应用概率统计　第 33 卷
第 6 期　2017 年 12 月

Chinese Journal of Applied Probability and Statistics
Dec., 2017, Vol. 33, No. 6, pp. 642-654
doi: 10.3969/j.issn.1001-4268.2017.06.007

Decay Rates of Weak Solutions to Non-Local Cauchy Problems *

LIN HuoNan

(*College of Mathematics and Informatics, Fujian Normal University, Fuzhou, 350117, China*)

Abstract: In this paper, the asymptotic behavior of the weak solution $(u_t)_{t\geqslant 0}$ to the non-local Cauchy problems as stated in (1) is considered. Only using lower bounds of jumping kernel $J(x,y)$ for large $|x-y|$, it is obtained that $\|u_t\|_p \leqslant c(t)\|u_0\|_q$ with any $1\leqslant q<p<\infty$ and large t. Explicit and sharp formulas for $c(t)$ are also given.

Keywords: non-local Cauchy problem; jump kernel; weak-Poincaré inequality

2010 Mathematics Subject Classification: 60G51; 60G52; 60J25; 60J75

Citation: Lin H N. Decay rates of weak solutions to non-local Cauchy problems [J]. Chinese J. Appl. Probab. Statist., 2017, 33(6): 642–654.

§1. Introduction and Main Results

Let $J(x,y)$ be a non-negative symmetric function on $R^d\times R^d$ such that $J(x,y)=J(y,x)$ for all $x,y\in R^d$. In this paper, it is aim to study the asymptotic behavior of weak solution to the following non-local Cauchy problems:

$$\begin{aligned}\partial_t u(t,x) &= \lim_{\varepsilon>0}\int_{y\in R^d:|y-x|\geqslant\varepsilon}(u(t,y)-u(t,x))J(x,y)\mathrm{d}y\\ &= \mathrm{p.v.}\int_{R^d}(u(t,y)-u(t,x))J(x,y)\mathrm{d}y \qquad \text{in } R_+\times R^d \end{aligned}\tag{1}$$

with the initial condition $u(0,x)=u_0(x)$ satisfying $u_0\in L^q(R^d)\cap L^p(R^d)$ for $1\leqslant q<p<\infty$. Here, p.v. $\int_{R^d}\cdots\mathrm{d}y$ indicates the Cauchy principal value. For example, (1) includes the case that $J(x,y)=c/|x-y|^{d+\alpha}$ with $\alpha\in(0,2)$ and $c>0$, which is the kernel corresponds to the fractional Laplacian. Throughout this paper, we assume that J is a

*The project was supported by the Fujian Province Natural Science Foundation Project (Grant Nos: 2014J01001; 2015J01238), the National Natural Science Foundation of China (Grant No. 11601083) and the Science and Technology Project of the Education Department of Fujian Province (Grant No. JA12060).

E-mail: linhn@fjnu.edu.cn.

Received July 28, 2017. Revised August 3, 2017.

Lévy-type kernel; this is, J satisfies that $J(x,x)=0$ for all $x\in R^d$ and

$$\sup_{x\in R^d}\int(1\wedge|x-y|^2)J(x,y)\mathrm{d}y<\infty.$$

For a smooth kernel J with compact support, it is proved in [1] that the solution u of the equation (1) has decay estimate

$$\|u_t\|_{L^q(R^d)}\leqslant ct^{-(d/2)(1-1/q)}\|u_0\|_1 \tag{2}$$

for any $q\in[1,\infty)$ and t large. Note that this decay rate is the same as the one that holds for solutions of the classical diffusion heat equation. When the jump kernel $J(x,y)$ has lower bound of the form

$$J(x,y)\geqslant c_1|x-y|^{-(d+2\sigma)}$$

for all $|x-y|>c_2$ with $\sigma\in(0,1)$ and some constants $c_1,c_2>0$, it is proved in [2] that

$$\|u_t\|_{L^q(R^d)}\leqslant ct^{-(d/(2\sigma))(1-1/q)}\|u_0\|_1 \tag{3}$$

for any $q\in[1,\infty)$ and t large. Both papers mentioned above adopt the so-called energy method by combining with Sobolev type inequalities. In particular, in [2] the fractional Sobolev-type inequality is used to derive (3). The motivation of this paper is twofold. First we want to see how energy methods can be applied to non-local Cauchy problems with more general kernels J, and second we establish weak Poincaré inequalities for non-local operators, which yield a more direct approach than these of [1, 2].

To taste main contribution of this paper, we present the following statement.

