应用概率统计 第 33 卷
第 4 期 2017 年 8 月

Chinese Journal of Applied Probability and Statistics
Aug., 2017, Vol. 33, No. 4, pp. 331-339
doi: 10.3969/j.issn.1001-4268.2017.04.001

Another Borel-Cantelli Lemma for Capacities *

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Abstract: In this paper, we will prove another Borel-Cantelli lemma for capacities induced by sublinear expectations.

Keywords: Borel-Cantelli lemma; capacity; sublinear expectation

2010 Mathematics Subject Classification: 60F99

Citation: Song L, Wu P Y. Another Borel-Cantelli lemma for capacities [J]. Chinese J. Appl. Probab. Statist., 2017, 33(4): 331–339.

§1. Introduction

In the classical probability theory, Borel-Cantelli lemma is a very important result: If $\{A_i\}_{i=1}^{\infty}$ is a sequence of events on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and $\sum\limits_{i=1}^{\infty} \mathsf{P}(A_i) < \infty$, then $\mathsf{P}\big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\big) = 0$; if $\{A_i\}_{i=1}^{\infty}$ is a sequence of independent events and $\sum\limits_{i=1}^{\infty} \mathsf{P}(A_i) = \infty$, then $\mathsf{P}\big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\big) = 1$. Many attempts have been made to weaken the independent condition in the second part of the Borel-Cantelli lemma (see [1–4]).

It is well-known that, motivated by the risk measures and stochastic volatility problems in finance, Peng [5] has introduced a new kind of nonlinear expectation. Hu [6], Hu and Zhang [7] have obtained Cramér's theorem and the central limit theorem for capacities induced by sublinear expectations. Chen et al. [8] have obtained a strong law of large numbers for capacities. In this paper, the authors derive a Borel-Cantelli lemma for capacities induced by sublinear expectations, supposing $\{A_n\}_{n=1}^{\infty}$ are mutually independent with respect to $\overline{v}$, i.e. $\overline{v}\big(\bigcap\limits_{i=n}^{\infty} A_i^c\big) = \prod\limits_{i=n}^{\infty} \overline{v}(A_i^c)$. The natural question is: In the framework of sublinear expectations, will Borel-Cantelli lemma still hold true under a weaker condition? Song [9] has achieved a Borel-Cantelli lemma for capacities. In this paper, we will

*The project was supported by the National Natural Science Foundation of China (Grant No. 11601280).
Received June 5, 2015. Revised October 26, 2015.

prove another Borel-Cantelli lemma for those capacities which covers the previous results induced by [9].

This paper is organized as follows: In Section 2, we will give some basic notions and lemma which will be used in the following section. In Section 3, we state and prove the main result of this paper.

§2. Preliminaries

Let $(\Omega, \mathscr{F})$ be a measurable space and $\mathscr{H}$ be a linear space of real valued functions defined on Ω. Firstly, we give the definition of sublinear expectation (see [5, 10–12]).

Definition 1 ([5]) A sublinear expectation is a functional $\widehat{\mathsf{E}}: \mathscr{H} \mapsto \mathbf{R}$ satisfying for all $X, Y \in \mathscr{H}$,

(a) monotonicity: $\widehat{\mathsf{E}}[X] \geqslant \widehat{\mathsf{E}}[Y]$, if $X \geqslant Y$;

(b) constant preserving: $\widehat{\mathsf{E}}[c] = c$, for $c \in \mathbf{R}$;

(c) sub-additivity: $\widehat{\mathsf{E}}[X+Y] \leqslant \widehat{\mathsf{E}}[X] + \widehat{\mathsf{E}}[Y]$;

(d) positive homogeneity: $\widehat{\mathsf{E}}[\lambda X] = \lambda \widehat{\mathsf{E}}[X]$, for $\lambda \geqslant 0$.

The triple $(\Omega, \mathscr{H}, \widehat{\mathsf{E}})$ is called a sublinear expectation space.

