\lambda)\widehat{f}(\lambda)^{\mathsf{T}}$$

hold. Fourier transform is nothing but the case of $V=0$, namely $H=-\Delta$.

The generalized eigenfunctions of H is defined as follows. Set

$$\tau(\mathrm{d}\lambda)=\mathrm{tr}\Sigma(\mathrm{d}\lambda).$$

Then, every element of $\Sigma(\mathrm{d}\lambda)$ is absolutely continuous with respect to σ, since $\Sigma(\mathrm{d}\lambda)$ is nonnegative definite, and its density is denoted by $f_{ij}(\lambda)$. Let $\mu_1(\lambda)$, $\mu_2(\lambda)$ (≥ 0) be the eigenvalues and $\boldsymbol{e}_1(\lambda)$, $\boldsymbol{e}_2(\lambda)$ be the eigenvectors of the matrix $(f_{ij}(\lambda))$. Then, $\phi_j(x,\lambda)=\sqrt{\mu_j(\lambda)}(\boldsymbol{\phi}(x,\lambda),\boldsymbol{e}_j(\lambda))$ $(j=1,2)$ are called **generalized eigenfunctions** for H, and it holds that

$$\boldsymbol{\phi}(x,\lambda)\Sigma(\mathrm{d}\lambda)\boldsymbol{\phi}(x,\lambda)^{\mathsf{T}}=\big(\phi_1(x,\lambda)^2+\phi_2(x,\lambda)^2\big)\tau(\mathrm{d}\lambda).$$

The definition of τ implies that at least one of $\{\mu_1(\lambda),\mu_2(\lambda)\}$ is nondegenerate for a.e. $\lambda\in\mathbb{R}$ with respect to τ. One can obtain the polynomial growth of the generalized eigenfunctions in this case also.

Lemma 40 For a.e. $\lambda \in \mathbb{R}$ with respect to τ generalized eigenfunctions $\phi_j(x,\lambda)$ $(j=1,2)$ of H satisfy

$$\phi_j(x,\lambda) = O\big(|x|^\rho\big) \qquad \text{as } |x| \to \infty$$

for any $\rho > 2$.

For a general $V \in L^1_{\rm loc}(\mathbb{R})$ one can not expect to have Lemma 37. However, the Weyl's limit circle and limit point method makes it possible to perform a similar argument, and the established result is called the **Weyl-Stone-Titchmarsh-Kodaira generalized expansion theorem**.

B Herglotz Functions

A Herglotz function m is a analytic function on $\mathbb{C}_+$ satisfying $\operatorname{Im} m(z) > 0$. In this section we prepare several basic facts which are necessary for our theorems.

Lemma 41 m has a representation

$$m(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big)\sigma(\mathrm{d}\lambda) \tag{76}$$

with

$$\alpha \in \mathbb{R}, \qquad \beta \geq 0, \qquad \int_{-\infty}^{\infty} \frac{1}{1+\lambda^2}\sigma(\mathrm{d}\lambda) < \infty.$$

Proof The upper half plane $\mathbb{C}_+$ is transformed onto the unit disc $\mathbb{D}$ by

$$\zeta = \frac{z-i}{z+i} \Big(\Longrightarrow z = i\frac{1+\zeta}{1-\zeta}\Big).$$

Since $m(\zeta)$ is analytic on $\mathbb{D}$ and satisfies $\operatorname{Im} m > 0$,

$$\operatorname{Im} m(\zeta) = \operatorname{Im} \frac{i}{2\pi}\int_{[0,2\pi)} \frac{\mathrm{e}^{i\theta}+\zeta}{\mathrm{e}^{i\theta}-\zeta}\tau(\mathrm{d}\theta)$$

with a finite measure τ. Hence, for some $\alpha \in \mathbb{R}$

$$m(\zeta) = \alpha + \frac{i}{2\pi}\int_{[0,2\pi)} \frac{\mathrm{e}^{i\theta}+\zeta}{\mathrm{e}^{i\theta}-\zeta}\tau(\mathrm{d}\theta) = \alpha + \beta z + \frac{i}{2\pi}\int_{(0,2\pi)} \frac{\mathrm{e}^{i\theta}+\zeta}{\mathrm{e}^{i\theta}-\zeta}\tau(\mathrm{d}\theta)$$

holds, where $\beta = \sigma(\{0\})/2\pi \geq 0$. Setting

$$\lambda = -\cot\frac{\theta}{2} = i\frac{1+\mathrm{e}^{i\theta}}{1-\mathrm{e}^{i\theta}} \Big(\Longrightarrow i\frac{\mathrm{e}^{i\theta}+\zeta}{\mathrm{e}^{i\theta}-\zeta} = \frac{\lambda z+1}{\lambda - z}\Big),$$

we have

$$m(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \frac{\lambda z+1}{\lambda - z}\tau(\mathrm{d}\lambda) = \alpha + \beta z + \int_{-\infty}^{\infty} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2+1}\Big)\sigma(\mathrm{d}\lambda),$$

where $\sigma(\mathrm{d}\lambda)=(\lambda^2+1)\tau(\mathrm{d}\lambda)$. $\square$

Let σ be a measure appearing in (76) for a Herglotz function m and

$$\sigma=\sigma_{\rm ac}+\sigma_{\rm s}$$

be the Lebesgue decomposition of σ.

