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Optimization of Investment-Dividend Problem in a Diffusion Model with Transaction Costs and Investment Constraints *

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Abstract: This paper investigates the investment-dividend optimization problem for a corporation with transaction costs and investment constraints. The main feature is that we assume general constraints on investments including the special case of short-sale and borrowing constraints. This results in a regular-impulse stochastic control problem. The nontrivial case is that the investment can't meet the loss of wealth due to discounting. In this case, delicate analysis is carried out on QVI w.r.t. three possible situations, leading to an explicit construction of the value functions together with the optimal policies. We also give explicit conclusion of the trivial case at last.

Keywords: investment constraints; transaction costs; regular-impulse control; quasi-variational inequalities

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§1. Introduction

In mathematical finance in the optimization of the continuous time models the issues of constraints on controls loom largely especially in the consumption/investment models. There exists an extensive literature on the constraint issue in the models of optimal

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consumption/investment behavior of a smaller investor, in particular *the investment constraints*. An excellent reference of investment constraints can be found in [1], which considers trading constraints, limited borrowing, and no bankruptcy binding. Other references are given in [2–4] and so on.

In the context of the investment/dividend control optimization, there have been several papers with deterministic investment strategy, which study optimal dynamic control of dividends [5,6]. In addition to dividends and investments, reinsurance or transaction cost can be allowed in the models [7]. However the problem of constraints on investments in the continuous insurance model has not been studied earnest, except [8] dealing with this problem for the diffusion model when no-borrowing constraint is considered. For other application of control theory in constrained insurance mathematics see [9] and [10] with solvency constraints, [11] with constraints on risk control.

Surprisingly if the risk is not controlled in the diffusion model, the problem of the investment/dividend control optimization become less meaningful, since the optimal amount of investments is zero. While if the constraints on investments are considered, this problem will be significantly more meaningful. In this paper, we investigate the linear diffusion process as an insurance corporation's reserve. We assume the corporation is a smaller investor in the Black-Scholes market, and does not pay transaction fees when trading, but there is a fixed cost each time the dividends are paid out. By the term *the investment constraints*, we consider the corporation's amount of investing in Stock is modeled by a control process $b_t \in [\min(\delta_1 X_t, c), \delta_2 X_t]$, $t \geqslant 0$. Here $0 < \delta_1 < \delta_2$, $c > 0$ are exogenous parameters. In this regard, short-sale and borrowing constraints, as well as incomplete markets, can be modeled as the limit case of $\delta_1 \to 0+$ and $\delta_2 = 1$ of the type of constraints we consider [8]. The constraints may include limits on the allocation to specific assets, the ability to access certain funds and even prohibitions on certain investments. The mathematical setup results in a mixed regular-impulse stochastic control problem, in which one maximizes the expected total discounted dividend distribution prior to the bankruptcy time.

In those earlier works, in contrast to our study, either the investment control without constraints or the transaction cost is absence in distributing dividends. One difficulty in this study is to distinguish the different situations of marginal investing reserve state for the nontrivial case (the investment can't meet the loss of wealth due to discounting), since there exists explicit constraints on investments. We overcome this difficulty by defining an auxiliary set to determine the situations, which depends on the exogenous parameters in our model.

The paper is organized as follows. We formulate the problem in Section 2, and characterize the value function in Section 3, and also associate this to the quasi-variational inequalities (QVIs). For the nontrivial case, a solution of QVI is constructed w.r.t. three different situations as well as the construction of the optimal policies in Section 4. At last, we give a brief description of the conclusion for the trivial case.

§2. Problem Formulation

To give a mathematical formulation of the optimization problem treated in this paper, we start with a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, endowed with a filtration $\mathscr{F} = \{\mathscr{F}_t\}_{t\geqslant 0}$, and a standard Brownian motions ω^0 adapted to $\mathscr{F}$. The reserve process $R = \{R_t, t \geqslant 0\}$ represents the liquid assets of a corporation, according to

$$\mathrm{d}R_t = \mu \mathrm{d}t + \sigma \mathrm{d}\omega_t^0,$$

with $\mu, \sigma > 0$.

Furthermore we have a classical Black-Scholes market, that is we have a risk free asset whose price process is governed by

$$\mathrm{d}P_t^0 = r_0 P_t^0 \mathrm{d}t,$$

and a risky asset whose price process is governed by

$$\mathrm{d}P_t^1 = r_1 P_t^1 \mathrm{d}t + \sigma_P P_t^1 \mathrm{d}\omega_t^1,$$

where r_1, r_0, σ_P are positive and $\{\omega_t^1\}$ is a standard Brownian motion w.r.t. $\{\mathscr{F}_t\}$ and independent of $\{\omega_t^0\}$. We denote b_t the amount invested in the risky asset at time t. The distribution of dividends is described by a sequence of increasing stopping times $\{\tau_i; i = 1, 2, \ldots\}$ and a sequence of random variables $\{\xi_i; i = 1, 2, \ldots\}$, which are associated with times and amounts of dividends paid out to shareholders.

Thus the dynamics of the controlled reserve process is given by

$$sX_t = x + \int_0^t [\mu + r_0(X_s - b_s) + r_1 b_s]\mathrm{d}s + \int_0^t \sigma \mathrm{d}\omega_s^0 + \int_0^t b_s \sigma_P \mathrm{d}\omega_s^1 - \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} \xi_i, \quad (1)$$

where $X_0 = x > 0$ is the initial reserve.

Definition 1 For given scalars $0 < \delta_1 < \delta_2$, $c > 0$, a triple

$$\pi := (b, \mathscr{T}, \xi) = (b; \tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

is called an admissible control or an admissible policy if $b : \Omega \times [0, \infty) \mapsto [\min(\delta_1 x, c), \delta_2 x] = \mathscr{Q}_x$ is an $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted process, $0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$, a.s. is a sequence of increasing stopping times w.r.t. $\{\mathscr{F}_t\}_{t\geqslant 0}$, and random variables ξ_i, $i = 1, 2, \ldots$ is $\mathscr{F}_{\tau_i}$ measurable with $0 \leqslant \xi_i \leqslant X_{\tau_i -}$. The class of all admissible controls is denoted by $\mathscr{A}(x)$.

The time of bankruptcy is defined by

$$\tau \equiv \tau^{\pi} := \inf\{t \geqslant 0 : X_t = 0\}.$$

We only deal with the optimization problem during the time interval $[0, \tau)$, we shall assume that X_t vanishes for $t \geqslant \tau$. Since both ω_t^0, ω_t^1 have independent increments, it follows from [12] that X_t is a homogeneous strong Markov process, and X_t has the same distribution as $\widetilde{X}_t$, where

$$\widetilde{X}_t = \begin{cases} x + \displaystyle\int_0^t [\mu + r_0(\widetilde{X}_s - b_s) + r_1 b_s] \mathrm{d}s \\ \quad + \displaystyle\int_0^t \sqrt{\sigma^2 + b_s^2 \sigma_P^2}\, \mathrm{d}\omega_s - \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} \xi_i, & \text{if } t < \tau; \\ 0, & \text{if } t \geqslant \tau. \end{cases} \tag{2}$$

Here ω is a standard Brownian motion. We drop the tilde and write X_t for (1) in the remainder sections. We assume that the standard absence of speculative bubbles condition is met and consider only the reserve process with finite expected cumulative present values.