From [12], we have the following representation theorem for sublinear expectations.

Lemma 2 ([12]) Let $\widehat{\mathsf{E}}$ be a sublinear functional defined on $\mathscr{H}$, i.e., (c) and (d) hold for $\widehat{\mathsf{E}}$. Then there exists a family $\{\mathsf{E}_\theta, \theta \in \Theta\}$ of linear functionals on $\mathscr{H}$, such that

$$\widehat{\mathsf{E}}[X] = \max_{\theta \in \Theta} \mathsf{E}_\theta[X], \qquad \text{for } X \in \mathscr{H}.$$

If (a) and (b) also hold, then E_θ is linear expectation for $\theta \in \Theta$. If we make furthermore the following assumption: (H1) For each sequence $\{X_n\}_{n=1}^{\infty} \subset \mathscr{H}$ such that $X_n(\omega) \downarrow 0$ for ω, we have $\widehat{\mathsf{E}}[X_n] \downarrow 0$. Then for each $\theta \in \Theta$, there exists a unique (σ-additive) probability measure P_θ defined on $\sigma(\mathscr{H})$ such that

$$\mathsf{E}_\theta[X] = \int_\Omega X(\omega) \mathrm{d}\mathsf{P}_\theta(\omega), \qquad \text{for } X \in \mathscr{H}.$$

In this paper, we are interested in the following sublinear expectation:

$$\overline{\mathsf{E}}[\cdot] = \sup_{\mathsf{P} \in \mathscr{P}} \mathsf{E}_{\mathsf{P}}[\cdot],$$

where $\mathscr{P}$ is a set of probability measures.

For this $\mathscr{P}$, we define

$$\overline{V}(A) := \overline{\mathsf{E}}[I_A] = \sup_{\mathsf{P}\in\mathscr{P}} \mathsf{E}_{\mathsf{P}}[I_A], \qquad \forall\, A \in \mathscr{F},$$

$$\overline{v}(A) := -\overline{\mathsf{E}}[-I_A] = \inf_{\mathsf{P}\in\mathscr{P}} \mathsf{E}_{\mathsf{P}}[I_A], \qquad \forall\, A \in \mathscr{F},$$

then, $\overline{V}$ and $\overline{v}$ are two capacities. It is easy to check that the pair of capacities satisfies

$$\overline{V}(A) + \overline{v}(A^c) = 1, \qquad \forall\, A \in \mathscr{F},$$

where A^c is the complement set of A.

§3. Main Result

Theorem 3 (Borel-Cantelli lemma for capacities) Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of events in $\mathscr{F}$, $(\overline{V}, \overline{v})$ be a pair of capacities induced by sublinear expectation $\overline{\mathsf{E}}$.

(i) If $\sum\limits_{i=1}^{\infty} \overline{V}(A_i) < \infty$, then $\overline{V}\big(\bigcap\limits_{m=1}^{\infty} \bigcup\limits_{i=m}^{\infty} A_i\big) = 0$;

(ii) Let H be an arbitrart real constant, set

$$\alpha_H = \liminf_{n\to\infty} \frac{\sum\limits_{1\leqslant i<j\leqslant n} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=1}^{n} \overline{v}(A_i)\Big)^2}. \tag{1}$$

If

$$\sum_{i=1}^{\infty} \overline{v}(A_i) = \infty, \quad \text{and} \quad \lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \overline{V}(A_i)}{\sum\limits_{i=1}^{n} \overline{v}(A_i)} = a, \tag{2}$$

then $a^2H + 2\alpha_H \geqslant 1$ and $\overline{v}\big(\bigcap\limits_{m=1}^{\infty} \bigcup\limits_{i=m}^{\infty} A_i\big) \geqslant (a^2H + 2\alpha_H)^{-1}$.