Lemma 42 σ, $\sigma_{\rm ac}$, $\sigma_{\rm s}$ are obtained from m as follows.

(i)

$$\sigma(I)=\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_I \operatorname{Im} m(\lambda+i\epsilon)\mathrm{d}\lambda \tag{77}$$

for any finite open interval I such that $\sigma(\partial I)=0$.

(ii)

$$\lim_{\epsilon\downarrow 0}\operatorname{Im} m(\lambda+i\epsilon)=\pi\lim_{\epsilon\downarrow 0}\frac{\sigma(\lambda+\epsilon)-\sigma(\lambda-\epsilon)}{2\epsilon}\qquad(\sigma(\lambda)\equiv\sigma([0,\lambda])) \tag{78}$$

holds if one of the sides exists finitely, hence the density $\sigma'(\lambda)$ of $\sigma_{\rm ac}$ is obtained by

$$\sigma'(\lambda)=\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\operatorname{Im} m(\lambda+i\epsilon) \tag{79}$$

for a.e. $\lambda\in\mathbb{R}$.

(iii) Let

$$S\equiv\Big\{\lambda\in\mathbb{R}:\lim_{\epsilon\downarrow 0}\operatorname{Im} m(\lambda+i\epsilon)=\infty\Big\}.$$

Then $\sigma_{\rm s}(\mathbb{R}\backslash S)=0$.

Proof Note

$$\int_I \operatorname{Im} m(\lambda+i\epsilon)\mathrm{d}\lambda=\int_{-\infty}^{\infty}\Big(\int_I\frac{\epsilon}{(\lambda-x)^2+\epsilon^2}\mathrm{d}\lambda\Big)\mathrm{d}\sigma(x)$$

and

$$\lim_{\epsilon\downarrow 0}\int_I\frac{\epsilon}{(\lambda-x)^2+\epsilon^2}\mathrm{d}\lambda=\pi\begin{cases}1 & \text{if } x\in I;\\ \dfrac{1}{2} & \text{if } x\in\partial I;\\ 0 & \text{if } x\in\mathbb{R}\backslash I,\end{cases}$$

which implies (i). Moreover

$$\operatorname{Im} m(\lambda+i\epsilon)=\int_0^\infty\frac{\epsilon}{u+\epsilon^2}\mathrm{d}s(u)$$

holds with

$$s(u)\equiv\frac{1}{\pi}\int_{x-\sqrt{u}}^{x+\sqrt{u}}\sigma(\mathrm{d}\lambda)\qquad\text{for } u\geq 0.$$

Then Hardy-Littlewood Taubelian theorem (and Abelian theorem) concludes (ii). (iii) was proved by Vallée Poussin. $\square$

The lemma below is due to a theorem holding for the **Nevanlinna class** N. An analytic function f on $\mathbb{C}_+$ belongs to N, if $\log^+|f(z)|$ has a harmonic majorant, where

$$\log^+ x = \max\{\log x, 0\} \qquad \text{for } x \geq 0.$$

If f is a nontrivial Herglotz function, then one can define $h = \log(-if)$ as an analytic function satisfying $0 < |\mathrm{Im}\, h| < \pi/2$, hence, for $p \in (0,1)$

$$\mathrm{Re}(-if)^p = \mathrm{Re}\, \mathrm{e}^{ph} = \mathrm{e}^{p\,\mathrm{Re}\, h}\cos(p\,\mathrm{Im}\, h) \geq \mathrm{e}^{p\,\mathrm{Re}\, h}\cos(p\pi/2),$$

and

$$|f(z)|^p = \mathrm{e}^{p\,\mathrm{Re}\, h(z)} \leq \mathrm{Re}(-if(z))^p$$

which implies

$$\log^+|f(z)| \leq \frac{1}{p}|f(z)|^p \leq (p\cos(p\pi/2))^{-1}\mathrm{Re}(-if(z))^p.$$

Since the right hand side is harmonic, f is an element of N. Clearly N is a linear space. Therefore, from theorem 5.3 in page 67 of Garnett [12] we have

Lemma 43 Herglotz functions satisfy the following properties.

(i) For a Herglotz function m there exists a finite limit $m(\lambda + i0)$ for a.e. $\lambda \in \mathbb{R}$.

(ii) If two Herglotz functions m_1, m_2 satisfy

$$m_1(\lambda + i0) = m_2(\lambda + i0) \qquad \text{for a.e. } \lambda \in A$$

for a measurable set A with positive Lebesgue measure, then $m_1 = m_2$ identically.

