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Asymptotic Finite-Time Ruin Probability for Multivariate Heavy-Tailed Claims *

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Abstract: This note deals with an insurance company with multiple lines of business. In the context of heavy-tailed heterogeneous claim amounts with the 1st upper-orthant tail dependence, based on the so-called k-out-of-n ruin set, we can exhibit the Radon measure μ and derive the asymptotic ruin probability for some of all lines business to ruin in a finite time. One numerical example is also presented to illustrate our main results.

Keywords: multivariate regular variation; k-out-of-n ruin set; upper-orthant tail dependence

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§1. Introduction

Insurance companies in the business of risks pool together risks faced by individuals or business companies and compensate the potential loss so as to offset their financial burden. To ensure the promised obligations, the insurance company sets aside amount called the reserve or surplus from which it can draw when claims are due. The ruin happens whenever the reserve fails to cover the claim amounts. So it plays the important role to evaluate the probability to ruin in insurance industry. The ruin probability with respect to the univariate risk has been intensively investigated in the past decades, readers may refer to [1], a comprehensive monograph, and the references therein for more details in literature.

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This paper studies the insurance company with multiple lines of business, which are exposed to catastrophic risks such as earthquakes, floods, hurricanes or terrorist attacks etc. Since such risks usually have an substantial effect on all lines of business at the same time, the statistical dependence among claims from these lines should be considered. To the best of our knowledge, fewer studies on the ruin probability with respect to multivariate risks were conducted in 10 years. To name a few, Collamore [2] was one among the first to introduce the high dimensional general ruin set problem with respect to light-tailed claims, we also refer readers to [3] and [4] and references therein for light-tailed claims. For the multi-dimensional heavy-tailed claim processes, Hult et al. [5] firstly studied the ruin probability in the context of multivariate regularly varying random walks and presented sharp asymptotic general ruin boundaries. Afterward, for the claim amount vector with multivariate regularly varying distributions, Hult and Lindskog [6] derived the asymptotic decay of the ruin probability as the initial capital is large. And then, Biard et al. [7] relaxed the independence and stationarity in the renewal risk model and successfully extended some asymptotic results on finite-time ruin probabilities of heavy-tailed claim amounts. Hereafter, Biard [8] again obtained the asymptotic finite-time ruin probability in the context of homogeneous regularly varying marginal claim amounts with some dependence structures. Recently, under the framework of heterogeneous heavy-tailed marginal claim amounts with certain dependence structures, Li et al. [9] derived the asymptotic finite-time ruin probability and developed the optimal allocation of the global initial reserve in the sense of minimizing the asymptotic ruin probability.

On the other hand, recent research works on the finite-time ruin problem in the literature were developed only for multivariate tail-independent claim amount. However, it is not uncommon to confront with claim amount vectors which are not tail-independent. So, along this line, in this study we consider the multivariate ruin set based on the well known k-out-of-n fault tolerant reliability system, and our research work here focuses on the multivariate finite-time k-out-of-n ruin probability in the framework of heterogeneous heavy-tailed marginal claim amounts with upper-orthant tail dependence. The rest of this paper is organized as follows: In Section 2, we present some important notions to be used throughout this paper. Section 3 introduces the k-out-of-n ruin set and the r-th upper-orthant tail dependence. In Section 4 we present the asymptotic multivariate finite-time ruin probability of the the k-out-of-n ruin set.

Throughout the paper, random vectors and real vectors are denoted by bold English letters in upper and lower cases, respectively. For example, $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)'$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)'$, the L_1 norm $\|\boldsymbol{X}\| = |X_1| + |X_2| + \cdots + |X_n|$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) > (b_1, b_2, \ldots, b_n) = \boldsymbol{b}$ means $a_i > b_i$ for all $i = 1, 2, \ldots, n$ and $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)'$. We also denote $\mathbf{1} = (1, 1, \ldots, 1)'$, $\mathbf{0} = (0, 0, \cdots, 0)'$, $\boldsymbol{\infty} = (\infty, \infty, \cdots, \infty)'$

and $\overline{\mathbb{R}} = [-\infty, +\infty]$.