Remark 4 It is obvious that $a \geqslant 1$. In fact, when capacities $\overline{V}$ and $\overline{v}$ satisfy $\overline{V} = \overline{v}$, the result is the same as that in [3]. Therefore, Theorem 3 generalizes previous result.

In order to prove Theorem 3, we need the following lemmas.

Lemma 5 Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of events in $\mathscr{F}$, $(\overline{V}, \overline{v})$ be a pair of capacities induced by sublinear expectation $\overline{\mathsf{E}}$. Then, for each n, we have

$$\overline{v}\Big(\bigcup_{i=1}^{n} A_i\Big) \geqslant \frac{\Big(\sum\limits_{i=1}^{n} \overline{v}(A_i)\Big)^2}{\sum\limits_{i,j=1}^{n} \overline{V}(A_iA_j)}.$$

Proof For arbitrary probability measure $\mathsf{P} \in \mathscr{P}$, applying inequality of Chung and Erdös (see [13]), we have

$$\mathsf{P}\Big(\bigcup_{i=1}^{n} A_i\Big) \geqslant \frac{\Big(\sum\limits_{i=1}^{n} \mathsf{P}(A_i)\Big)^2}{\sum\limits_{i,j=1}^{n} \mathsf{P}(A_iA_j)}.$$

So we get

$$\inf_{\mathsf{P}\in\mathscr{P}} \mathsf{P}\Big(\bigcup_{i=1}^{n} A_i\Big) \geqslant \frac{\inf\limits_{\mathsf{P}\in\mathscr{P}}\Big(\sum\limits_{i=1}^{n} \mathsf{P}(A_i)\Big)^2}{\sup\limits_{\mathsf{P}\in\mathscr{P}} \sum\limits_{i,j=1}^{n} \mathsf{P}(A_iA_j)} \geqslant \frac{\Big(\sum\limits_{i=1}^{n} \inf\limits_{\mathsf{P}\in\mathscr{P}} \mathsf{P}(A_i)\Big)^2}{\sum\limits_{i,j=1}^{n} \sup\limits_{\mathsf{P}\in\mathscr{P}} \mathsf{P}(A_iA_j)}.$$

By the definition of $\overline{V}$ and $\overline{v}$, it is easy to obtain

$$\overline{v}\Big(\bigcup_{i=1}^{n} A_i\Big) \geqslant \frac{\Big(\sum\limits_{i=1}^{n} \overline{v}(A_i)\Big)^2}{\sum\limits_{i,j=1}^{n} \overline{V}(A_iA_j)}.$$

The proof is completed. $\square$

Lemma 6 Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of events in $\mathscr{F}$, $(\overline{V}, \overline{v})$ be a pair of capacities induced by sublinear expectation $\overline{\mathsf{E}}$. Assume that $(\overline{V}, \overline{v})$ satisfies condition (2). For every positive integer $m \geqslant 1$, set

$$\alpha_H^{(m)} = \liminf_{n\to\infty} \frac{\sum\limits_{m\leqslant i<j\leqslant n} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2},$$

then, $\alpha_H = \alpha_H^{(m)}$ for every fixed real number H and every fixed positive integer m.

Proof Let $m < n$. For real number a_{ij} $(1 \leqslant i < j \leqslant n)$, we have

$$\sum_{1\leqslant i<j\leqslant n} a_{ij} = \sum_{1\leqslant i<j\leqslant m} a_{ij} + \sum_{m\leqslant i<j\leqslant n} a_{ij} + \sum_{1\leqslant i<m<j\leqslant n} a_{ij}.$$