A sequence of Herglotz functions m_n is said to converge to a Herglotz function m if

$$m_n(z) \to m(z) \qquad \text{for any } z \in \mathbb{C}_+.$$

From this convergence we would like to obtain a convergence of their boundary values $m_n(\lambda + i0)$. For this purpose, first, for $f \in L^2(\mathbb{R})$ and $z \in \mathbb{C}_+$ set

$$S(f)(z) = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} f(\lambda)\mathrm{d}\lambda.$$

Defining the Fourier transform $\widehat{f}$ of f by

$$\widehat{f}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi i\lambda t} f(\lambda)\mathrm{d}\lambda,$$

we have

$$S(f)(z) = 2\pi i \int_0^\infty \mathrm{e}^{2\pi i z t} \widehat{f}(t) \mathrm{d}t,$$

which shows

$$S(f)(\lambda) \equiv \lim_{\epsilon \searrow 0} S(f)(\lambda + i\epsilon)$$

exists for a.e. $\lambda \in \mathbb{R}$ and

$$\|S(f)\| \le 2\pi \|f\|. \tag{80}$$

Lemma 44 For $f_n \in L^2(\mathbb{R})$ assume there exists a constant C such that

$$\|f_n\| \le C \tag{81}$$

holds for any $n \ge 1$. For an $f \in L^2(\mathbb{R})$, if $S(f_n)(z)$ converges to $S(f)(z)$ for any $z \in \mathbb{C}_+$, then $S(f_n)(\lambda)$ converges to $S(f)(\lambda)$ weakly in $L^2(\mathbb{R})$.

Proof Let

$$\mathscr{L} = \Big\{\varphi; \varphi(\lambda) = \sum_{j:\,\text{finite sum}} \frac{c_j}{\lambda - z_j}, c_j \in \mathbb{C}, z_j \in \mathbb{C}_-\Big\}.$$

Then, $\mathscr{L}$ is dense in $L^2(\mathbb{R})$, and the assumption implies

$$\lim_{n\to\infty} \int_{-\infty}^\infty f_n(\lambda)\overline{\varphi(\lambda)} \mathrm{d}\lambda = \int_{-\infty}^\infty f(\lambda)\overline{\varphi(\lambda)} \mathrm{d}\lambda$$

for any $\varphi \in \mathscr{L}$. This combined with (81) yields the weak convergence of $\{f_n\}$ to f. The weak convergence of $\{S(f_n)\}$ is immediate from (80). □

We discuss one property about reflectionlessness of (43). For a Herglotz function m one has $\operatorname{Im} \log m = \arg m \in (0, \pi)$, hence $\log m$ is a Herglotz function. Since $\operatorname{Im} \log m = \arg m$ is uniformly bounded, in the expression (76) for $\log m$, we do not have the β-term, hence

$$\frac{\log m(z) - \log m(i)}{z - i} = \frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{\lambda - z} \frac{\arg m(\lambda)}{\lambda - i} \mathrm{d}\lambda. \tag{82}$$

Lemma 45 Suppose two sequences of Herglotz functions $m_n^\pm$ are reflectionless on A, and $m_n^\pm$ converge to Herglotz functions $m^\pm$ respectively. Then, $m^\pm$ are reflectionless on A as well.

Proof Set

$$f_n(\lambda) = \frac{1}{\pi} \frac{\arg m_n(\lambda)}{\lambda - i}.$$

Then, $\{f_n\}$ satisfies the two conditions in Lemma 44, and the identity (82) shows the weak convergence of

$$\frac{\log m_n(\lambda + i0) - \log m_n(i)}{\lambda - i}$$

in $L^2(\mathbb{R})$, which implies that of $(\log m_n(\lambda+i0))/(\lambda-i)$, since $\log m_n(i)\to\log m(i)$. Since the reflectionless property of $m_n^{\pm}$ on A is equivalent to

$$\log m_n^+(\lambda+i0)=\overline{\log m_n^-(\lambda+i0)}+\pi i \qquad \text{for a.e. } \lambda\in A, \tag{83}$$

and the weak convergence preserves the property (83), the proof is complete. □

C Inverse Spectral Problem

The next theme is the **inverse spectral problem**, which was developed by Gelfand-Levitan and Marchenko. We defined m_+ from a potential V (on $[0,\infty)$) through the operator $H=-\Delta+V$. The inverse spectral problem considers the problem of recovering V from m_+. Historically this problem has its origin in the classical **moment problem**, that is, the correspondence between a measure σ on $\mathbb{R}$ and its moments $\{m_k\}_{k\ge 0}$

$$m_k=\int_{\mathbb{R}}\lambda^k\sigma(\mathrm{d}\lambda).$$

We explain the inverse spectral problem in discrete operators, which is related to the moment problem. In the discrete case let H_+ be

$$(H_+u)(x)=\begin{cases}\displaystyle\sum_{y\in\mathbb{Z}:|x-y|=1}u(y)+V(x)u(x), & \text{if } x\ge 2;\\ u(2)+V(1)u(1), & \text{if } x=1.\end{cases}$$

Since H_+ is a self-adjoint operator on $\ell^2(\mathbb{Z}_+)$, the inverse $(H_+-z)^{-1}$ exists as a bounded operator for $z\in\mathbb{C}\backslash\mathbb{R}$. Set $f=(H_+-z)^{-1}\delta_1\in\ell^2(\mathbb{Z}_+)$. Then f satisfies

$$\begin{cases}\displaystyle\sum_{y\in\mathbb{Z}:|x-y|=1}f(y)+V(x)f(x)=zf(x)+\delta_1(x)=zf(x) & \text{for } x\ge 2,\\ f(2)+V(1)f(1)=1+zf(1).\end{cases}$$

Extend f to $\mathbb{Z}_-\cup\{0\}$ so that f satisfies

$$\sum_{y\in\mathbb{Z}:|x-y|=1}f(y)+V(x)f(x)=zf(x)$$

for all $x\in\mathbb{Z}$. For instance $f(0)$ should be the value determined by solving

$$f(2)+f(0)+V(1)f(1)=zf(1),$$

which means

$$f(0)=1.$$

Then $f_+(x,z) = f(x)/f(0)$ and

$$m_+(z) = \frac{f(1)-f(0)}{f(0)} = f(1) - 1 = ((H_+ - z)^{-1}\delta_1, \delta_1) - 1.$$