Theorem 1 Assume that there are constants $c_0>0$, $\alpha>0$ and $\beta\in R$ such that for any $x,y\in R^d$ with $|x-y|$ large enough,

$$J(x,y)\geqslant c_0\Big(1\wedge\frac{\ln^\beta(1+|x-y|)}{|x-y|^{d+\alpha}}\Big).$$

Let u be the weak solution of (1) associated to an initial condition $u_0\in L^q(R^d)\cap L^p(R^d)$ with $1\leqslant q<p<\infty$. Then, there exist positive constants c_1, t_0 such that for all $t\geqslant t_0$,

$$\|u_t\|_p\leqslant\begin{cases}c_1\big[t^{-d/\alpha}\ln^{-\beta d/\alpha}t\big]^{1/q-1/p}\|u_0\|_q, & \alpha\in(0,2),\ \beta\in R;\\ c_1\big[t^{-d/2}\ln^{-(1+\beta)d/2}t\big]^{1/q-1/p}\|u_0\|_q, & \alpha=2,\ \beta\geqslant-1;\\ c_1t^{-d(1/q-1/p)/2}\|u_0\|_q, & \alpha>2 \text{ or } \alpha=2,\ \beta<-1.\end{cases}$$

It is clear that the assertion above for the case $\alpha \in (0, 2)$ and $\beta \in R$ extends (3) proved in [2]. The assertion for the case $\alpha > 2$ (or $\alpha = 2$ and $\beta < -1$) indicates that (3) holds for a large class of jump kernels J which satisfies that $\sup_{x\in R^d} \int |x-y|^2 J(x,y)\mathrm{d}y < \infty$. The estimates in Theorem 1 are more delicate, e.g. see the case $\alpha = 2$ and $\beta \geqslant -1$. We believe that Theorem 1 could not be obtained by the fractional Sobolev-type inequality, due to the appearing of the logarithmic factor.

At the end of this section, we shall briefly present the idea of the proof. The symmetry assumption on J allows us to use the energy approach, as done in [3]. For simplicity, we will avoid the dependence on t of the function u. For $p > 1$, we multiply the equation (1) by $p|u|^{p-2}u$ and integrate, obtaining that

$$\begin{aligned}\partial_t \|u\|_p^p &= \langle \partial_t u, p|u|^{p-2}u\rangle_{L^2(R^d)}\\ &= \int p|u(t,x)|^{p-2}u(t,x)\Big[\text{p.v.}\int_{R^d}(u(t,y)-u(t,x))J(x,y)\mathrm{d}y\Big]\mathrm{d}x\\ &= -\frac{p}{2}\iint (u(y)-u(x))(|u(y)|^{p-2}u(y)-|u(x)|^{p-2}u(x))J(x,y)\mathrm{d}y\mathrm{d}x. \qquad (4)\end{aligned}$$

Now, we recall the following inequality: let $p > 1$ and $a, b \neq 0$, then, there exists a constant C depending only on p, such that

$$(a-b)(|a|^{p-2}a - |b|^{p-2}b) \geqslant C|a-b|^p, \qquad (5)$$

see, e.g. [2; Appendix]. Hence, combining with both the conclusions above, we obtain

$$\partial_t \|u\|_p^p \leqslant -c_p \iint |u(x)-u(y)|^p J(x,y)\mathrm{d}x\mathrm{d}y =: -c_p D_p(u,u). \qquad (6)$$

Therefore, the main task of our arguments is to present good estimates for $D_p(u,u)$.

§2. Preliminary Properties of Weak Solution

Throughout this paper, we assume that there exists a unique weak solution $u(t, x)$ to the equation (1). For simplicity, we denote by $u(t, x)$ this unique solution. This section is mainly concerned about some properties for this unique weak solution $u(t, x)$. As it is common in the literature, we adopt the weak solution for the problem (1) by formally multiplying the equation by a suitable test function and then integrating by parts, see [4; Section 2]. Define the following symmetric bilinear form

$$D(u,v) = \frac{1}{2}\int (u(x)-u(y))^2 J(x,y)\mathrm{d}x\mathrm{d}y, \qquad u, v \in \mathscr{F},$$

where

$$\mathscr{F} = \{u \in L^2(R^d) : D(u,u) < \infty\}.$$

Let $C_c^{\mathrm{lip}}(R^d)$ be the totality of Lipschitz continuous functions on R^d with compact support. Since