By condition (2), we get

$$\frac{\sum\limits_{i=m}^{n} \overline{v}(A_i)}{\sum\limits_{i=1}^{n} \overline{v}(A_i)} = 1 - \frac{\sum\limits_{i=1}^{m-1} \overline{v}(A_i)}{\sum\limits_{i=1}^{n} \overline{v}(A_i)} \to 1 \qquad (n \to \infty).$$

$$\alpha_H = \liminf_{n\to\infty} \frac{\sum\limits_{1\leqslant i<j\leqslant n} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=1}^{n} \overline{v}(A_i)\Big)^2}$$

$$= \liminf_{n\to\infty} \frac{\sum\limits_{1\leqslant i<j\leqslant n} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} \frac{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2}{\Big(\sum\limits_{i=1}^{n} \overline{v}(A_i)\Big)^2}$$

$$= \liminf_{n\to\infty} \frac{\sum\limits_{1\leqslant i<j\leqslant n} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2}$$

$$= \liminf_{n\to\infty} (I_1 + I_2 + I_3), \tag{3}$$

where

$$I_1 = \frac{\sum\limits_{1\leqslant i<j\leqslant m} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2},$$

$$I_2 = \frac{\sum\limits_{m\leqslant i<j\leqslant n} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2},$$

$$I_3 = \frac{\sum\limits_{1\leqslant i<m<j\leqslant n} (\overline{V}(A_iA_j) - H\overline{V}(A_i)\overline{V}(A_j))}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2}.$$

For fixed m, from condition (2), it follows that $\lim\limits_{n\to\infty} I_1 = 0$, and

$$|I_3| \leqslant \frac{m \sum\limits_{m<j\leqslant n} (\overline{V}(A_j) + |H|\overline{V}(A_j))}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} = \frac{m(1+|H|) \sum\limits_{m<i\leqslant n} \overline{V}(A_i)}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} \to 0 \qquad (n\to\infty). \tag{4}$$

From (3) and (4), we have $\alpha_H = \alpha_H^{(m)}$. The proof is completed. □

Now we give the proof of Theorem 3.

Proof of Theorem 3 (i) First, for each probability $\mathsf{P} \in \mathscr{P}$, from the classical Borel-Cantelli lemma, we get

$$\mathsf{P}\Big(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i\Big) = 0.$$

By the definition of $\overline{V}$, it is easy to obtain

$$\overline{V}\Big(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i\Big) = 0.$$

We now prove (ii). We begin with proving $a^2H + 2\alpha_H \geqslant 1$. Obviously

$$2 \sum_{1\leqslant i<j\leqslant n} \overline{V}(A_iA_j) = \sum_{i,j=1}^{n} \overline{V}(A_iA_j) - \sum_{i=1}^{n} \overline{V}(A_i),$$

$$2\sum_{1\leqslant i<j\leqslant n}\overline{V}(A_i)\overline{V}(A_j)=\Big(\sum_{i=1}^{n}\overline{V}(A_i)\Big)^2-\sum_{i=1}^{n}(\overline{V}(A_i))^2.$$

Therefore, by the above equalities and Lemma 5, we find

$$\begin{aligned}
2\alpha_H&=\liminf_{n\to\infty}\frac{\sum\limits_{i,j=1}^{n}\overline{V}(A_iA_j)-\sum\limits_{i=1}^{n}\overline{V}(A_i)-H\Big[\Big(\sum\limits_{i=1}^{n}\overline{V}(A_i)\Big)^2-\sum\limits_{i=1}^{n}(\overline{V}(A_i))^2\Big]}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}\\
&=\liminf_{n\to\infty}\left[\frac{\sum\limits_{i,j=1}^{n}\overline{V}(A_iA_j)}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}-\frac{\sum\limits_{i=1}^{n}\overline{V}(A_i)}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}-H\frac{\Big(\sum\limits_{i=1}^{n}\overline{V}(A_i)\Big)^2}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}+H\frac{\sum\limits_{i=1}^{n}(\overline{V}(A_i))^2}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}\right]\\
&\geqslant 1+\liminf_{n\to\infty}\left[-\frac{\sum\limits_{i=1}^{n}\overline{V}(A_i)}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}-H\frac{\Big(\sum\limits_{i=1}^{n}\overline{V}(A_i)\Big)^2}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}+H\frac{\sum\limits_{i=1}^{n}(\overline{V}(A_i))^2}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}\right]. \qquad (5)
\end{aligned}$$