§2. Preliminaries

The regularly varying random variables are usually employed to describe losses due to catastrophic risks with heavy-tails. For a comprehensive presentation of theory, methods and applications of heavy-tailed distributions, we refer readers to [10].

A real function L is said to be *regularly varying* with order $\alpha > 0$ (written as $\mathscr{R}_\alpha$) if $L(tx)/L(t) \to x^{-\alpha}$ for all $x > 0$ as $t \to \infty$. In this note we will study multivariate claim amounts and thus the multivariate version of regular variation is at the center of our discussion.

Definition 1 (Multivariate regular variation) A multivariate random vector $\boldsymbol{X} \in \mathbb{R}^n$ with an unbounded support is said to be *regularly varying* if there exists some nonzero Radon measure μ such that

$$\lim_{t\to\infty} \frac{\mathsf{P}(\boldsymbol{X} \in tA)}{\mathsf{P}(\|\boldsymbol{X}\| > t)} = \mu(A), \tag{1}$$

for any Borel set $A \in \mathbb{R}^n$ with $\mu(\partial A) = 0$ and bounded away from $\mathbf{0}$.

Since for the Radon measure μ in Definition 1, there exists some $\alpha > 0$ such that $\mu(tA) = t^{-\alpha}\mu(A)$ holds for any $t > 0$ and each Borel set $A \subset \mathbb{R}^n$ bounded away from $\mathbf{0}$, $\boldsymbol{X}$ is also said to be *regularly varying with index* α and *limiting measure* μ (denoted as $\boldsymbol{X} \in \mathscr{M}\mathscr{R}_{\alpha,\mu}$). Also the following equivalent definition for the multivariate regular variation is quite convenient in some occasions.

Definition 2 [10] A multivariate random vector $\boldsymbol{X} \in \mathbb{R}^n$ with an unbounded support is *regularly varying* if there exists a Radon measure ν on $[\mathbf{0}, \infty]\backslash\{\mathbf{0}\}$ such that

$$\lim_{t\to\infty} \frac{\mathsf{P}(\boldsymbol{X}/t \in [\mathbf{0}, \boldsymbol{x}]^c)}{\mathsf{P}(\boldsymbol{X}/t \in [\mathbf{0}, \mathbf{1}]^c)} = \nu([\mathbf{0}, \boldsymbol{x}]^c), \tag{2}$$

for any $\boldsymbol{x} \in [\mathbf{0}, \infty)\backslash\{\mathbf{0}\}$, here $\nu([\mathbf{0}, \cdot]^c)$ is continuous.

§3. Ruin Set and Upper Tail Dependence

To describe the reserve of an insurance company with n lines of business, we consider a multivariate risk process

$$\boldsymbol{R}_t = u\boldsymbol{a} + t\boldsymbol{c} - \boldsymbol{S}_t, \tag{3}$$

where u is the *global initial reserve*, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)' \in (0, 1)^n$ with $a_1 + a_2 + \cdots + a_n = 1$ is the vector according to which the initial capital u is allocated to all n lines and

$\boldsymbol{c} = (c_1, c_2, \ldots, c_n)' \in (0, \infty)^n$ is the vector of *premium rates*, $\boldsymbol{S}_t = \sum\limits_{i=1}^{N(t)} \boldsymbol{X}_i$ gives the vector of aggregate claim amounts, $N(t)$ is a Poisson process with $\lambda > 0$, and $\boldsymbol{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in})'$ is a sequence of i.i.d. random claim vectors, which are independent of $N(t)$. From now on, we write $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)'$ for a generic element of $\boldsymbol{X}_i$'s and denote $\overline{F}_i = 1 - F_i$ be the survival function of the marginal X_i, $i = 1, 2, \ldots, n$.

The univariate finite-time ruin probability is defined as, for $u, t^* > 0$,

$$\begin{aligned}\psi(u, t^*) &= \mathsf{P}(R_t < 0 \text{ for some } t \in [0, t^*] \,|\, R_0 = u) \\ &= \mathsf{P}\Big(\sup_{[0,t^*]} \{S_t - ct\} > u\Big).\end{aligned}$$

In the multivariate case, the ruin may occur in many different ways as a result of various concerns on the safety of the insurance company. To name a few, we list here some versions of the finite-time ruin probability based on the corresponding ruin sets. One may see for example [6, 9, 11, 12] for more details.