$$\sup_{x \in R^d} \int (1 \wedge |x-y|^2) J(x,y)\mathrm{d}y < \infty,$$

according to [5; Example 1.2.4], $C_c^{\mathrm{lip}}(R^d) \subset \mathscr{F}$ and the pair $(D, C_c^{\mathrm{lip}}(R^d))$ is a closable Markovian symmetric form on $L^2(R^d)$. Let $\mathscr{D}$ be the closure of $C_c^{\mathrm{lip}}(R^d)$ with respect to the norm $\|f\|_{D_1} := \sqrt{D(f,f) + \|f\|_2^2}$. Then, $(D, \mathscr{D})$ is a regular Dirichlet form on $L^2(R^d)$. Let $u_0 \in L^2(R^d)$, a function $u \in L^2((0,\infty) \times R^d)$ is called a weak solution of the problem (1), if for any $t_0 > 0$,

$$\int_0^{t_0} \int_{R^d} u \partial_t \varphi \mathrm{d}x\mathrm{d}t + \int_{R^d} \varphi(0,x) u_0(x)\mathrm{d}x = \int_0^{t_0} D(u(t), \varphi(t))\mathrm{d}t$$

for every $\varphi \in C_c^1((0,\infty) \times R^d)$.

The main result of this section is as follows:

Proposition 2 Let $u(t,x)$ be the unique weak solution $u(t,x)$ to the equation (1). Then, we have the following two statements:

(i) If $u_0 \in L^p(R^d)$ for some $1 \leqslant p \leqslant \infty$, then $u(t) \in L^p(R^d)$ for every $t > 0$, and $\|u(t)\|_p \leqslant \|u_0\|_p$. Even more, if $u_0 \geqslant 0$ then $\int_{R^d} u(t,x)\mathrm{d}x = \int_{R^d} u_0(x)\mathrm{d}x$.

(ii) For any $u_0 \in L^2(R^d)$, $\partial_t u(t) \in L^2(R^d)$ for every $t > 0$.

Proof Let $(T_t)_{t>0}$ and $(L, \mathscr{D}(L))$ be the Markovian semigroup and the generator associated with the regular Dirichlet form $(D, \mathscr{D})$, respectively. For any $u_0 \in L^2(R^d)$, define $u(t,x) = T_t u_0(x)$. Then, $u(t) \in \mathscr{D}(L)$ such that for any $v \in C_c^{\mathrm{lip}}(R^d)$ and $t > 0$,

$$\int Lu(t) v \mathrm{d}x = -D(u(t), v).$$

In particular, for any $\varphi \in C_c^1((0,\infty) \times R^d)$ and any $t_0 > 0$,

$$\int_0^{t_0} \int_{R^d} \partial_t u \varphi \mathrm{d}x\mathrm{d}t = \int_0^{t_0} \int_{R^d} Lu\varphi \mathrm{d}x\mathrm{d}t = -\int_0^{t_0} D(u(t), \varphi(t))\mathrm{d}t,$$

which implies that $u(t,x)$ is a weak solution to the equation (1).

By the assumption that there exists a unique weak solution $u(t,x)$ to the equation (1), we can assume that $u(t,x)$ satisfies the properties above. Then, assertion (ii) immediately follows from the argument above. On the other hand, it is well known that $(T_t)_{\geqslant 0}$ can be extended or restricted to a Markovian semigroup on $L^p(R^d)$ with $p \in [1,\infty]$, see e.g. [5] or [6]. By [7; Theorem 1.1], the Dirichlet form $(D,\mathscr{D})$ is conservative, i.e. $T_t 1 = 1$, a.e. for all $t>0$. Having these facts at hand, we also prove assertion (i). □

§3. General Results of Decay Rates for Non-Local Problems

In order to get the decay rates for non-local Cauchy problems, we will study weak-Poincaré inequalities for energy form $D_p(u,u)$. There may be two ways to establish such functional inequalities. The first one is based on the information on the jump kernels J. This approach is more direct, but the assumption that $J(x,y)$ is positive everywhere is required. For the second argument we make use of the Fourier analysis and the comparison with the operator of the convolution form. Despite of the complexity, such idea can give us sharper estimates than those yielded by the first one. We remark that Fourier methods are not directly applicable to the jump kernels which are not in convolution form.