Using condition (2), we get

$$\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{n}\overline{V}(A_i)}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}=0,\qquad \lim_{n\to\infty}H\frac{\Big(\sum\limits_{i=1}^{n}\overline{V}(A_i)\Big)^2}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}=a^2H. \qquad (6)$$

For any event A_i, we have $(\overline{V}(A_i))^2\leqslant\overline{V}(A_i)$. Hence

$$0\leqslant\frac{\sum\limits_{i=1}^{n}(\overline{V}(A_i))^2}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}\leqslant\frac{\sum\limits_{i=1}^{n}\overline{V}(A_i)}{\Big(\sum\limits_{i=1}^{n}\overline{v}(A_i)\Big)^2}\to 0\qquad (n\to\infty). \qquad (7)$$

One can easily deduce from (5), (6) and (7) that $a^2H+2\alpha_H\geqslant 1$.

We are now ready to prove $\overline{v}\big(\bigcap\limits_{m=1}^{\infty}\bigcup\limits_{i=m}^{\infty}A_i\big)\geqslant(a^2H+2\alpha_H)^{-1}$.

Let $B_m=\bigcup\limits_{i=m}^{\infty}A_i$, then $B_1\supset B_2\supset\cdots\supset B_m\supset B_{m+1}\cdots$, and $\bigcap\limits_{m=1}^{\infty}\bigcup\limits_{i=m}^{\infty}A_i=\bigcap\limits_{m=1}^{\infty}B_m$. We only need to prove $\overline{v}\big(\bigcap\limits_{m=1}^{\infty}B_m\big)\geqslant(a^2H+2\alpha_H)^{-1}$. Let $m<n$, from Lemma 5, we have

$$\overline{v}\Big(\bigcup_{i=m}^{n}A_i\Big)\geqslant\frac{\Big(\sum\limits_{i=m}^{n}\overline{v}(A_i)\Big)^2}{\sum\limits_{i,j=m}^{n}\overline{V}(A_iA_j)}. \qquad (8)$$

On the other hand, it is easy to check that

$$\sum_{i,j=m}^{n}\overline{V}(A_iA_j)=\sum_{i=m}^{n}\overline{V}(A_i)+2\sum_{m\leqslant i<j\leqslant n}\overline{V}(A_iA_j)=\sum_{i=m}^{n}\overline{V}(A_i)+J_1+J_2, \qquad (9)$$

where

$$J_1 = 2 \sum_{m \leqslant i < j \leqslant n}^{n} (\overline{V}(A_i A_j) - H\overline{V}(A_i)\overline{V}(A_j)), \qquad J_2 = 2H \sum_{m \leqslant i < j \leqslant n}^{n} \overline{V}(A_i)\overline{V}(A_j).$$

By (8) and (9), we can deduce that

$$\begin{aligned}\overline{v}\Big(\bigcup_{i=m}^{n} A_i\Big) &\geqslant \frac{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2}{\sum\limits_{i=m}^{n} \overline{V}(A_i) + J_1 + J_2} \\ &= \left[\frac{\sum\limits_{i=m}^{n} \overline{V}(A_i)}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} + \frac{J_1}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} + \frac{J_2}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2}\right]^{-1}. \end{aligned} \tag{10}$$

Since

$$0 \leqslant \frac{\sum\limits_{i=m}^{n} (\overline{V}(A_i))^2}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} \leqslant \frac{\sum\limits_{i=m}^{n} \overline{V}(A_i)}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2}. \tag{11}$$