- The probability that the sum of all line reserves becomes negative before t^*,

$$\psi_+(u, t^*) = \mathsf{P}\Big(\sup_{[0,t^*]} \Big\{\sum_{j=1}^{n} (S_t^{(j)} - c_j t)\Big\} > u\Big);$$

- The probability that at least one line's reserve becomes negative before t^*,

$$\psi_\vee(u, t^*) = \mathsf{P}\Big(\bigcup_{j=1}^{n} \Big\{\sup_{[0,t^*]} (S_t^{(j)} - c_j t) > a_j u\Big\}\Big);$$

- The probability that different fraction of each positive line fail to cover the negative position of other lines somewhere before t^*,

$$\psi_{\boldsymbol{\omega}}(u, t^*) = \mathsf{P}(\boldsymbol{R}_t \in \Gamma_{\boldsymbol{\omega}} \text{ for some } t \in [0, t^*]),$$

  where, for $\boldsymbol{\omega'} = (\omega_1, \omega_2, \ldots, \omega_n) \in [0, 1]^n$,

$$\Gamma_{\boldsymbol{\omega}} = \Big\{\boldsymbol{x} : \sum_{k=1}^{n} \omega_k (x_k)_+ < \sum_{k=1}^{n} (-x_k)_+\Big\}.$$

Motivated by the well known k-out-of-n fault tolerant reliability system [13], we introduce the ruin set

$$\Gamma_k = \Big\{\boldsymbol{x} : \sum_{i=1}^{n} I(x_i) \leqslant k\Big\}, \qquad \text{for } k = 0, 1, \ldots, n-1. \tag{4}$$

where $I(x) = 1$ or 0 according to $x \geqslant 0$ or $x < 0$, respectively. Suppose the ruin occurs for a multivariate risk process whenever the risk reserve $\boldsymbol{R}_t$ hits Γ_k at some time $t \leqslant t^*$, a predetermined time point. Then, the ruin probability may be explicitly expressed as

$$\phi_k(u, t^*) = \mathsf{P}(\boldsymbol{R}_t \in \Gamma_k \text{ for some } t \in [0, t^*]). \tag{5}$$

Evidently, the ruin set Γ_k corresponds to at most k lines' nonnegative reserves. For example, Γ_0 means that all lines' reserves are negative, Γ_{n-1} means that at least one line's reserve is negative, and it holds that, for all t^* and u,

$$\psi_+(u,t^*) = \psi_{\mathbf{1}}(u,t^*) \geqslant \phi_0(u,t^*), \qquad \psi_\vee(u,t^*) = \psi_{\mathbf{0}}(u,t^*) = \phi_{n-1}(u,t^*).$$

For ease of reference, we state one important proposition, which plays a key rule in deriving the asymptotic ruin probability in Section 4.

Proposition 3[6] For the risk process $\boldsymbol{R}_t$ in (3) with the claim amounts vector $\boldsymbol{X} \in \mathscr{MR}_{\alpha,\mu}$ for some $\alpha > 1$, we have

$$\lim_{u\to\infty} \frac{\phi_k(u,t^*)}{\mathsf{P}(\|\boldsymbol{X}\| > u)} = \lambda t^* \mu(\boldsymbol{a} - \Gamma_k), \qquad \text{for } t^* > 0, \tag{6}$$

where $\boldsymbol{a} - \Gamma_k = \{\boldsymbol{x} : \boldsymbol{x} = \boldsymbol{a} - \boldsymbol{y},\, \boldsymbol{y} \in \Gamma_k\}$.