The corporation wants to find a control $\pi = (b, \mathscr{T}, \xi) \in \mathscr{A}$ that maximizes the performance function J defined by

$$J(x, \pi) := \mathsf{E}_x \Big[\sum_{i=1}^{\infty} \mathrm{e}^{-r\tau_i} g(\xi_i) I_{\{\tau_i < \tau\}} \Big],$$

which represents the expected present value of dividends received by the shareholder until the time of bankruptcy. In the performance function J, we assume the function $g : [0, \infty) \mapsto (-\infty, +\infty)$ is given by

$$g(\eta) := k\eta - K, \tag{3}$$

where $k \in (0, 1)$ and $K \in (0, \infty)$ are constants.

A few remarks on the control component b_t are in order. Taking investment constraints into account, it is still a limiting case of $\delta_1 \to 0+$ and $\delta_2 = 1$, though reinsurance is not considered in our study. It is certainly meaningful to relax this constraint to one with any arbitrary upper and lower bounds. For example, for the financial company case, $\delta_2 > 1$ would mean that the corporation can borrow money to invest in Stock. Furthermore, our formulation can model investment control problems for corporations with stochastic income other than insurance ones.

§3. Characterization of the Value Function

In this section, we will establish the quasi-variational inequalities associated with the stochastic control problem, and derive the properties of the value function.

Let us first denote the value function by V, for $x \geqslant 0$,

$$V(x) := \sup\{J(x,\pi); \pi \in \mathscr{A}(x)\} = \sup_{\pi \in \mathscr{A}(x)} \mathsf{E}_x \Big[\sum_{i=0}^{\infty} \mathrm{e}^{-r\tau_i} g(\xi_i) I_{\{\tau_i < \tau\}} \Big]. \tag{4}$$

Then the optimal control $\pi^* = (b^*, \mathscr{T}^*, \xi^*)$ is a policy satisfying

$$V(x) = J(x, \pi^*).$$

The main tools for solving this problem are dynamic programming and QVI, as well as relevant discussions [13]. To analyze the value function, we need the following lemma.

Lemma 2 Let Y_t be Itô's process on a positive half line,

$$Y_t = x + \int_0^t [\mu + r_0 Y_s + (r_1 - r_0) b_s] \mathrm{d}s + \int_0^t \sqrt{\sigma^2 + \sigma_P^2 b_s^2}\, \mathrm{d}\omega_s,$$

where $b_t \in [\min(\delta_1 Y_t, c), \delta_2 Y_t]$. Let $h > 0$ and $\zeta_h = \inf\{t \geqslant 0 : Y_t = h\}$. Then for any fixed $t > 0$,

$$\mathsf{P}(\zeta_0 < \zeta_h \wedge t) \to 1, \qquad \mathsf{E}_x \Big(\max_{0 \leqslant s \leqslant \zeta_0 \wedge t} Y_s \Big) \to 0, \tag{5}$$

as $x \downarrow 0$ uniformly over all the processes Y_t.

Proof of Lemma 2 see Appendix. Similarly to [11], we can show that the value function V has the following basic properties.

Proposition 3 The value function V is a continuous, nondecreasing function subject to

$$V(0+) = 0.$$

With Lemma 2, the proof of Proposition 3 is similar to Proposition 1 of [11], and we ignore it here. Then the following dynamic programming principle holds:

$$V(x) = \sup_{\pi \in \mathscr{A}} \Big[\mathsf{E}_x \Big(\sum_i I_{\{\tau_i < \tau \wedge \theta\}} g(\xi_i) \Big) + \mathsf{E}\, \mathrm{e}^{-r(\tau \wedge \theta)} V(X_{\tau \wedge \theta}) \Big]$$

for every $x \geqslant 0$ and $\mathscr{F}_t$-stopping time θ [13]. As in [14], we also define the net appreciation rate $\rho : R^+ \times \mathscr{Q}_x \mapsto R$ as

$$\rho(x, b) = \mu + r_0 x + (r_1 - r_0) b - rx.$$

Under our assumption that finite expected cumulative present values on X_t, the cash flow $\rho(X, b) := \{\rho(X_t)\}_{t \in [0,\infty)}$ has a finite expected cumulative present value. Based on this mapping with the value function V, an interesting result is summarized as follows.

Proposition 4 The value function V defined by (4) satisfies for every $x \in [0, \infty)$,

$$V(x) \leqslant k \Big[x + \sup_{\pi \in \mathscr{A}} \mathsf{E}_x \int_0^{\pi} \mathrm{e}^{-rs} \rho(X_s, b_s) \mathrm{d}s \Big]. \tag{6}$$

Proof Applying the generalized Itô theorem to the identity mapping $x \mapsto x$ yields

$$\mathsf{E}_x[\mathrm{e}^{-r\tau_N} X_{\tau_N}] = x + \mathsf{E}_x \int_0^{\tau_N} \mathrm{e}^{-rs}\rho(X_s, b_s)\mathrm{d}s - \mathsf{E}_x\Big[\sum_{\tau_i \leqslant \tau_N} \mathrm{e}^{-r\tau_i}\xi_i\Big],$$

where $\tau_N = N \wedge \tau \wedge \inf\{t \geqslant 0 : X_t \geqslant N\}$ is an increasing sequence of almost surely finite stopping times tending towards τ. Reordering terms, invoking the nonnegativity of the controlled jump-diffusion, and letting $N \mapsto \infty$ yields by dominated convergence theorem obviously

$$J(x, \pi) \leqslant k\mathsf{E}_x\Big[\sum_{i=0}^{\infty} \mathrm{e}^{-r\tau_i}\xi_i I_{\{\tau_i < \tau\}}\Big] \leqslant k\Big[x + \mathsf{E}_x \int_0^{\tau} \mathrm{e}^{-rs}\rho(X_s, b_s)\mathrm{d}s\Big].$$

The proposition follows from the above inequality. □

Remark 5 Similarly to the analysis in [14], the value of the optimal policy can grow at most at a linear rate for large reserve, if the net appreciation rate is bounded. Take $r_0 \leqslant r_1 \leqslant r$ for example, $\rho(x, b) \leqslant \mu + (r_0 - r)x + (r_1 - r_0)\delta_2 x \leqslant \mu$. Then $V(x) \leqslant k(x + \mu/r)$, that is, the maximal linear growth rate of the value function V is k in this case.

For a function $\phi : [0, \infty) \mapsto R$, we define the maximum utility operator M by

$$M\phi(x) := \sup\{\phi(x - \eta) + g(\eta) : x \geqslant \eta > 0\},$$

where g is given by (3).

Definition 6 A function $W : [0, \infty) \mapsto [0, \infty)$ satisfies the QVIs of the control problem if for every $b \in \mathscr{Q}_x$ and $x \in [0, \infty)$,

$$\mathscr{L}^b W(x) \leqslant 0, \tag{7}$$

$$W(x) \geqslant MW(x), \tag{8}$$

$$(W(x) - MW(x))\Big(\max_{b \in \mathscr{Q}_x} \mathscr{L}^b W(x)\Big) = 0, \tag{9}$$

$$W(0) = 0. \tag{10}$$

We observe that a solution W of the QVI separates the interval $(0, \infty)$ into two disjoint regions: a continuation region

$$\mathscr{C} := \Big\{x \in (0, \infty) : MW(x) < W(x) \text{ and } \max_{b \in \mathscr{Q}_x} \mathscr{L}^b W(x) = 0\Big\}$$

and an intervention region

$$\Sigma := \Big\{x \in (0, \infty) : MW(x) = W(x) \text{ and } \max_{b \in \mathscr{Q}_x} \mathscr{L}^b W(x) < 0\Big\}.$$

Given a solution to the QVI, we define the following policy associated with this solution.