For fixed positive integer m, from condition (2) and inequality (11), it follows that

$$\lim_{n\to\infty} \frac{\sum\limits_{i=m}^{n} (\overline{V}(A_i))^2}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} = 0, \tag{12}$$

$$\lim_{n\to\infty} \frac{J_2}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} = H \lim_{n\to\infty} \frac{\Big(\sum\limits_{i=m}^{n} \overline{V}(A_i)\Big)^2 - \sum\limits_{i=m}^{n} (\overline{V}(A_i))^2}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2} = a^2 H. \tag{13}$$

For fixed m, from (10), (12), (13) and Lemma 6, we have

$$\overline{v}\Big(\bigcup_{i=m}^{\infty} A_i\Big) \geqslant \left[a^2 H + \liminf_{n\to\infty} \frac{J_1}{\Big(\sum\limits_{i=m}^{n} \overline{v}(A_i)\Big)^2}\right]^{-1}.$$

That is to say $\overline{v}(B_m) \geqslant (a^2 H + 2\alpha_H)^{-1}$, for every fixed m. This yields that $\overline{v}\Big(\bigcap\limits_{m=1}^{\infty} \bigcup\limits_{i=m}^{\infty} A_i\Big) \geqslant (a^2 H + 2\alpha_H)^{-1}$. We complete the proof. $\square$

From Theorem 3, we can easy obtain the following corollary.

Corollary 7 Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of events in $\mathscr{F}$, $(\overline{V}, \overline{v})$ be a pair of capacities induced by sublinear expectation $\overline{\mathsf{E}}$. Assume that $\overline{V}(A_i A_j) = \overline{V}(A_i)\overline{V}(A_j)$ for all $i, j > L$

such that $i \neq j$ and for constant L. If

$$\sum_{i=1}^{\infty} \overline{v}(A_i) = \infty, \quad \text{and} \quad \lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \overline{V}(A_i)}{\sum\limits_{i=1}^{n} \overline{v}(A_i)} = a,$$

then $\overline{v}\big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\big) \geqslant 1/a^2$.

Finally, let us introduce the following Example 8.

Example 8 Let $\Omega = \mathbf{N}$, $\mathscr{P} = \{\mathsf{P}_n, n \in \mathbf{N}\}$, where $\mathsf{P}_1(\{1\}) = 1$ and $\mathsf{P}_n(\{1\}) = 1 - 1/n^2$, $\mathsf{P}_n(\{kn\}) = 1/n^3$, $k = 1, 2, \ldots, n$, for $n = 2, 3, \ldots$. Define capacities $\overline{V}(A) = \sup\limits_{\mathsf{P}\in\mathscr{P}} \mathsf{P}(A)$, $\overline{v}(A) = \inf\limits_{\mathsf{P}\in\mathscr{P}} \mathsf{P}(A)$, for $A \in \mathscr{F}$. Let event $B = \{\omega : \omega = 1\}$, event $C = \{\omega : 2 \leqslant \omega \leqslant 4\}$, then $\overline{V}(B) = 1$, $\overline{v}(B) = 3/4$, $\overline{V}(C) = 1/4$, $\overline{v}(C) = 0$, $\overline{V}(B\cap C) = 0$, $\overline{V}(B\cup C) = 1$, $\overline{v}(B \cup C) = 25/27$. For arbitrary positive integer k, put $A_{2k-1} = B$, $A_{2k} = C$, then

$$a = \lim_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \overline{V}(A_i)}{\sum\limits_{i=1}^{n} \overline{v}(A_i)} = \frac{5}{3}, \quad \text{and} \quad \overline{v}\Big(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\Big) = \overline{v}(B \cup C) = \frac{25}{27} \geqslant \frac{1}{a^2 H + 2\alpha_H} = \frac{9}{20}.$$

Acknowledgements The authors would like to thank anonymous referees for their careful reading and valuable suggestions.

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关于容度的一个Borel-Cantelli引理

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摘 要: 在次线性期望理论框架下, 本文得到了关于容度的另一个Borel-Cantelli引理.

关键词: Borel-Cantelli引理; 溶度; 次线性期望

中图分类号: O211.63