Actually, after assigning the dependence structure among claim amounts from all lines of business, we can exhibit μ and then derive the asymptotic ruin probability. As forementioned in Section 1, the asymptotic tail independence especially plays an important role in measuring the interdependence among risk claims. In literature, the upper tail dependence of bivariate copulas has been discussed extensively ever since it was introduced by Joe[14], and a multivariate version of tail dependence was introduced and studied by Definition 7.1 of [15] and Definition 1.1 of [16]. Inspired by their work, we introduce the r-th upper-orthant tail dependence as to further refine the tail dependence. Let $N = \{1, 2, \ldots, n\}$ and $|J|$ be the number of elements in the subset $J \subseteq N$.

Definition 4 (r-th upper-orthant tail dependence) A multivariate random vector $\boldsymbol{X} \in \mathbb{R}^n$ is said to be r-th upper-orthant tail dependent ($1 \leqslant r \leqslant n-1$), if there exists one $i \in N$ and some nonempty $J \subseteq N$ with $|J| = r$ such that, for all $\boldsymbol{x} > \boldsymbol{0}$,

$$\tau_r(x_j \,|\, x_i : j \in J) = \lim_{t\to\infty} \mathsf{P}\Big(\bigcap_{j\in J, i\notin J} \{X_j > tx_j\} \,|\, X_i > tx_i \Big) > 0,$$

and for all $i \in N$ and $J_1 \subseteq N$ with $|J_1| = r+1$,

$$\tau_{r+1}(x_j \,|\, x_i : j \in J_1) = \lim_{t\to\infty} \mathsf{P}\Big(\bigcap_{j\in J_1, i\notin J_1} \{X_j > tx_j\} \,|\, X_i > tx_i \Big) = 0.$$

As $r = 0$, it holds that, for all $j \neq i$,

$$\tau_1(x_j \,|\, x_i) = \lim_{t\to\infty} \mathsf{P}(X_j > tx_j \,|\, X_i > tx_i) = 0,$$

$\boldsymbol{X}$ is 0-th upper-orthant tail dependent, this is just the so-called asymptotic tail independence in [17]. For more recent work about the upper-orthant tail dependence, readers may also refer to [16] and references. Here we present some examples for the r-th upper-orthant tail dependence.

Example 5 (Farlie-Gumbel-Morgenstern copula) Suppose a random vector $\boldsymbol{X}$ with the joint distribution function

$$G(\boldsymbol{x}) = \Big(1 + \theta \prod_{i=1}^{n} \overline{F}_i(x_i)\Big) \prod_{i=1}^{n} F_i(x_i), \qquad -1 \leqslant \theta \leqslant 1.$$

After some routine calculus, we have

$$\tau_1(x_j \,|\, x_i) = \lim_{t\to\infty} \mathsf{P}(X_j > tx_j \,|\, X_i > tx_i) = 0,$$

for all $j \neq i$ and $(x_i, x_j) > \mathbf{0}$, then $\boldsymbol{X}$ is 0-th upper-orthant tail dependence.

Example 6 (Gumbel copula) Suppose that $\boldsymbol{X} = (X_1, X_2)$ has the joint distribution function

$$G(x_1, x_2) = \exp\{-[(-\ln F_1(x_1))^{1/\beta} + (-\ln F_2(x_2))^{1/\beta}]^{\beta}\}, \qquad 0 < \beta < 1.$$

It is well known that Gumbel copula is an extreme value copula. Assume that $\overline{F}_2(tx_2)/\overline{F}_1(tx_1)$ has a positive limit as $t \to \infty$. By some standard calculus, we have

$$\tau_1(x_2 \,|\, x_1) > 0, \qquad \text{for all } (x_1, x_2) > \mathbf{0}.$$

Then $\boldsymbol{X}$ is 1-st upper-orthant tail dependent.

In a similar manner, for $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ with Gumbel-Hougaard copula

$$C(u_1, u_2, \ldots, u_n) = \exp\Big\{ - \Big(\sum_{i=1}^{n} (-\ln u_i)^{1/\beta} \Big)^{\beta} \Big\}, \qquad 0 < \beta < 1,$$

where the parameter β, measuring the degree of dependence, ranges from comonotonicity ($\beta \to 0+$) to mutual independence ($\beta \to 1-$). Also we can verify that $\boldsymbol{X}$ is $(n-1)$-th upper-orthant tail dependent, in the context that $\overline{F}_i(tx_i)/\overline{F}_1(tx_1)$ has positive limit as $t \to \infty$ for $i = 2, 3, \ldots, n$ and all $(x_1, x_2, \ldots, x_n) > \mathbf{0}$.