Definition 7 The control $\pi^W = (b^W; \tau_1^W, \tau_2^W, \ldots, \tau_n^W, \ldots; \xi_1^W, \xi_2^W, \ldots, \xi_n^W, \ldots)$ is called the QVI control associated with W if the associated process X^W given by (2) satisfies

$$\begin{aligned}
&\mathsf{P}\Big\{b^W(t) \neq \arg\max_{b\in\mathscr{Q}_x} \mathscr{L}^W W(X_t^W), X_t^W \in \mathscr{C}\Big\} = 0,\\
&\tau_1^W := \inf\{t \geqslant 0 : W(X^W(t)) = MW(X^W(t))\},\\
&\xi_1^W := \arg \sup_{0<\eta\leqslant X^W(\tau_1^W)} \{W(X(\tau_1^W) - \eta) + g(\eta)\},
\end{aligned}$$

and for every $n \geqslant 2$,

$$\begin{aligned}
&\tau_n^W := \inf\{t \geqslant \tau_{n-1} : W(X^W(t)) = MW(X^W(t))\},\\
&\xi_n^W := \arg \sup_{0<\eta\leqslant X^W(\tau_n^W)} \{W(X(\tau_n^W) - \eta) + g(\eta)\},
\end{aligned}$$

with $\tau_0^W := 0$, $\xi_0^W := 0$.

Theorem 8 Let $W \in C^1(0,\infty)$ be a solution of the QVI (7)–(9). Suppose there exists $u > 0$ such that W is twice continuously differentiable on $(0, u)$ and W is linear on $[u,\infty)$. Then for $x \in (0,\infty)$,

$$V(x) \leqslant W(x).$$

Moreover, if the QVI control π^W associated with W is admissible, then W coincides with the value function and the QVI control associated with W is the optimal policy, i.e.,

$$V(x) = W(x) = J(x, \pi^W).$$

Proof The idea of proving this theorem is similar to Theorem 3.4 of [15]. □

§4. Construction of $W(x)$ for the Case of $r > r_1 > r_0$

For such a function W define

$$L := \inf\{x \geqslant 0 : MW(x) = W(x)\}. \tag{11}$$

Then from (9) follows:

$$\max_{b\in\mathscr{Q}_x} \Big\{\frac{1}{2}[\sigma^2 + \sigma_P^2 b^2]W''(x) + [\mu + r_0 x + (r_1 - r_0)b]W'(x) - rW(x)\Big\} = 0, \tag{12}$$

for $x < L$.

Define

$$m_p = \frac{r_1 - r_0}{\sigma_P^2}, \qquad \gamma_1 = \frac{(r_1 - r_0)m_p}{2}.$$

We call m_p the market price of financial market or risk premiums. Then by differentiation we can find the maximizing function

$$b(x) = -m_p \frac{W'(x)}{W''(x)}. \tag{13}$$

According to [8], $W(x)$ is given by

$$W(x) = k_1 \int_{x_1}^{x} \mathrm{e}^{-\int_{x_1}^{z} [m_p/b(y)]\mathrm{d}y} \mathrm{d}z + k_2, \tag{14}$$

where

$$b'(x) = \frac{(\gamma_1 + r - r_0)b^2(x) + (\mu + r_0 x)m_p b(x) - \sigma^2 m_p^2/2}{[(r_1 - r_0)/2] \cdot (\sigma^2/\sigma_P^2) + [(r_1 - r_0)/2] \cdot b^2(x)}, \tag{15}$$

and k_1, k_2 are free constants. Then the investment strategy satisfies

$$b(x_1+) = b(x_1-) = b_0, \tag{16}$$

and x_1, b_0 are not given a priori and have to be determined later.

Combining (15), (16), we get that $b'(0) < 0$, which contradicts the constraint on $b(0+)$. Then the investment strategy should satisfy $b(x) = \delta_1 x$ on interval $[0, x_0)$, where x_0 is determined later. In this case W must satisfy

$$\frac{1}{2}[\sigma^2 + \sigma_P^2 \delta_1^2 x^2]W''(x) + [\mu + (r_0 + (r_1 - r_0)\delta_1)x]W'(x) - rW(x) = 0. \tag{17}$$

In [5], it is shown that a solution of (17), (10) is

$$W(x) = k_3[E(x, \alpha(\delta_1) + 1, \delta_1) + T_1(\delta_1) D(x, \alpha(\delta_1) + 1, \delta_1)], \tag{18}$$

where k_3 is a free constant and

$$\begin{aligned}
&\lambda(x) = \frac{2(r_0 + (r_1 - r_0)x)}{\sigma_P^2 x^2}, \qquad T_1(x) = -\frac{E(0, \alpha(x) + 1, x)}{D(0, \alpha(x) + 1, x)}, \\
&\alpha(x) = \frac{1}{2}\Big(\sqrt{(\lambda(x) - 1)^2 + \frac{8r}{\sigma_P^2 x^2}} - (1 + \lambda(x))\Big), \\
&D(x, \nu, y) = \int_x^{\infty} (t - x)^{\nu} K(y, t)\mathrm{d}t, \qquad -1 < \nu < 1 + 2\alpha(y) + \lambda(y), \\
&E(x, \nu, y) = \int_{-\infty}^{x} (x - t)^{\nu} K(y, t)\mathrm{d}t, \qquad -1 < \nu < 1 + 2\alpha(y) + \lambda(y),
\end{aligned} \tag{19}$$

with

$$K(x, t) = \Big(\sigma_P^2 t^2 + \frac{\sigma^2}{x^2}\Big)^{-(\alpha(x)+1+\lambda(x))} \exp\Big\{-\frac{2\mu}{x\sigma\sigma_P} \arctan\Big(\frac{\sigma_P}{\sigma} xt\Big)\Big\}. \tag{20}$$

Definition 9 We assume that $b(x) = \delta_1 x = c$ at $x^* = c/\delta_1$, and $\widetilde{x}$ is the smallest positive solution of $b'(x) = 0$ in (15) and $b(x) \in \mathscr{Q}_x$.

By calculation, the existence of $\widetilde{x}$ follows from the fact that $0 < \widetilde{x} < x^*$ and $H(\delta_1\widetilde{x}, \widetilde{x}) \leqslant 0 \leqslant H(\delta_2\widetilde{x}, \widetilde{x})$, or $\widetilde{x} > x^*$ and $H(c, \widetilde{x}) \leqslant 0 \leqslant H(\delta_2\widetilde{x}, \widetilde{x})$, where

$$H(b(x), x) = (\gamma_1 + r - r_0)b^2(x) + (\mu + r_0 x)m_p b(x) - \frac{\sigma^2 m_p^2}{2}.$$

From now on, we assume that the parameters setting ensure the existence of $\widetilde{x}$.

Now we discuss some properties of $b(x)$ defined by (15) and (16), which will be needed later on. Similarly to [8], the following lemmas hold.