3.1 The Case that $J(x,y)$ is Positive Everywhere

Proposition 3 For any $1 \leqslant q < p < \infty$, there exist two constants $c_1, c_2 > 0$ such that for all $r > 0$,

$$\|f\|_p^p \leqslant c_1 \beta(c_2 r) D_p(f,f) + r\|f\|_q^p, \qquad f \in L^q(R^d) \cap L^p(R^d), \tag{7}$$

where

$$\beta(r) = r^{q/(p-q)} \sup_{|x'-y'| \leqslant r^{-q/(d(p-q))}} J(x',y')^{-1}. \tag{8}$$

Proof For any $f \in L^q(R^d) \cap L^p(R^d)$ with $1 \leqslant q < p < \infty$ and for any $s > 0$, define

$$f_s(x) := \frac{1}{|B(0,s)|} \int_{B(x,s)} f(z)\mathrm{d}z.$$

We have, by the Hölder inequality,

$$\begin{aligned} \|f_s\|_\infty &\leqslant \frac{1}{|B(0,s)|} \sup_x \int_{B(x,s)} |f(z)|\mathrm{d}z \\ &\leqslant \sup_x \Big(\frac{1}{|B(0,s)|} \int_{B(x,s)} |f(z)|^q \mathrm{d}z \Big)^{1/q} \\ &\leqslant |B(0,s)|^{-1/q} \|f\|_q, \end{aligned}$$

and so, by Jensen's inequality and Fubini's theorem, for any $1 \leqslant q < p < \infty$,

$$\begin{aligned}
\|f_s\|_p^p &\leqslant \|f_s\|_\infty^{p-q} \int |f_s(z)|^q \mathrm{d}z \\
&= \|f_s\|_\infty^{p-q} \int \Big| \frac{1}{|B(0,s)|} \int_{B(z,s)} f(y)\mathrm{d}y \Big|^q \mathrm{d}z \\
&\leqslant \|f_s\|_\infty^{p-q} \int \frac{1}{|B(0,s)|} \int_{B(z,s)} |f(y)|^q \mathrm{d}y \mathrm{d}z \\
&= \|f_s\|_\infty^{p-q} \frac{1}{|B(0,s)|} \int |f(y)|^q \mathrm{d}y \int_{B(y,s)} \mathrm{d}z \\
&= \|f_s\|_\infty^{p-q} \|f\|_q^q \\
&\leqslant |B(0,s)|^{-(p-q)/q} \|f\|_q^p.
\end{aligned}$$

Therefore, for any $f \in L^q(R^d) \cap L^p(R^d)$ with $1 \leqslant q < p < \infty$ and for any $s > 0$, by the inequality that $(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$ for all $a, b \geqslant 0$ and Hölder's inequality,

$$\begin{aligned}
\|f\|_p^p &\leqslant 2^{p-1}\|f - f_s\|_p^p + 2^{p-1}\|f_s\|_p^p \\
&\leqslant 2^{p-1} \int \Big(\frac{1}{|B(0,s)|} \int_{B(x,s)} |f(x) - f(y)|\mathrm{d}y \Big)^p \mathrm{d}x + 2^{p-1}|B(0,s)|^{-(p-q)/q}\|f\|_q^p \\
&= 2^{p-1} \int \Big(\frac{1}{|B(x,s)|} \int_{B(x,s)} |f(x) - f(y)|\mathrm{d}y \Big)^p \mathrm{d}x + 2^{p-1}|B(0,s)|^{-(p-q)/q}\|f\|_q^p \\
&\leqslant 2^{p-1} \int \Big(\frac{1}{|B(0,s)|} \int_{B(x,s)} |f(x) - f(y)|^p \mathrm{d}y \Big) \mathrm{d}x + 2^{p-1}|B(0,s)|^{-(p-q)/q}\|f\|_q^p \\
&\leqslant \left(\frac{2^{p-1} \sup\limits_{0<|x'-y'|\leqslant s} J(x', y')^{-1}}{|B(0,s)|} \right) \int \int_{B(x,s)} |f(x) - f(y)|^p J(x,y) \mathrm{d}y \mathrm{d}x \\
&\quad + 2^{p-1}|B(0,s)|^{-(p-q)/q}\|f\|_q^p \\
&\leqslant \left(\frac{2^{p-1} \sup\limits_{0<|x'-y'|\leqslant s} J(x', y')^{-1}}{|B(0,s)|} \right) D_p(f,f) + 2^{p-1}|B(0,s)|^{-(p-q)/q}\|f\|_q^p.
\end{aligned}$$

This yields the desired assertion by setting $r = 2^{p-1}|B(0,s)|^{-(p-q)/q}$ in the inequality above. $\square$

The inequality (7) in Proposition 3 is the so-called $L_p \to L_q$-weak Poincaré inequality in the literatures, which was first established in [8] for the case $p = 2$ and was used to describe various decay of Markov semigroups. Note that, the function $\beta(r) < \infty$ only when $J(x, y)$ is positive everywhere. On the other hand, since $D_p(f, f) \geqslant 0$, without loss of generality we may and do assume that β is decreasing; otherwise we can use $\inf\limits_{s \leqslant r} \beta(s)$ to replace $\beta(r)$. We will use the following statement.