Here note

$$((H_+ - z)^{-1}\delta_1, \delta_1) = -z^{-1}\sum_{k=0}^{\infty}((z^{-1}H_+)^k\delta_1, \delta_1) = -\sum_{k=0}^{\infty} z^{-k-1}(H_+^k\delta_1, \delta_1)$$

for z such that $|z| > \|H_+\|$. Therefore, knowing $m_+(z)$ is equivalent to knowing $(H_+^k\delta_1, \delta_1)$ for all $k \geq 0$. From

$$(H_+\delta_1, \delta_1) = V(1)$$

one can know $V(1)$. Assume one can know $\{V(x)\}_{1\leq x\leq n}$ from $\{(H_+^k\delta_1, \delta_1)\}_{1\leq k\leq n}$. Since

$$(H_+^{n+1}\delta_1, \delta_1) = V(n+1) + \text{a function of } \{V(x)\}_{1\leq x\leq n},$$

one can know $V(n)$, which shows that $m_+(z)$ can recover completely $\{V(x)\}_{x\geq 1}$.

In the continuous case the problem was not simple and we had to wait for the works by Gelfand-Levitan and Marchenko. We give the lemma without proof. For the proof refer to [29].

Lemma 46 Suppose a potential V is bounded. Then, $V|_{\mathbb{R}_+}$ can be completely recovered from m_+.

The boundedness of V is unessential. However, for a general $V \in L^1_{\text{loc}}(\mathbb{R}_+)$, if the boundary is of limit circle type, one has to impose a boundary condition to define m_+. From this m_+ one can recover $V|_{\mathbb{R}_+}$ as well as the boundary condition at ∞.

D Oseledec Theorem (Deterministic Version)

Suppose $\{A(n)\}_{n\geq 1}$ be a family of invertible $l\times l$ matrices satisfying

$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{n}\log\|A(n)\| = 0 \\ \lim\limits_{n\to\infty} \dfrac{1}{n}\log\|A(n)^{-1}\| = 0 \end{cases} \tag{84}$$

and set

$$T(n) = A(n)A(n-1)\cdots A(1).$$

Let $\{\mu_1(n) \leq \mu_2(n) \leq \cdots \leq \mu_l(n)\}$ be the eigenvalues of $(T(n)^*T(n))^{1/2}$ and assume

$$\lim_{n\to\infty}\frac{1}{n}\log\mu_i(n) = \gamma_i \qquad \text{for } i = 1, 2, \ldots, l \tag{85}$$

exists. $\{\gamma_i\}_{1\le i\le l}$ are called **Lyapounov exponents** for $\{T(n)\}_{n\ge 1}$ and let $\{\widehat{\gamma}_i\}_{1\le i\le r}$ be the distinct set of $\{\gamma_i\}_{1\le i\le l}$ arranged so that $\gamma_1<\gamma_2<\cdots<\gamma_r$. Set

$$\begin{aligned}\Gamma_i &= \{1\le j\le l; \gamma_j=\widehat{\gamma}_i\}, \qquad 1\le i\le r,\\ k_i &= \#\Gamma_i.\end{aligned}$$

Denoting by $f_i(n)$ the normalized eigenvector for $(T(n)^*T(n))^{1/2}$ corresponding to $\mu_i(n)$, we define

$$L_i(n)=\mathrm{span}\{f_j(n); j\in\Gamma_i\}.$$

Denote by P_L the orthogonal projection to a subspace L.

Theorem 47 We assume the conditions (84) and (85).

(i) For $1\le i\le r$ there exists a subspace L_i such that the following limit exists.

$$\lim_{n\to\infty} P_{L_i(n)}=P_{L_i}.$$

(ii) $\{L_i\}_{1\le i\le r}$ are orthogonal and $\dim L_i=k_i$. Moreover we have

$$(T(n)^*T(n))^{1/2n}\to\sum_{i=1}^r k_i \mathrm{e}^{\gamma_i}P_{L_i}.$$

Set

$$V_i=L_1+L_2+\cdots+L_i, \qquad V_0=\{0\}.$$

(iii) For $f\in V_i\backslash V_{i-1}$ it holds

$$\lim_{n\to\infty}\frac{1}{n}\log\|T(n)f\|=\gamma_i.$$

Proof First note

$$\begin{aligned}(T(n+1)f_i(n+1),T(n+1)f_j(n)) &= (T(n+1)^*T(n+1)f_i(n+1),f_j(n))\\ &=\mu_i(n+1)^2(f_i(n+1),f_j(n)).\end{aligned}$$

Then, observing

$$\|T(n)f_i(n)\|=(T(n)f_i(n),T(n)f_i(n))^{1/2}=(T(n)^*T(n)f_i(n),f_i(n))^{1/2}=\mu_i(n),$$

we see

$$\begin{aligned}|(T(n+1)f_i(n+1),T(n+1)f_j(n))| &\le \|T(n+1)f_i(n+1)\|\|T(n+1)f_j(n)\|\\ &\le \mu_i(n+1)\|T(n+1)f_j(n)\|\\ &\le \mu_i(n+1)\|A(n+1)\|\|T(n)f_j(n)\|\\ &=\mu_i(n+1)\mu_j(n)\|A(n+1)\|.\end{aligned}$$