Lemma 10 The function $b(x)$ is strictly increasing on $(\widetilde{x}, \infty)$ with $b(\infty) = \infty$.

Proof First consider the special case $x = \widetilde{x}+$. Then we need to show that $b'(\widetilde{x}+) = \lim\limits_{\epsilon\to 0+} b'(x+\epsilon) > 0$. Since $b'(\widetilde{x}) = 0$, that is, $H(b(\widetilde{x}), \widetilde{x}) = 0$, $b(\widetilde{x}) > 0$ is obvious from the investment constraints. Then for arbitrary $\epsilon > 0$,

$$H(b(\widetilde{x}+\epsilon), \widetilde{x}+\epsilon) = r_0 m_p b(\widetilde{x})\epsilon + o(\epsilon) > 0.$$

The required positivity follows from the above inequality. In other words, $b'(x) > 0$ for all $x > \widetilde{x}$. The conclusion $b(\infty) = \infty$ is obtained from the strictly increasing property. □

Lemma 11 The equation $b(x) = \delta_2 x$ has a unique solution $x_2 \geqslant \widetilde{x}$ if $\delta_2 \leqslant 2(r - r_0)/(r_1 - r_0) + m_p$. Furthermore $b'(x) \geqslant \delta_2$ for all $x \geqslant x_2$.

Proof Similarly to the proof in [8], integrating both sides of (15) from $\widetilde{x}$ to x and divided by x, we have

$$\frac{b(x) - b(\widetilde{x})}{x} = \frac{1}{x}\int_{\widetilde{x}}^{x} \frac{H(b(y), y)}{[(r_1 - r_0)/2]\cdot(\sigma^2/\sigma_P^2) + [(r_1 - r_0)/2]\cdot b^2(y)}\mathrm{d}y.$$

We also take Cesaro averages and get that

$$\lim\sup_{x\to\infty}\frac{b(x)}{x} \geqslant 2\frac{\gamma_1 + r - r_0}{r_1 - r_0} + \frac{2m_p r_0}{r_1 - r_0}\lim\inf_{x\to\infty}\frac{x}{b(x)} \geqslant 2\frac{r - r_0}{r_1 - r_0} + m_p > 2,$$

for $r > \max(r_0, r_1)$. Since $b(\widetilde{x})/\widetilde{x} \leqslant \delta_2$ from the definition of $\widetilde{x}$, there exists $x_2 = \sup\{x \geqslant \widetilde{x} : b(x)/x_2 \leqslant \delta_2\}$.

Then there exist some (maybe all) $x \in (\widetilde{x}, x_2]$ such that $b'(x) \geqslant \delta_2$, which is from the fact that x_2 exists and is finite. We also have

$$H(b(x), x) - \delta_2\frac{r_1 - r_0}{2}\Big(\frac{\sigma^2}{\sigma_P^2} + b^2(x)\Big) \geqslant 0,$$

We can conclude that the left hand side of the above equality is an increasing function at x from $r > r_1 > r_0$ and $b'(x) \geqslant \delta_2$. The assertion $b'(y) \geqslant \delta_2$ holds for all $y \geqslant x$. □

For later use in this section, we assume $\delta_2 \leqslant 2(r-r_0)/(r_1-r_0)+m_p$. Due to (13) and (18), we definite a set $\mathscr{O}$ satisfying

$$\mathscr{O} = \Big\{x \in (\widetilde{x}, x^*) : \delta_1 x \leqslant -\frac{r_1-r_0}{\sigma_P^2 \alpha(\delta_1)} \frac{E(x,\alpha(\delta_1),\delta_1) - T_1 D(x,\alpha(\delta_1),\delta_1)}{E(x,\alpha(\delta_1)-1,\delta_1) + T_1 D(x,\alpha(\delta_1)-1,\delta_1)} \leqslant \delta_2 x\Big\}.$$

Therefore we must differentiate among three possible cases in the next subsections.

4.1 Case of $\mathscr{O} \neq \Phi$

In this case, define

$$\overline{x} = \inf\{x : x \in \mathscr{O}\}. \tag{21}$$

$\widetilde{x} < x^*$ is obvious, and $b'(x) > \delta_1$ holds for all $x > \overline{x}$ from Lemma 11. Then $b(x) = \delta_1 x$ must hold on interval $[0, \overline{x}]$ from the investment constraints. From the definition of $\overline{x}$, we have $b(x) \geqslant \delta_1 x$ on $[\overline{x}, L)$, and $x_0 = x_1 = \overline{x}$.

On $[\overline{x}, L)$, (14) satisfies and $b(\overline{x}) = b_0 = \delta_1 \overline{x}$. Referring to Lemmas 10 and 11, $b(\overline{x}) = \delta_1 \overline{x} < \delta_2 \overline{x}$ holds, then there definitely exists x_2 such that

$$b(x_2) = \delta_2 x_2, \tag{22}$$

where $b(x)$ is defined as (15). This results in that $b(x) = \delta_2 x$ on $[x_2, L)$, and (12) becomes

$$\frac{1}{2}[\sigma^2 + \sigma_P^2 \delta_2^2 x^2] W''(x) + [\mu + (r_0 + (r_1 - r_0)\delta_2)x] W'(x) - rW(x) = 0. \tag{23}$$

According to [8], we must find $u_1 > x_2$ and a solution W to (23) such that $W''(u_1) = 0$, $W'(u_1) = 1$, which ensures a twice continuously differentiable solution. We will first give a lemma on the property of function $W(x)$.

Lemma 12 There exist constants $C_1 > 0$, k_1, k_2, k_3 and $u_1 > x_2$ and a twice continuously differentiable function W_1 such that $W_1(x)$ is given by (18), (14) for $x < x_2$ and satisfies (23) for $x_2 < u_1$ with $W_1''(u_1) = 0$, and $W_1'(u_1) = 1$.

Proof of Lemma 12 is similar to Lemma 4.4 of [8], and we omit it here. Then $W_1(x)$ is given similarly as (18) for $x \in [x_2, L)$,

$$W_1(x) = k_4 E(x, \alpha(\delta_2) + 1, \delta_2) + k_5 D(x, \alpha(\delta_2) + 1, \delta_2),$$

where $E(x, \nu, y)$, $D(x, \nu, y)$, $\alpha(y)$ are defined as (19), (20). Continuity of the function W_1 and its derivative W_1' at the point $\overline{x}$, x_2 implies that we can write the solution to (11), (12) in the following form

$$W_1(x) = \begin{cases} C_1 k_3^1 [T_1(\delta_1) D(x, \alpha(\delta_1)+1, \delta_1) + E(x, \alpha(\delta_1)+1, \delta_1)], & 0 \leqslant x < \overline{x}; \\ C_1 k_3^1 \Big[T_3(\overline{x}) \displaystyle\int_{\overline{x}}^{x} \mathrm{e}^{-\int_{\overline{x}}^{z} [m_p/b(y)] \mathrm{d}y} \mathrm{d}z + T_2(\overline{x})\Big], & \overline{x} \leqslant x < x_2; \\ C_1 [T_4 D(x, \alpha(\delta_2)+1, \delta_2) + E(x, \alpha(\delta_2)+1, \delta_2)], & x_2 \leqslant x < L, \end{cases} \tag{24}$$