Theorem 4 Let β be the function given by (8). Assume that the function β is decreasing. Then, for any $1 \leqslant q < p < \infty$, there exist positive constants c_1, t_0 such that for all $t \geqslant t_0$,

$$\|u_t\|_p \leqslant G^{-1}(c_1 t)^{1/p}\|u_0\|_q,$$

where

$$G(r) = \int_r^1 \frac{\beta(s)}{s}\mathrm{d}s$$

and $G^{-1}(r) = \inf\{s > 0 : G(s) \leqslant r\}$.

Proof According to (6) and Proposition 3,

$$\partial_t\|u_t\|_p^p \leqslant -c_p D_p(u_t, u_t) \leqslant \frac{-\|u_t\|_p^p + r\|u_t\|_q^p}{c_3\beta(c_2 r)}, \qquad r > 0.$$

Using the fact that $\|u_t\|_q \leqslant \|u_0\|_q$ for all $t \geqslant 0$, and taking $r = \|u_t\|_p^p/(2\|u_0\|_q^p)$ in the inequality above, we get that

$$\partial_t\|u_t\|_p^p \leqslant -c_4\|u_t\|_p^p\Big(\beta\frac{c_2\|u_t\|_p^p}{2\|u_0\|_q^p}\Big)^{-1}.$$

If $\|u_t\|_p^p \geqslant 2\|u_0\|_q^p/c_2$ for every $t > 0$, then the inequality above along with the fact that β is decreasing gives us

$$\partial_t\|u_t\|_p^p \leqslant -c_5\|u_t\|_p^p, \qquad t > 0,$$

which implies that $\lim\limits_{t\to\infty}\|u_t\|_p^p = 0$ and this is a contradiction. Hence, there exists $t_0 \in [0,\infty)$ such that $\|u_t\|_p^p \leqslant 2\|u_0\|_q^p/c_2$ for $t = t_0$. Due to (6), the function $t \mapsto \|u_t\|_p^p$ is decreasing, and so $\|u_t\|_p^p \leqslant 2\|u_0\|_q^p/c_2$ for any $t \geqslant t_0$.

In the following, for any $t \geqslant t_0$ and $r > 0$, set $f(t) = c_2\|u_t\|_p^p/(2\|u_0\|_q^p)$ and $\varphi(r) = r/\beta(r)$. Then, we have

$$\begin{cases} f'(t) \leqslant -c_6\varphi(f(t)), & t > t_0, \\ f(t_0) \leqslant 1. \end{cases}$$

Furthermore, using the fact that $f(t)$ is a decreasing function,

$$\begin{aligned} G(f(t)) &= \int_{f(t)}^1 \frac{1}{\varphi(s)}\mathrm{d}s \\ &= \int_{f(t_0)}^1 \frac{1}{\varphi(s)}\mathrm{d}s + \int_{f(t)}^{f(t_0)} \frac{1}{\varphi(s)}\mathrm{d}s \\ &= \int_{f(t_0)}^1 \frac{1}{\varphi(s)}\mathrm{d}s - \int_{t_0}^t \frac{1}{\varphi(f(s))}\mathrm{d}f(s) \\ &\geqslant c_7 + c_6(t - t_0), \end{aligned}$$

which along with the decreasing property of G yields that

$$f(t) \leqslant G^{-1}(c_7 + c_6(t - t_0)).$$

In particular, there exist two positive constants c_8, t_1 such that

$$f(t) \leqslant G^{-1}(c_8 t), \qquad t \geqslant t_1.$$

The required assertion immediately follows from the inequality above. $\square$

Example 5 Assume that there are constants c_0, $\alpha > 0$ and $\beta \in R$ such that for any $x, y \in R^d$,

$$J(x,y) \geqslant c_0\Big(1 \wedge \frac{\ln^{\beta}(1+|x-y|)}{|x-y|^{d+\alpha}}\Big).$$

Then, for any $1 \leqslant q < p < \infty$, there exist positive constants c_1, t_0 such that

$$\|u_t\|_p \leqslant c_1[t^{-1}\ln^{-\beta} t]^{(d/\alpha)(1/q-1/p)}\|u_0\|_q, \qquad t \geqslant t_0.$$

Proof According to the assumption on $J(x,y)$, there is a constant $c_1 > 0$ such that for $r \in (0,1)$,

$$\beta(r) = r^{-\alpha q/(d(p-q))}\ln^{-\beta}(1/r).$$

Then, for any $r \in (0,1)$,

$$G(r) = c_2 r^{-\alpha q/(d(p-q))}\ln^{-\beta}(1/r).$$

Therefore, for any $r > 0$ large enough,

$$G^{-1}(r) = c_3[r^{-1}\ln^{-\beta} r]^{d(p-q)/(\alpha q)}.$$

The proof is complete. $\square$

3.2 The General Case that $J(x,y)$ is Only Non-Negative

To consider the general case that $J(x,y)$ is only non-negative, we will compare with an equation in the convolution form. We note that the convolution form of the equation allows the use of Fourier analysis to obtain decay bounds.