Hence we have

$$|(f_i(n+1), f_j(n))| \leq \frac{\mu_j(n)}{\mu_i(n+1)} \|A(n+1)\|.$$

Similarly if we start from $(T(n)f_i(n+1), T(n)f_j(n))$, we have an estimate

$$|(f_i(n+1), f_j(n))| \leq \frac{\mu_i(n+1)}{\mu_j(n)} \|A(n+1)^{-1}\|.$$

From the conditions (84) and (85), it follows that there exist $\delta_{\pm}(n)$ satisfying $\delta_{\pm}(n) \to 0$ as $n \to \infty$ and

$$\begin{cases} \dfrac{\mu_j(n)}{\mu_i(n+1)} \|A(n+1)\| = \mathrm{e}^{(\gamma_j - \gamma_i + \delta_+(n))n}, \\ \dfrac{\mu_i(n+1)}{\mu_j(n)} \|A(n+1)^{-1}\| = \mathrm{e}^{(\gamma_i - \gamma_j + \delta_-(n))n}. \end{cases}$$

Therefore if $i \notin \Gamma_j$, then with $\delta(n) \to 0$

$$|(f_i(n+1), f_j(n))| \leq \mathrm{e}^{-(|\gamma_i - \gamma_j| + \delta(n))n} \tag{86}$$

holds. Setting

$$P_n(k) = P_{\sum\limits_{k \leq i \leq r} L_i(n)}, \qquad Q_n(k) = P_{\sum\limits_{1 \leq i \leq k} L_i(n)},$$

we see from (86)

$$\|P_{n+1}(k')Q_n(k)\| + \|P_n(k')Q_{n+1}(k)\| \leq \mathrm{e}^{-(\gamma_{k'} - \gamma_k + \delta(n))n}$$

if $k' > k$. Observe

$$\begin{aligned} &\|P_{n+h}(k+1)Q_n(k)\| \\ =\ &\|P_{n+h}(k+1)P_{n+h-1}(k+1)Q_n(k) + P_{n+h}(k+1)Q_{n+h-1}(k)Q_n(k)\| \\ \leq\ &\|P_{n+h-1}(k+1)Q_n(k)\| + \|P_{n+h}(k+1)Q_{n+h-1}(k)Q_n(k)\| \\ \leq\ &\|P_{n+h-1}(k+1)Q_n(k)\| + \mathrm{e}^{-(\gamma_{k+1} - \gamma_k + \delta(n))(n+h-1)}. \end{aligned}$$

Repeating this argument until n, we have

$$\|P_{n+h}(k+1)Q_n(k)\| \leq \sum_{j=0}^{h-1} \mathrm{e}^{-(\gamma_{k+1} - \gamma_k + \delta(n))(n+j)} \leq \mathrm{e}^{-(\gamma_{k+1} - \gamma_k + \delta(n))n}$$

with a different $\delta(n)$. Similarly we see

$$\begin{aligned} &\|P_{n+h}(k+2)Q_n(k)\| \\ \leq\ &\sum_{j=0}^{h-1} \mathrm{e}^{-(\gamma_{k+2} - \gamma_k + \delta(n))(n+j)} + \sum_{j=0}^{h-1} \mathrm{e}^{-(\gamma_{k+2} - \gamma_{k+1} + \delta(n))(n+j)} \mathrm{e}^{-(\gamma_{k+1} - \gamma_k + \delta(n))n} \\ \leq\ &\mathrm{e}^{-(\gamma_{k+2} - \gamma_k + \delta(n))n}. \end{aligned}$$

In general it holds if $k' > k$

$$\|P_{n+h}(k')Q_n(k)\| + \|Q_{n+h}(k'-1)P_n(k+1)\| \le \mathrm{e}^{-(\gamma_{k'}-\gamma_k+\delta(n))n} \tag{87}$$

for any $h > 0$. Now we see

$$\begin{aligned}\|P_{n+h}(k) - P_n(k)\| &\le \|(P_{n+h}(k) - P_n(k))P_n(k)\| + \|(P_{n+h}(k) - P_n(k))Q_n(k-1)\| \\ &= \|Q_{n+h}(k-1)P_n(k)\| + \|P_{n+h}(k)Q_n(k-1)\| \\ &\le 2\mathrm{e}^{-(\gamma_{k+1}-\gamma_k+\delta(n))n},\end{aligned}$$

which implies $P_n(k)$ converges as $n \to \infty$ for any k. Therefore, each $P_{L_i(n)}$ converges and we denote the limit by $P_i = P_{L_i}$. Letting $h \to \infty$ in (87) we have

$$\Big\|\Big(\sum_{i\ge k'} P_{L_i}\Big)\Big(\sum_{i\le k} P_{L_i(n)}\Big)\Big\| + \Big\|\Big(\sum_{i\ge k'} P_{L_i(n)}\Big)\Big(\sum_{i\le k} P_{L_i}\Big)\Big\| \le \mathrm{e}^{-(\gamma_{k'}-\gamma_k+\delta(n))n}.$$