where $\overline{x}$ is given by (21), x_2 by (22), $T_1(\delta_1)$ by (19), and

$$\begin{aligned}
T_2(x) &= T_1(\delta_1)D(x,\alpha(\delta_1)+1,\delta_1)+E(x,\alpha(\delta_1)+1,\delta_1),\\
T_3(x) &= (\alpha(\delta_1)+1)(-T_1(\delta_1)D(x,\alpha(\delta_1),\delta_1)+E(x,\alpha(\delta_1),\delta_1)),\\
T_4 &= \frac{E(x_2,\alpha(\delta_2),\delta_2)+[\alpha(\delta_2)\delta_2x_2/m_p]E(x_2,\alpha(\delta_2)-1,\delta_2)}{D(x_2,\alpha(\delta_2),\delta_2)-[\alpha(\delta_2)\delta_2x_2/m_p]D(x_2,\alpha(\delta_2)-1,\delta_2)},\\
k_3^1 &= \frac{-T_4D(x_2,\alpha(\delta_2),\delta_2)+E(x_2,\alpha(\delta_2),\delta_2)}{-T_1(\delta_1)D(x^*,\alpha(\delta_1),\delta_1)+E(x^*,\alpha(\delta_1),\delta_1)}\mathrm{e}^{\int_{\overline{x}}^{x_2}[m_p/b(y)]\mathrm{d}y}.
\end{aligned}$$

The corresponding optimal control b_1^* is given by

$$b_1^*(t)=\begin{cases}\delta_1X_t, & 0\leqslant X_t<\overline{x};\\ b(X_t), & \overline{x}\leqslant X_t<x_2;\\ \delta_2X_t, & x_2\leqslant X_t\leqslant L.\end{cases}\tag{25}$$

4.2 Case of $\mathscr{O}=\Phi$ and $x^*<\widetilde{x}$

In this case $b(x)=\delta_1x$ on interval $[0,x^*)$, and $x_0=x^*=c/\delta_1$, (17) holds. Next, we consider the value function on $[x^*,L)$.

With $x^*<\widetilde{x}$ and the definition of $\widetilde{x}$, $b'(x^*)<0$ holds, which indicates that $b(x)$ is a decreasing function on $[x^*,\widetilde{x})$. We should keep $b(x)=b^*=c$ until x_1 satisfying $b'(x_1)=0$, that is

$$x_1=\frac{\sigma^2m_p^2/2-(\gamma_1+r-r_0)(b^*)^2}{m_pr_0b^*}-\frac{\mu}{r_0}.\tag{26}$$

Inserting $b^*=c$ into (12), we get

$$\frac{1}{2}[\sigma^2+\sigma_P^2c^2]W''(x)+[\mu+r_0x+(r_1-r_0)c]W'(x)-rW(x)=0.\tag{27}$$

In [7] it is shown that a solution of (27) is

$$W(x)=k_6g(x)+k_7h(x),\tag{28}$$

where k_6, k_7 are free constants, $g(x)$, $h(x)$ are defined by

$$\begin{aligned}
g(x) &= \mathrm{e}^{-(\mu+(r_1-r_0)c+r_0x)^2/[r_0(\sigma^2+\sigma_P^2c^2)]}U\Big(\frac{1}{2}+\frac{r}{2r_0},\frac{1}{2},\frac{(\mu+(r_1-r_0)c+r_0x)^2}{r_0(\sigma^2+\sigma_P^2c^2)}\Big),\\
h(x) &= \sqrt{\frac{1}{r_0}}\frac{\mu+(r_1-r_0)c+r_0}{\sqrt{\sigma^2+\sigma_P^2c^2}}\mathrm{e}^{-(\mu+(r_1-r_0)c+r_0x)^2/[r_0(\sigma^2+\sigma_P^2c^2)]}\\
&\quad\cdot M\Big(1+\frac{r}{2r_0},\frac{3}{2},\frac{(\mu+(r_1-r_0)c+r_0x)^2}{r_0\sigma^2}\Big)
\end{aligned}$$

and $M(p,q,x)$, $U(a,c,x)$ are the confluent hypergeometric functions of the first and second kind, respectively [16]. On $x\in[x_1,L)$, (14) satisfies, $b(x_1)=b_0=c$, x_2 by (22). By construction the equation (23) holds for $x_2<x<u_1$. As in the previous subsection we need:

Lemma 13 There exist constants $C_2>0$, k_1, k_2, k_6, k_7 and $u_1>x_2$ and a twice continuously differentiable function W_2 such that $W_2(x)$ is given by (18), (14), (28) for $x<x_2$ and satisfies (23) for $x_2<u_1$ with $W_2''(u_1)=0$, and $W_2'(u_1)=1$.

Similarly to Section 4.1, continuity of the function W_2 and its derivative W_2' at the points x^*, x_1, x_2 imply that we can write the solution to (11), (12) in the following form

$$W_2(x)=\begin{cases} C_2k_3^2[T_1(\delta_1)D(x,\alpha(\delta_1)+1,\delta_1)+E(x,\alpha(\delta_1)+1,\delta_1)], & 0\leqslant x<x^*;\\ C_2k_3^2[k_4g(x)+k_5h(x)], & x^*\leqslant x<x_1;\\ C_2k_3^2\Big[(k_4g'(x_1)+k_5h'(x_1))\displaystyle\int_{x_1}^{x}\mathrm{e}^{-\int_{x_1}^{z}[m_p/b(y)]\mathrm{d}y}\mathrm{d}z & \\ \quad +k_4g(x_1)+k_5h(x_1)\Big], & x_1\leqslant x<x_2;\\ C_2[T_4D(x,\alpha(\delta_2)+1,\delta_2)+E(x,\alpha(\delta_2)+1,\delta_2)], & x_2\leqslant x<L, \end{cases}\tag{29}$$

where x_1 is given by (26), x_2 by (22), and

$$k_4=\frac{T_2(x^*)g'(x^*)-T_3(x^*)g(x^*)}{h(x^*)g'(x^*)-h'(x^*)g(x^*)},\qquad k_5=\frac{T_2(x^*)h'(x^*)-T_3(x^*)h(x^*)}{h'(x^*)g(x^*)-h(x^*)g'(x^*)},$$
$$k_3^2=(\alpha(\delta_2)+1)\frac{-T_4D(x_2,\alpha(\delta_2),\delta_2)+E(x_2,\alpha(\delta_2),\delta_2)}{k_4g'(x_1)+k_5h'(x_1)}\mathrm{e}^{\int_{x_1}^{x_2}[m_p/b(y)]\mathrm{d}y}.$$

The corresponding optimal control b_2^* is given by

$$b_2^*(t)=\begin{cases}\delta_1X_t, & 0\leqslant X_t<x^*;\\ c, & x^*\leqslant X_t<x_1;\\ b(X_t), & x_1\leqslant X_t<x_2;\\ \delta_2X_t, & x_2\leqslant X_t\leqslant L.\end{cases}\tag{30}$$

4.3 Case of $\mathscr{O}=\Phi$ and $x^*\geqslant\widetilde{x}$

Similar to Section 4.2, in this case $b(x)=\delta_1x$ on interval $[0,x^*)$, and $x_0=x^*=c/\delta_1$, then (17) holds.