Proposition 6 Assume that

$$J(x,y) \geqslant J_0(x-y), \qquad x, y \in R^d,$$

where J_0 is a non-negative measurable function on R^d. Then, there exists a constant $c > 0$ such that for any $r > 0$,

$$\|f\|_2^2 \leqslant \Phi(cr^{1/d}) D_2(f,f) + r\|f\|_1^2, \qquad f \in L^1(R^d) \cap L^2(R^d),$$

where

$$\Phi(r) = \frac{1}{\inf_{|\xi|\geqslant r} \phi(\xi)} \qquad \text{and} \qquad \phi(\xi) = \int (1 - \cos\langle z, \xi\rangle) J_0(z) \mathrm{d}z.$$

Proof Define

$$D_{0,2}(f,g) = \int (f(x) - f(y))(g(x) - g(y)) J_0(x-y) \mathrm{d}x \mathrm{d}y, \qquad f, g \in C_c^\infty(R^d).$$

By the Fourier transform, we have

$$D_{0,2}(f,g) = \int \widehat{g}(\xi) \overline{\widehat{f}}(\xi) \phi(\xi) \mathrm{d}\xi,$$

where

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int \mathrm{e}^{-\langle \xi, z\rangle} f(z) \mathrm{d}z$$

is the Fourier transform of f. Thus, for every $s > 0$,

$$\int_{\{|\xi|\geqslant s\}} |\widehat{f}(\xi)|^2 \mathrm{d}\xi \leqslant \Phi(s) \int |\widehat{f}(\xi)|^2 \phi(\xi) \mathrm{d}\xi \leqslant \Phi(s) D_{0,2}(f,f).$$

Therefore, for any $s > 0$,

$$\begin{aligned} \|f\|_2^2 &= \int |\widehat{f}(\xi)|^2 \mathrm{d}\xi \\ &= \int_{\{|\xi|\geqslant s\}} |\widehat{f}(\xi)|^2 \mathrm{d}\xi + \int_{\{|\xi|< s\}} |\widehat{f}(\xi)|^2 \mathrm{d}\xi \\ &\leqslant \Phi(s) D_{0,2}(f,f) + cs^d \|f\|_1^2. \end{aligned}$$

By setting $r = cs^d$ in the inequality above, we prove the desired assertion by the fact that $D_{0,2}(f,f) \leqslant D_2(f,f)$ for all $f \in C_c^\infty(R^d)$. □

Theorem 7 For any $1 \leqslant q < p < \infty$, there exist positive constants c_0 and t_0 such that for all $t \geqslant t_0$,

$$\|u_t\|_p \leqslant (G^{-1}(c_0 t))^{1/q - 1/p} \|u_0\|_q,$$

where

$$G(r) = \int_r^1 \frac{\Phi(s^{1/d})}{s} \mathrm{d}s$$

and $\Phi(r)$ is defined in Proposition 6.

Proof According to Proposition 6 and the proof of Theorem 4, there exist positive constants c_1 and t_0 such that for all $t \geqslant t_0$,

$$\|u_t\|_2 \leqslant G^{-1}(c_1 t)^{1/2}\|u_0\|_1.$$

Using this and the iteration argument, we can arrive that for any $k \geqslant 1$ there exist $c_2, t_1 > 0$ such that for all $t \geqslant t_1$,

$$\|u_t\|_{2^k q} \leqslant \prod_{i=1}^{k}(G^{-1}(c_2 t))^{1/(2^i q)}\|u_0\|_q = (G^{-1}(c_2 t))^{(1/q)(1-1/2^k)}\|u_0\|_q.$$

Now, for any $1 \leqslant q < p$, we take $k \geqslant 1$ such that $2^k q \geqslant p$. Setting

$$\theta = \Big(1 - \frac{q}{p}\Big)\Big(1 - \frac{1}{2^k}\Big)^{-1} \in (0,1),$$

and using the interpolation and the fact that $\|u_t\|_q \leqslant \|u_0\|_q$ for all $t > 0$, we get that for all $t \geqslant t_1$,

$$\begin{aligned}\|u_t\|_p &\leqslant \|u_t\|_{2^k q}^{\theta}\|u_t\|_q^{1-\theta}\\ &\leqslant \big((G^{-1}(c_2 t))^{(1/q)(1-1/2^k)}\|u_0\|_q\big)^{\theta}\|u_0\|_q^{1-\theta}\\ &\leqslant (G^{-1}(c_2 t))^{1/q-1/p}\|u_0\|_q.\end{aligned}$$