Then

$$\begin{aligned}\Big\|\Big(\sum_{i\ge k'} P_{L_i}\Big)P_{L_k(n)}\Big\| &= \Big\|\Big(\sum_{i\ge k'} P_{L_i}\Big)\Big(\sum_{i\le k} P_{L_i(n)} - \sum_{i\le k-1} P_{L_i(n)}\Big)\Big\| \\ &\le \Big\|\Big(\sum_{i\ge k'} P_{L_i}\Big)\Big(\sum_{i\le k} P_{L_i(n)}\Big)\Big\| + \Big\|\Big(\sum_{i\ge k'} P_{L_i}\Big)\Big(\sum_{i\le k-1} P_{L_i(n)}\Big)\Big\| \\ &\le \mathrm{e}^{-(\gamma_{k'}-\gamma_k+\delta(n))n} + \mathrm{e}^{-(\gamma_{k'}-\gamma_{k-1}+\delta(n))n} \le 2\mathrm{e}^{-(\gamma_{k'}-\gamma_k+\delta(n))n}.\end{aligned}$$

Similarly we have

$$\Big\|P_{L_{k'}(n)}\Big(\sum_{i\le k} P_{L_i}\Big)\Big\| \le 2\mathrm{e}^{-(\gamma_{k'}-\gamma_k+\delta(n))n}. \tag{88}$$

Then from (88) it follows

$$\begin{aligned}\Big\|T(n)\Big(\sum_{j\le i} P_{L_j}\Big)\Big\| &\le \Big\|T(n)\Big(\sum_{j\ge i+1} P_{L_j(n)}\Big)\Big(\sum_{j\le i} P_{L_j}\Big)\Big\| + \Big\|T(n)\Big(\sum_{j\le i} P_{L_j(n)}\Big)\Big(\sum_{j\le i} P_{L_j}\Big)\Big\| \\ &\le \sum_{j\ge i+1} \mu_j(n)\Big\|P_{L_j(n)}\Big(\sum_{j\le i} P_{L_j}\Big)\Big\| + \sum_{j\le i} \mu_j(n)\Big\|P_{L_j(n)}\Big(\sum_{j\le i} P_{L_j}\Big)\Big\| \\ &\le 2\sum_{j\ge i+1} \mathrm{e}^{(\gamma_j+\delta(n))n}\mathrm{e}^{-(\gamma_j-\gamma_i+\delta(n))n} + \sum_{j\le i} \mathrm{e}^{(\gamma_j+\delta(n))n},\end{aligned}$$

hence

$$\limsup_{n\to\infty} \frac{1}{n}\log\Big\|T(n)\Big(\sum_{j\le i} P_{L_j}\Big)\Big\| \le \gamma_i.$$

On the other hand we see if $f \in V_i \backslash V_{i-1}$

$$\begin{aligned}\|T(n)f\|^2 &= \sum_{j\ge i+1} \mu_j(n)^2\Big\|P_{L_j(n)}\Big(\sum_{j\le i} P_{L_j}\Big)f\Big\|^2 + \sum_{j\le i} \mu_j(n)^2\Big\|P_{L_j(n)}\Big(\sum_{j\le i} P_{L_j}\Big)f\Big\|^2 \\ &\ge \mu_i(n)^2\Big\|P_{L_i(n)}\Big(\sum_{j\le i} P_{L_j}\Big)f\Big\|^2,\end{aligned}$$

hence

$$\liminf_{n\to\infty}\frac{1}{n}\log\|T(n)f\| \geq \gamma_i + \liminf_{n\to\infty}\frac{1}{n}\log\Big\|P_{L_i(n)}\sum_{j\leq i}P_{L_j}f\Big\|$$
$$= \gamma_i + \liminf_{n\to\infty}\frac{1}{n}\log\Big\|P_{L_i}\sum_{j\leq i}P_{L_j}f\Big\| = \gamma_i,$$

since $P_{L_i}\sum_{j\leq i}P_{L_j}f = P_{L_i}f \neq 0$, which concludes the proof. □

E L^2-Convergence of Lyapounov Exponent

In this section we prove the L^2-convergence of Lyapounov exponents for uniquely ergodic potentials from the locally uniform convergence of IDS. The key idea, which goes back to Johnson-Moser [15], is to observe that Lyapounov exponent and IDS are the real part and the imaginary part of a certain analytic function on $\mathbb{C}_+$.

For $z\in\mathbb{C}$ let $\psi(x,z)$ be a unique solution to

$$-u''(x)+V(x)u(x)=zu(x), \qquad u(0)=0, \qquad u'(0)=1,$$

and set

$$f_x(\kappa)=\psi'(x,\kappa^2)-i\kappa\psi(x,\kappa^2).$$

Then, f_x is an entire function of κ for each fixed x satisfying

$$f_x(\kappa)=\overline{f_x(-\overline{\kappa})}. \tag{89}$$

f_x is an analog of $\mathrm{e}^{-ix\kappa}$ in the case $V\equiv 0$, $\phi=0$.

Lemma 48 If $V(x)\geq 0$, then $f_x(\kappa)\neq 0$ on $\mathbb{C}_+\cup\mathbb{R}$.