With $x^*\geqslant\widetilde{x}$ and (15), $b'(x^*)\geqslant0$ holds, which indicates that $b(x)$ is an increasing function on $[x^*,L)$. Then $x_1=x^*$, (14) satisfies, and $b(x_1)=b_0=c$. x_2 is defined as (22), and $W(x)$ is similar to Section 4.2 on $[x_2,L)$. By construction, the equation (23) holds for $x_2<x<u_1$. As in the previous subsections we also need:

Lemma 14 There exist constants $C_3 > 0$, k_1, k_2, k_3 and $u_1 > x_2$ and a twice continuously differentiable function W_3 such that $W_3(x)$ is given by (18), (14) for $x < x_2$ and satisfies (23) for $x_2 < u_1$ with $W_3''(u_1) = 0$, and $W_3'(u_1) = 1$.

Similarly to (24) we can write the solution as

$$W_3(x) = \begin{cases} C_3 k_3^3 [T_1(\delta_1) D(x, \alpha(\delta_1)+1, \delta_1) + E(x, \alpha(\delta_1)+1, \delta_1)], & 0 \leqslant x < x^*; \\ C_3 k_3^3 \Big[T_3(x^*) \displaystyle\int_{x^*}^{x} \mathrm{e}^{-\int_{x^*}^{z}[m_p/b(y)]\mathrm{d}y} \mathrm{d}z + T_2(x^*)\Big], & x^* \leqslant x < x_2; \\ C_3 [T_4 D(x, \alpha(\delta_2)+1, \delta_2) + E(x, \alpha(\delta_2)+1, \delta_2)], & x_2 \leqslant x < L, \end{cases} \tag{31}$$

where

$$k_3^3 = \frac{-T_4 D(x_2, \alpha(\delta_2), \delta_2) + E(x_2, \alpha(\delta_2), \delta_2)}{-T_1(\delta_1) D(x^*, \alpha(\delta_1), \delta_1) + E(x^*, \alpha(\delta_1), \delta_1)} \mathrm{e}^{\int_{x^*}^{x_2}[m_p/b(y)]\mathrm{d}y},$$

and other constants are defined the same as in the previous subsections. The corresponding optimal control b_3^* is given by

$$b_3^*(t) = \begin{cases} \delta_1 X_t, & 0 \leqslant X_t < x^*; \\ b(X_t), & x^* \leqslant X_t < x_2; \\ \delta_2 X_t, & x_2 \leqslant X_t \leqslant L. \end{cases} \tag{32}$$

4.4 The Structure of W_i $(i = 1, 2, 3)$

Next, we analyze the structure of the function W_i $(i = 1, 2, 3)$ for $x \geqslant L$. We start with showing that $W_i''(x) > 0$ for $x > u_1$.

Proposition 15 Suppose W_i $(i = 1, 2, 3)$ is respectively given by (24), (29), (31) w.r.t. three different cases, then there exists $u_1 > x_2$ such that $W_i''(u_1) = 0$ and for $x > u_1$

$$W_i''(x) > 0.$$

Proof The existence of u_1 follows from Lemmas 12, 13, and 14. Since $W_i(x)$ is a solution of (23), differentiating (23) we have

$$\begin{aligned} &\frac{1}{2}[\sigma^2 + \sigma_P^2 \delta_2^2 x^2] W_i'''(x) + [\mu + (r_0 + (r_1 - r_0)\delta_2 + \sigma_P^2)x] W_i''(x) \\ &= [r - r_0 - (r_1 - r_0)\delta_2] W_i'(x). \end{aligned}$$

Let $x = u_1$ in the above equation. It follows that

$$\frac{1}{2}[\sigma^2 + \sigma_P^2 \delta_2^2 u_1^2] W_i'''(u_1) - [r - r_0 - (r_1 - r_0)\delta_2] W_i'(u_1) = 0.$$

Then $W_i'''(u_1) > 0$ from $r - r_0 - (r_1 - r_0)\delta_2 > 0$. Thus $W_i''(x) > 0$ on (u_1, u_1+). If a constant $\widehat{x}$ exists such that $W_i''(\widehat{x}) \leqslant 0$, then $\widehat{x} = \inf\{x : W_i''(x) \leqslant 0\} < \infty$. Substituting

$x = \widehat{x}$ into the above equation, $W_i'''(\widehat{x}) > 0$ holds instantly. Therefore $W_i''(x) < 0$ on $(\widehat{x}-, \widehat{x})$, which contradicts with the definition of $\widehat{x}$. Thus $\widehat{x} = \infty$ holds and we have the assertion. $\square$

From Proposition 15, we easily obtain for $i = 1, 2, 3$

$$W_i''(x) \begin{cases} < 0, & x < u_1; \\ > 0, & u_1 < x < L. \end{cases} \tag{33}$$

Thus we have the following corollary.

Corollary 16 Let L defined by (9). Then, the function W_i' $(i = 1, 2, 3)$ strictly decreases on $(0, u_1)$ and strictly increases on (u_1, L). And on $(0, L)$ there exists only one root of the equation

$$W_i'(x) = k.$$

Moreover, if the solution to the QVI is unique, then

$$W_i(x) = W_i(\beta) + k(x - \beta) - K = W_i(L) + k(x - L), \qquad x \geqslant L. \tag{34}$$

Proof From (33), we must have $W_i'(x)$ $(i = 1, 2, 3)$ strictly decreases on $(0, u_1)$ and strictly increases on (u_1, L).

Note that T_4 must be negative. Let u_1 be as in Proposition 15 and let $\alpha_i C_i = W_i'(u_1)$. By (33) and $C_i > 0$, then $W_i'(x)/C_i$ strictly decreases on $(0, u_1)$ and strictly increases on $[u_1, +\infty)$. Thus if $0 < C_i < k/\alpha_i$, two points $\beta^{C_i} < u_1 < L^{C_i}$ must exist such that $W_i'(\beta^{C_i}) = W_i'(L^{C_i}) = k$. Other, if $C_i = k/\alpha_i$, then $\beta^{C_i} = u_1 = L^{C_i}$. It is easy to hold that β^{C_i} is increasing in C_i, while L^{C_i} is decreasing in C_i, for $C_i \in (0, k/\alpha_i)$.

Define

$$I(C_i) := \int_{\beta^{C_i}}^{L^{C_i}} (k - W_i'(y)) \mathrm{d}y.$$

Since the limits in the integral and the integrand are continuous in C_i, the function $I(C_i)$ is also continuous in C_i. The fact that both the integrand and the interval $[\beta^{C_i}, L^{C_i}]$ are decreasing w.r.t. C_i, implies that $I(C_i)$ is decreasing in C_i. Due to the fact that $C_i F_i(x) \to 0$ uniformly on any compact set of $(0, +\infty)$, we have $L^{C_i} \to +\infty$ and $(k - C_i F_i(y)) \to k$ as $C_i \to 0$. Therefore, $I(C_i) \to +\infty$ as $C_i \to 0$. Because $I(k/\alpha_i) = 0$ there exists $C_i < k/\alpha_i$, such that

$$W_i(L^{\widetilde{C}_i}) - W_i(\beta^{\widetilde{C}_i}) = k(L^{\widetilde{C}_i} - \beta^{\widetilde{C}_i}) - K,$$

the second equality holds. $\square$

4.5 Solution to the QVI and the Optimal Policy

According to Corollary 16, L is the root of

$$\frac{E(L,\alpha(\delta_2)-1,\delta_2)}{D(L,\alpha(\delta_2)-1,\delta_2)} = -T_4$$

and C_i $(i=1,2,3)$ is given by

$$C_i = \frac{k}{(1+\alpha(\delta_2))[E(L,\alpha(\delta_2),\delta_2) - T_4 D(L,\alpha(\delta_2),\delta_2)]} = C > 0.$$

Theorem 17 The function W_i $(i=1,2,3)$ given by (24), (29), (31) respectively is continuously differentiable on $(0,\infty)$ and is twice continuously differentiable on $(0,L)\cup(L,\infty)$. The function is the solution to (7)−(9) subject to the growth condition (6).