The proof is complete. □

In the following, we assume that J_0 is radial and will establish some estimates for this case. For any $\xi \in R^d$, using the fact that J_0 is radial and the inequality that

$$1 - \cos r \geqslant \frac{\cos 1}{2} r^2, \qquad r \in (-1,1),$$

we obtain

$$\begin{aligned}\phi(\xi) &= \int (1 - \cos\langle \xi, z\rangle) J_0(|z|)\mathrm{d}z\\ &= \int (1 - \cos(|\xi| z_1)) J_0(|z|)\mathrm{d}z\\ &\geqslant \frac{\cos 1}{2}|\xi|^2 \int_{\{|\xi||z_1| \leqslant 1\}} |z_1|^2 J_0(|z|)\mathrm{d}z\\ &\geqslant \frac{\cos 1}{2d}|\xi|^2 \int_{\{|z| \leqslant 1/|\xi|\}} |z|^2 J_0(|z|)\mathrm{d}z.\end{aligned} \tag{9}$$

Recall that a measurable and positive function $l : (1,\infty) \to (0,\infty)$ is said to vary regularly at infinite with index α if for every $\lambda > 0$,

$$\lim_{r\to\infty} \frac{l(\lambda r)}{l(r)} = \lambda^{\rho}.$$

Proposition 8 The following statements hold.

(i) Assume that $J_0(|z|) \geqslant l(|z|)/|z|^d$, where l varies regularly at infinite with index $-\alpha \in (-2,0]$. Then, there is a constant $c_1 > 0$ such that for all $\xi \in R^d$ with $|\xi|$ small enough, $\phi(\xi) \geqslant c_1 l(1/|\xi|)$. In particular, if there are $\alpha \in [0,2)$, $\beta \in R$ and $c_2 > 0$ such that for $z \in R^d$ with $|z|$ large enough,

$$J_0(|z|) \geqslant \frac{c_2}{|z|^{d+\alpha}} \ln^\beta |z|.$$

Then there is a constant $c_3 > 0$ such that for all $\xi \in R^d$ with $|\xi|$ small enough,

$$\phi(\xi) \geqslant c_3 |\xi|^\alpha \ln^\beta(1/|\xi|).$$

(ii) Assume that there are two constants $c_4 > 0$ and $\beta > -1$ such that for $z \in R^d$ with $|z|$ large enough,

$$J_0(|z|) \geqslant \frac{c_4}{|z|^{d+2}} \ln^\beta |z|.$$

Then, there is a constant $c_5 > 0$ such that for all $\xi \in R^d$ with $|\xi|$ small enough,

$$\phi(\xi) \geqslant c_5 |\xi|^2 \ln^{1+\beta}(1/|\xi|).$$

(iii) If $\int_{R^d} |z|^2 J_0(|z|) \mathrm{d}z < \infty$, then there is a constant $c_6 > 0$ such that for all $\xi \in R^d$ with $|\xi|$ small enough, $\phi(\xi) \geqslant c_6 |\xi|^2$.

Proof The assertion (i) is a consequence of Karamata's theorem ([9; Proposition 1.5.8]) and (9). By some calculations, the assertion (ii) follows from (9). It is also easy to see that the assertion (iii) immediately follows from (9) and the assumption that $\int_{R^d} |z|^2 J_0(|z|) \mathrm{d}z < \infty$. $\square$

As an application of all the results in this subsection, we present the following examples. In particular, the assertion in Example 9 (i) below improves that of Example 5.

Example 9 (i) Assume that there are constants $c_0 > 0$, $\alpha > 0$ and $\beta \in R$ such that for any $x, y \in R^d$ with $|x-y|$ large enough,

$$J(x,y) \geqslant c_0 \Big(1 \wedge \frac{\ln^\beta(1+|x-y|)}{|x-y|^{d+\alpha}}\Big).$$