Proof If $\kappa\in\mathbb{R}\backslash\{0\}$, then

$$f_x(\kappa)=0 \Longrightarrow \psi'(x,\kappa^2)=\psi(x,\kappa^2)=0,$$

which is impossible. For $z\in\mathbb{C}$ the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Im}\big(\psi(x,z)\overline{\psi'(x,z)}\big) = -(\mathrm{Im}\,z)|\psi(x,z)|^2, \qquad \mathrm{Im}\big(\psi(0,z)\overline{\psi'(0,z)}\big)=0$$

shows that for any $x>0$

$$\begin{cases}\mathrm{Im}\big(\psi(x,z)\overline{\psi'(x,z)}\big)<0 & \text{if } z\in\mathbb{C}_+;\\ \mathrm{Im}\big(\psi(x,z)\overline{\psi'(x,z)}\big)>0 & \text{if } z\in\mathbb{C}_-.\end{cases}$$

Therefore, for $\kappa \in \mathbb{C}_+$ such that $\mathrm{Re}\,\kappa > 0$, we have $\mathrm{Im}(\psi'(x,\kappa^2)/\psi(x,\kappa^2)) < 0$. Since

$$f_x(\kappa) = 0 \Longrightarrow \frac{\psi'(x,\kappa^2)}{\psi(x,\kappa^2)} = i\kappa,$$

which is impossible. hence $f_x(\kappa) \neq 0$. For $\kappa \in \mathbb{C}_+$ such that $\mathrm{Re}\,\kappa < 0$, the property $f_x(\kappa) \neq 0$ follows from (89). If $\kappa \in \mathbb{C}_+ \cup \mathbb{R}$ satisfies $\mathrm{Re}\,\kappa = 0$. Then $\kappa = i\alpha$ with $\alpha \geq 0$ and

$$f_x(\kappa) = 0 \Longleftrightarrow \psi'(x,-\alpha^2) + \alpha\psi(x,-\alpha^2) = 0.$$

However, $\psi(x,-\alpha^2)$ is determined by an integral equation:

$$\psi(x,-\alpha^2) = x + \int_0^x (x-y)\psi(y,-\alpha^2)(V(y)+\alpha^2)\mathrm{d}y.$$

Hence, $\psi(x,-\alpha^2) \geq 0$ and $\psi'(x,-\alpha^2) \geq 1$ hold, since $V \geq 0$, which shows $f_x(\kappa) \neq 0$. $\square$

Then one can define

$$\log f_x(\kappa) = \int_0^x \frac{\partial_y f_y(\kappa)}{f_y(\kappa)}\mathrm{d}y.$$

Lemma 49 $\log f_x(\kappa)$ satisfies

$$\sup_{\substack{\kappa\in\mathbb{C}_+\cup\mathbb{R}\\ x\in[0,L]}} |\log f_x(\kappa) + i\kappa x| < \infty$$

for each $L > 0$.

Proof First note there exists a C^1 function $K(x,y)$ such that

$$\psi(x,\kappa^2) = \frac{\sin\kappa x}{\kappa} + \int_0^x K(x,y)\frac{\sin\kappa y}{\kappa}\mathrm{d}y.$$

For the proof see Marchenko [29]. Then, an integration by part implies

$$\begin{cases} \psi(x,\kappa^2) = \dfrac{\sin\kappa x}{\kappa} + O\big(\kappa^{-2}\mathrm{e}^{-i\kappa x}\big), \\ \psi'(x,\kappa^2) = \cos\kappa x + O\big(\kappa^{-1}\mathrm{e}^{-i\kappa x}\big). \end{cases}$$

Therefore

$$\delta_x(\kappa) \equiv \mathrm{e}^{i\kappa x} f_x(\kappa) = 1 + O(\kappa^{-1}).$$

Define log on the unit disc $\mathbb{D} = \{|z-1| < 1\}$ with $\log 1 = 0$. Then, for sufficiently large κ, we have

$$\log f_x(\kappa) = -i\kappa x + 2\pi i n(x) + \log\delta_x(\kappa)$$

for some $\mathbb{Z}$-valued function $n(x)$. Since $n(x)$ should be continuous and $n(0) = 0$, we see $n(x) = 0$ identically. Clearly

$$\log \delta_x(\kappa) = O(\kappa^{-1})$$

holds, which completes the proof. $\square$

Remark 50 Without the condition $V(x) \geq 0$, $f_x(\kappa)$ may have zeroes on $i\mathbb{R}_+$, but at most finitely many.

Lemma 49 implies $\log f_x(\kappa) + i\kappa x$ is an element of the Hardy space $H^2(\mathbb{R})$. Therefore, its real part and imaginary part are unitarily related by Hilbert transform. To investigate the real part we estimate $\mathrm{Im}\, f_x(\kappa)$ for $\kappa \in \mathbb{R}$. Set

$$\psi'(x,\kappa^2) - i\kappa\psi(x,\kappa^2) = r\mathrm{e}^{-i\theta},$$

namely $\theta = -\mathrm{Im}\, f_x(\kappa)$. Then, θ satisfies

$$\theta' = \kappa - \frac{1}{\kappa}V(x)\sin^2\theta, \qquad \theta(0) = 0. \tag{90}$$

Now let $\{T_x\}_{x\in\mathbb{R}}$ be a continuous flow on a compact metric space Ω. Assume the flow is uniquely ergodic, namely the flow has a unique invariant probability measure μ. Then our probability space is $(\Omega, \mathscr{B}(\Omega), \mu)$. One can show the following

Lemma 51 Let $V(\omega)$ be a nonnegative continuous function on Ω, and $\theta_\kappa(x,\omega)$ be the solution to (90) with $V(x) = V(T_x\omega)$ for each fixed $\omega \in \Omega$.