Proof We just prove case 2, other two cases can be proved by the conclusion of this case.

By construction the function W_2 satisfies (15) on $[x_1,x_2)$ and there is a solution to (17), (27), (14) respectively on $(0,x^*)$, $[x^*,x_1)$, $[x_1,x_2)$. If $L > x > u_1$ then $W_2'(x) = CF_2(x)$ is increasing. On the interval $[x_2,u_1)$ the function W_2 is decreasing. $b_2(x) = -m_p W_2'(x)/W_2''(x)$ shows that it is an increasing function of x on $[x_2,u_1)$ wherefrom follows that $b_t = \delta_2 X_t$ is the maximizer of the left-hand side of (12). From the fact that W_2' is decreasing on $(0,u_1]$, it follows that $b_t = \delta_2 X_t$ is a maximizer of the left-hand side of (12). Then W_2 satisfies (12) on $[x_2,L)$, since it is a solution to (23) on this interval.

Since W_2' is decreasing on $(0,u_1]$, we have $W_2'(x) > k$ for $x \leqslant \beta$. Then $W_2(x-\eta) + k\eta - K$ is a decreasing function of η and arrives the supremum at $\eta = 0+$, that is, $W_2(x) - K < W_2(x)$. For $x > \beta$ we have that $MW_2(x) = W_2(\beta) + k(x-\beta) - K$ by differentiating. Since W_2' is decreasing on $(0,u_1]$ and increasing on (u_1,L) which equals k at the points L and β, we can write that

$$MW_2(x) = W_2(\beta) + k(x-\beta) - K = W_2(L) - k(L-x) < W_2(x).$$

This implies that $MW_2(x) < W_2(x)$ on $(0,L)$ and therefore W_2 satisfies (7)−(9) on this interval.

We need to repeat the proof argument of Proposition 4.1 of [15], relying on (34) and linearity of $W_2(x)$ for $x \geqslant L$.

The growth condition (6) is obvious for $W_2'(x) = k$ when $x > L$. Thus we complete the proof. □

Theorem 18 For $i=1,2,3$, the control

$$\pi_i^* = (b_i^*, \mathscr{T}_i^*, \xi_i^*) = (b_i^*; \tau_{i1}^*, \tau_{i2}^*, \ldots, \tau_{in}^*, \ldots; \xi_{i1}^*, \xi_{i2}^*, \ldots, \xi_{in}^*, \ldots)$$

defined by (25), (30), (32) respectively, and

$$\tau_{i1}^* = \inf\{t \geqslant 0 : X_t^* = L\}, \qquad \xi_{i1}^* := L - \beta, \tag{35}$$

and for every $j \geqslant 2$,

$$\tau_{ij}^* := \inf\{t \geqslant \tau_{i,j-1}^* : X_t^* = L\}, \qquad \xi_{ij}^* := L - \beta. \tag{36}$$

π_i^* is the QVI control associated with the function W_i defined by (24), (29), (31), respectively. This control is optimal and the function W_i coincides with the value function. That is,

$$V_i(x) = W_i(x) = J(x, \pi_i^*) = J(x; b_i^*, \mathscr{T}_i^*, \xi_i^*).$$

Proof In view of Theorem 17, the function W_i $(i = 1, 2, 3)$ defined by (24), (29), (31) satisfies all the conditions of Theorem 8. From Definition 7 and the discussion in Section 3 and 4, we know that the control π_i^* defined in (25), (30), (32) and (35) − (36) is the control associated with W_i. In addition, according to Definition 9, the control π_i^* is admissible. Therefore, applying Theorem 8, we conclude that the function W_i is the value function and π_i^* is the optimal policy, for $i = 1, 2, 3$ respectively. □

§5. The Case of $r \leqslant \max(r_0, r_1)$

The simplest case would be $r < \max(r_0, r_1)$. Regarding to an economic analysis with this case, it is shown that the optimal value function is finite and there should be infinitely many optimal investment strategies.

Lemma 19 If $r < \max(r_0, r_1)$, then $V(x) = \max(0, kx - K)$.

Proof Assume $r_0 \leqslant r < r_1$. Choose a policy π such that $b_\pi(t) = \delta_2 X_t$. Then the reserve process is

$$X_t = x + \int_0^t [\mu + (r_0 + (r_1 - r_0)\delta_2)X_t]\mathrm{d}t + \int_0^t \sqrt{\sigma^2 + \sigma_P^2\delta_2^2X_t^2}\mathrm{d}\omega_t - \sum_{i=1}^{\infty} I_{\{\tau_i<t\}}\xi_i.$$

The problem becomes the impulse control of dividends, that is, W must satisfies (23). Then a solution of (10), (23) is

$$W(x) = C[E(x, \alpha(\delta_2) + 1, \delta_2) + T_1(\delta_2)D(x, \alpha(\delta_2) + 1, \delta_2)],$$

where $C = k/\{(1 + \alpha(\delta_2))[E(L, \alpha(\delta_2), \delta_2) + T_1(\delta_2)D(L, \alpha(\delta_2), \delta_2)]\}$, and L is the root of

$$\frac{E(L, \alpha(\delta_2) - 1, \delta_2)}{D(L, \alpha(\delta_2) - 1, \delta_2)} = -T_1(\delta_2).$$

By L'Hospital rule and the above equation, it can be seen that $L \to 0$. Then $L = \beta = 0$. It suggests that if $kx - K > 0$ the optimal dividend policy is distributing all the initial reserve x, no matter which investment strategy the corporation will take. □

To see that there exists no unique policy giving the optimal return. It is noted that in the proof of Lemma 19, the choice of amount of investment is not unique.

Now we assume $r = \max(r_0, r_1)$ with $r_0 \neq r_1$. Then we try to repeat the calculations of Section 4, we have to find a constant x_2, a constant u_1 and a function $W(x)$ which satisfies (23) with $r_1 = r > r_0$, i.e.

$$\frac{1}{2}[\sigma^2 + \sigma_P^2\delta_2^2x^2]W''(x) + [\mu + (r_0 + (r - r_0)\delta_2)x]W'(x) - rW(x) = 0,$$

for $x \in (x_2, L)$ with $W'(u_1) = 1$ and $W''(u_1) = 0$ with $u_1 \in (x_2, L)$. Since $r > r_0(1-\delta_2) + r\delta_2$, $W(x)$ is the same as the case discussing in Section 4 on (x_2, L).

Otherwise the case $r = r_0 = r_1$ is much different from the above case. In this case the function $W(x)$ satisfies

$$\frac{1}{2}[\sigma^2 + \sigma_P^2\delta_2^2x^2]W''(x) + (\mu + rx)W'(x) - rW(x) = 0.$$

According to the analysis in [6], it suggests that u_1 does not exist or that $u_1 = \infty$, then the optimal policy does not exist. But we can try to obtain the value function by redefining it on $[0, x_2)$ as the method used in [6], and the finite property will also be easily verified.