Then, for any $1 \leqslant q < p < \infty$, there exist positive constants c_1, t_0 such that for all $t \geqslant t_0$,

$$\|u_t\|_p \leqslant \begin{cases} c_1[t^{-d/\alpha} \ln^{-\beta d/\alpha} t]^{1/q-1/p} \|u_0\|_q, & \alpha \in (0,2),\ \beta \in R; \\ c_1[t^{-d/2} \ln^{-(1+\beta)d/2} t]^{1/q-1/p} \|u_0\|_q, & \alpha = 2,\ \beta \geqslant -1; \\ c_1 t^{-(d/2)(1/q-1/p)} \|u_0\|_q, & \alpha > 2 \text{ or } \alpha = 2,\ \beta < -1. \end{cases}$$

(ii) Assume that there are constants $c_0 > 0$ and $\beta < -1$ such that for any $x, y \in R^d$ with $|x-y|$ large enough,

$$J(x,y) \geqslant c_0\Big(1 \wedge \frac{\ln^{\beta}(1+|x-y|)}{|x-y|^d}\Big).$$

Then, for any $1 \leqslant q < p < \infty$, there exist positive constants c_1, t_0 such that for all $t \geqslant t_0$,

$$\|u_t\|_p \leqslant \exp\Big[-c_1\Big(\frac{1}{q}-\frac{1}{p}\Big)t^{1/(1-\beta)}\Big]\|u_0\|_q.$$

Proof (i) When $\alpha \in (0,2)$ and $\beta \in R$, we apply Proposition 8 (i) and get that for $r > 0$ small enough,

$$G(r) \asymp r^{-\alpha/d}\ln^{-\beta}(1/r).$$

For the case that $\alpha = 2$ and $\beta \geqslant -1$, it follows from Proposition 8 (ii) that for $r > 0$ small enough,

$$G(r) \asymp r^{-2/d}\ln^{-(1+\beta)}(1/r).$$

Note that, if $\alpha > 2$ or $\alpha = 2$ with $\beta < -1$, then $\int_{R^d}|z|^2 J_0(|z|)\mathrm{d}z < \infty$. Hence, by Proposition 8 (iii), we find that

$$G(r) \asymp r^{-2/d}.$$

The proof is complete. $\square$

Acknowledgements The author would like to thank the editors and two anonymous referees for their helpful comments and suggestions. He is also grateful to Wang Jian for his very valuable suggestions and helpful comments.

References

[1] Ignat L I, Rossi J D. A nonlocal convection-diffusion equation [J]. *J. Funct. Anal.*, 2007, **251(2)**: 399–437.

[2] Chasseigne E, Felmer P, Rossi J D, et al. Fractional decay bounds for nonlocal zero order heat equations [J]. *Bull. Lond. Math. Soc.*, 2014, **46(5)**: 943–952.

[3] Ignat L I, Rossi J D. Decay estimates for nonlocal problems via energy methods [J]. *J. Math. Pures Appl.*, 2009, **92(2)**: 163–187.

[4] Brändle C, de Pablo A. Nonlocal heat equations: decay estimates and Nash inequalities [OL]. 2015 [2015-11-10]. https://arxiv.org/abs/1312.4661v4.

[5] Fukushima M, Oshima Y, Takeda M. *Dirichlet Forms and Symmetric Markov Processes* [M]. 2nd rev. and ext. ed. Berlin: Walter de Gruyter GmbH, 2011.

[6] Farkas W, Jacob N, Schilling R L. Feller semigroups, L^p-sub-Markovian semigroups, and applications to pseudo-differential operators with negative definite symbols [J]. *Forum Math.*, 2001, **13(1)**: 51–90.

[7] Masamune J, Uemura T, Wang J. On the conservativeness and the recurrence of symmetric jump-diffusions [J]. *J. Funct. Anal.*, 2012, **263(12)**: 3984–4008.

[8] Röckner M, Wang F Y. Weak Poincaré inequalities and L_2-convergence rates of Markov semigroups [J]. *J. Funct. Anal.*, 2001, **185(2)**: 564–603.

[9] Bingham N H, Goldie C M, Teugels J L. *Regular Variation* [M]. Cambridge: Cambridge University Press, 1987.

非局部柯西问题弱解的衰减速度

林 火 南

(福建师范大学数学与信息学院, 福州, 350117)

摘 要: 考虑如 (1) 所示的非局部柯西问题弱解 $(u_t)_{t\geqslant 0}$ 的渐近性质. 仅利用跳核 $J(x,y)$ 当 $|x-y|$ 充分大时下界的性质, 本文证明了对于任意 $1\leqslant q<p<\infty$ 及充分大 t, $\|u_t\|_p\leqslant c(t)\|u_0\|_q$ 成立, 同时给出 $c(t)$ 的最优显示估计式.

关键词: 非局部柯西问题; 跳核; 弱 Poincaré 不等式

中图分类号: O211.62; O211.63; O211.9