(i) There exists a constant C depending only on $M = \max|V(\omega)|$ such that

$$\Big| -\frac{1}{x}\theta_\kappa(x,\omega) + \kappa\Big| \leq \frac{C}{1+|\kappa|}.$$

(ii) Uniformly on $(\kappa,\omega) \in I \times \Omega$ (I is any compact interval)

$$\lim_{x\to\infty}\frac{1}{x}\theta_\kappa(x,\omega) = \pi N(\kappa^2).$$

Proof From (90)

$$\Big| -\frac{1}{x}\theta_\kappa(x,\omega) + \kappa\Big| \leq \frac{M}{|\kappa|}$$

follows. On the other hand, since zeroes of $\sin^2\theta$ are $\{n\pi\}$, $\theta_\kappa(x,\omega)$ never return to an interval $[n\pi,(n+1)\pi]$ once it leaves the interval, we have a bound

$$|\theta_\kappa(a,\omega)| \leq \pi \times \#\{y \in [0,a]; \theta_\kappa(y,\omega) = 0\}.$$

Moreover, Sturm's comparison theorem implies that this right hand side number differs from the number of eigenvalues of

$$-u'' + V(T_x\omega)u = \kappa^2 u, \qquad u(0) = u(a) = 0$$

by ± 1. Then, comparison of the eigenvalues with those of the same eigenvalues with constant potential yields a bound

$$|\theta_\kappa(a,\omega)| \le aC, \qquad \text{if } |\kappa| \le 1.$$

Thus one can prove the statement (i).

(ii) is essentially Theorem 4.5 of Johnson-Moser [15], although they proved it when the potential V is almost periodic and their definition of θ is

$$\theta = \operatorname{Im} \log(\psi'(x,\kappa^2) + i\psi(x,\kappa^2)).$$

We omit the proof. $\square$

Our theorem is as follows:

Theorem 52 Let $V(\omega)$ be a continuous function on Ω with minimum c. Then, for any fixed $\omega \in \Omega$

$$\frac{1}{x}\log(\psi'(x,\kappa^2+c) - i\kappa\psi(x,\kappa^2+c)) + i\kappa \underset{x\to\infty}{\to} i\kappa - w(\kappa^2+c)$$

in $H^2(\mathbb{R})$ holds. Especially we have

$$\lim_{x\to\infty}\int_c^\infty \Big|\frac{1}{x}\log\sqrt{\psi'(x,\lambda)^2 + (\lambda-c)\psi(x,\lambda)^2} - \gamma(\lambda)\Big|^2 \frac{\mathrm{d}\lambda}{\sqrt{\lambda-c}} = 0.$$

Proof We can assume $V(\omega) \ge 0$ by replacing V with $V-c$. For any fixed $x>0$, Lemma 49 shows

$$\frac{1}{x}\log(\psi'(x,\kappa^2) - i\kappa\psi(x,\kappa^2)) + i\kappa \in H^2(\mathbb{R}).$$

Since Lemma 51 implies

$$\lim_{x\to\infty} \operatorname{Im}\Big\{\frac{1}{x}\log(\psi'(x,\kappa^2) - i\kappa\psi(x,\kappa^2)) + i\kappa\Big\} = \lim_{x\to\infty}\Big(\kappa - \frac{1}{x}\theta_\kappa(x,\omega)\Big) = \kappa - \pi N(\kappa^2)$$

in $L^2(\mathbb{R})$, from the unitarity of Hilbert transform it follows that

$$\frac{1}{x}\log\sqrt{\psi'(x,\kappa^2)^2 + \kappa^2\psi(x,\kappa^2)^2} = \operatorname{Re}\Big\{\frac{1}{x}\log(\psi'(x,\kappa^2) - i\kappa\psi(x,\kappa^2)) + i\kappa\Big\} \to \gamma(\kappa^2)$$

in $L^2(\mathbb{R})$, which concludes the proof. $\square$

These three lemmas are valid also for another solution $\varphi_z(x,z)$ to

$$-\varphi''(x) + V(x)\varphi(x) = z\varphi(x), \qquad \varphi(0)=1, \qquad \varphi'(0)=0.$$

Moreover, for $\lambda < c$ the identity

$$\psi(x,\lambda) = \frac{\exp\Big(\int_0^x m_+(\lambda, T_y\omega)\Big)\mathrm{d}y + \exp\Big(-\int_0^x m_-(\lambda, T_y\omega)\Big)\mathrm{d}y}{m_+(\lambda,\omega) + m_-(\lambda,\omega)}$$

has a meaning, and the unique ergodicity implies immediately

$$\lim_{x\to\infty}\frac{1}{x}\log\psi(x,\lambda)=\gamma(\lambda)$$

for each ω.

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遍历Schrödinger算子的谱问题

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本文介绍一维Schrödinger算子的谱理论. 主要讨论作者所得到的定理, 并给出了详细证明.

关键词: 遍历Schrödinger算子, 反射势, 综合态密度, Lyapounov指数.

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