§6. Conclusion

We give a brief description of the conclusions in the previous sections. If $r < \max(r_0, r_1)$ then the investment will meet the loss of wealth due to discounting. In this case $V(x) = \max(0, kx - K)$ and there are infinitely many investment policies in the admissible set. For $r > r_1 > r_0$ or $r = \max(r_0, r_1)$ with $r_0 \neq r_1$, $V(x) < \infty$ and there exists a nontrivial policy whose return coincides with $V(x)$. The "borderline" case is $r = r_0 = r_1$, the corresponding value function $V(x)$ is finite but no optimal policy exists. That is, the different optimal policies appear due to different possible relationships between exogenous parameters, which show the multiplicity of different economic environments a corporation faces.

Appendix

Proof of Lemma 2 Let $t_n = \inf\{t : Y_t \geqslant n\}$, $n \in N$, then $t_0 < t_1 < \cdots < t_n < \cdots$ is a sequence of increasing stopping times w.r.t. $\{\mathscr{F}_t\}_{t \geqslant 0}$. Define

$$m(s) = \mu + r_0Y_s + (r_1 - r_0)b_s, \qquad S(s) = \sqrt{\sigma^2 + \sigma_P^2 b_s^2}.$$

Combining with constraint $b_t \in [\min\{\delta_1 Y_t, c\}, \delta_2 Y_t]$, it follows that

$$d(n) = \mu + r_0 n \leqslant m(s \wedge t_n) \leqslant \mu + [r_0 + (r_1 - r_0)\delta_2]n = g(n),$$
$$\sigma \leqslant S(s \wedge t_n) \leqslant \sqrt{\sigma^2 + \sigma_P^2 \delta_2^2 n^2} = c(n).$$

In view of Lemma 1 in [11], we can obtain that

$$\lim_{n\to\infty} \lim_{x\downarrow 0} \mathsf{P}(\zeta_0 < \zeta_h \wedge t \wedge t_n) = 1, \tag{37}$$
$$\lim_{n\to\infty} \lim_{x\downarrow 0} \mathsf{E}_x\Big(\max_{0\leqslant s\leqslant \zeta_0\wedge t\wedge t_n} Y_s\Big) = 0. \tag{38}$$

then the first limit holds from (37). Next we turn to prove the second limit in (5).

For any fixed $t > 0$, there exists t_j in $\{t_n : n = 0, 1, 2, \ldots\}$ such that $Y_{t_j} \geqslant Y_t$, since Y_t is a continuous process. If $n \geqslant j$, we rewrite (38) as $\lim_{x\downarrow 0} \mathsf{E}_x\big(\max_{0\leqslant s\leqslant \zeta_0\wedge t} Y_s\big) \downarrow 0$. On the other hand, if $0 \leqslant n < j$,

$$\begin{aligned}
&\mathsf{E}_x\Big(\max_{0\leqslant s\leqslant \zeta_0\wedge t\wedge t_n} Y_s\Big)\\
\leqslant{}& x + \mathsf{E}_x\Big[\max_{0\leqslant s\leqslant t} \int_0^s [r_0 Y_s + (r_1 - r_0) b_s]\mathrm{d}s\Big] + \mathsf{E}_x\Big[\max_{0\leqslant s\leqslant t} \int_0^s \sqrt{\sigma^2 + \sigma_P^2 b_s^2}\, \mathrm{d}\omega_s\Big]\\
\leqslant{}& x + [\mu + (r_0 + (r_1 - r_0)\delta_2) j]t + 4\sqrt{2t}\sqrt{\sigma^2 + \sigma_P^2 \delta_2^2 j^2}.
\end{aligned}$$

Here we use the constraints on b_s, Theorem 7.3 in [17] and $b_s \leqslant \delta_2 Y_s$. Hence the second limit (5) follows. $\square$

References

[1] Zariphopoulou T. Consumption-investment models with constraints [J]. *SIAM J. Control Optim.*, 1994, **32(1)**: 59–85.

[2] Yang Y H. Existence of optimal consumption and portfolio rules with portfolio constraints and stochastic income, durability and habit formation [J]. *J. Math. Econom.*, 2000, **33(2)**: 135–153.

[3] Pham H. Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints [J]. *Appl. Math. Optim.*, 2002, **46(1)**: 55–78.

[4] Müller S. Constrained portfolio optimization [D]. St. Gallen, Switzerland: University of St. Gallen, 2005.

[5] Paulsen J, Gjessing H K. Optimal choice of dividend barriers for a risk process with stochastic return on investments [J]. *Insurance Math. Econom.*, 1997, **20(3)**: 215–223.

[6] Højgaard B, Taksar M. Optimal risk control for a large corporation in the presence of returns on investments [J]. *Finance Stoch.*, 2001, **5(4)**: 527–547.

[7] Zhang X, Song M. Optimization of risk policy and dividends with fixed transaction costs under interest rate [J]. *Front. Math. China*, 2012, **7(4)**: 795–811.

[8] Højgaard B, Taksar M. Optimal dynamic portfolio selection for a corporation with controllable risk and dividend distribution policy [J]. *Quant. Finance*, 2004, **4(3)**: 315–327.

[9] Paulsen J. Optimal dividend payouts for diffusions with solvency constraints [J]. *Finance Stoch.*, 2003, **7(4)**: 457–473.

[10] Bai L H, Hunting M, Paulsen J. Optimal dividend policies for a class of growth-restricted diffusion processes under transaction costs and solvency constraints [J]. *Finance Stoch.*, 2012, **16(3)**: 477–511.

[11] Choulli T, Taksar M, Zhou X Y. A diffusion model for optimal dividend distribution for a company with constraints on risk control [J]. *SIAM J. Control Optim.*, 2003, **41(6)**: 1946–1979.

[12] Protter P. *Stochastic Integration and Differential Equations: A New Approach* [M]. 2nd ed. New York: Springer-Verlag, 1992.

[13] Fleming W H, Soner H M. *Controlled Markov Processes and Viscosity Solutions* [M]. New York: Springer-Verlag, 2005.

[14] Alvarez L H R, Rakkolainen T A. Optimal payout policy in presence of downside risk [J]. *Math. Methods Oper. Res.*, 2009, **69(1)**: 27–58.

[15] Cadenillas A, Choulli T, Taksar M, et al. Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm [J]. *Math. Finance*, 2006, **16(1)**: 181–202.

[16] Slater L J. *Confluent Hypergeometric Functions* [M]. Cambridge: University Press, 1960.

[17] Mao X R. *Stochastic Differential Equations and Their Applications* [M]. Chichester, UK: Horwood Publishing, 1997.

带交易费用和投资约束的扩散模型中最优投资 – 分红问题

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摘　要: 本文研究了带交易费用和投资约束的最优投资 – 分红问题. 假定公司投资受到包含卖空和借贷的一般性约束条件, 由此产生正则 – 脉冲随机控制问题. 本文重点研究了投资收入不能满足资本折扣损失的非平凡情形, 区分了三种不同可能状况下的拟变分不等式, 并构造了其对应的值函数和最优策略. 我们最后也给出了平凡情形下随机控制的具体结论.

关键词: 投资约束; 交易费用; 正则 – 脉冲控制; 拟变分不等式

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