Example 7 Suppose the nonnegative random vector $\boldsymbol{X} = (X_1, X_2, X_3)$ has the joint distribution function

$$G(x_1, x_2, x_3) = F_3(x_3) \exp\{-[(-\ln F_1(x_1))^{1/\beta} + (-\ln F_2(x_2))^{1/\beta}]^{\beta}\},$$

for all $(x_1, x_2, x_3) > \mathbf{0}$ and $\beta \in (0, 1)$. In the context of $\overline{F}_i(tx_i)/\overline{F}_1(tx_1)$ having positive limit as $t \to \infty$ for $i = 2, 3$, by some routine calculous we have

$$\tau_1(x_2 \,|\, x_1) > 0, \qquad \tau_2(x_2, x_3 \,|\, x_1) = 0,$$
$$\tau_2(x_1, x_3 \,|\, x_2) = 0, \qquad \tau_2(x_1, x_2 \,|\, x_3) = 0.$$

This implies that $\boldsymbol{X}$ has the 1-st upper-orthant tail dependence. In this case $\tau_1(0 \,|\, x_1) = 1$.

Notice that if X_i and X_j are tail-equivalent (see (7)), then $\overline{F}_i(tx_i)/\overline{F}_j(tx_j)$ having positive limit as $t \to \infty$.

§4. Approximate Finite-Time Ruin Probability

In this section work on the asymptotic ruin probability of the risk process in (3) in the following context.

A1 Tail equivalence: There exists some $L \in \mathscr{R}_\alpha$ with $\alpha > 1$ such that

$$\lim_{t\to\infty} \frac{\overline{F}_i(t)}{L(t)} = \gamma_i^\alpha > 0, \qquad i = 1, 2, \ldots, n. \tag{7}$$

In this case, $\overline{F}_i \in \mathscr{R}_\alpha$ for every $i = 1, 2, \ldots, n$, and $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_n$ are said to be *tail equivalent*. Readers may refer to [18] for more details.

A2 Asymptotic independence: For $X_1, X_2, \ldots, X_n$,

$$\lim_{t\to\infty} \mathsf{P}(X_i > t \,|\, X_j > t) = 0, \qquad i \neq j. \tag{8}$$

A3 The 1st upper-orthant tail dependence: For $X_1, X_2, \ldots, X_n$, the following conditions are satisfied:

- $\tau_1(x_j \,|\, x_i)$ exists for all $i \neq j$ and $\tau_1(x_j \,|\, x_i) > 0$ for some $i \neq j$, and
- for all $J \subseteq N$ with $|J| = 2$,

$$\tau_2(x_k \,|\, x_i : k \in J) = 0. \tag{9}$$

Now we are ready to derive the approximate ruin probability of the k-out-of-n ruin set for tail-equivalent claim amounts with upper-tail orthant tail dependence. Firstly, let us present some propositions to be utilized in developing approximation.

Proposition 8 [9] Under A1 and A2, the Radon measure μ in (1) satisfies $\mu([\boldsymbol{x}, \boldsymbol{\infty}]) = 0$ and for all $\boldsymbol{x} > \boldsymbol{0}$,

$$\mu\Big((x_i, \infty) \times \bigcap_{j=1, j\neq i}^{n} \{x_j = 0\}\Big) = \frac{\gamma_i^\alpha}{x_i^\alpha} \lim_{u\to\infty} \frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\| > u)}. \tag{10}$$

Proposition 9 Under A1 and A3, for all $\boldsymbol{x} > \boldsymbol{0}$, the Radon measure μ in (1) satisfies $\mu([\boldsymbol{x}, \boldsymbol{\infty}]) = 0$ for $n \geqslant 3$,

$$\mu\Big((x_i, \infty) \times \bigcap_{j=1, j\neq i}^{n} \{x_j = 0\}\Big) = \begin{cases} \dfrac{\gamma_i^\alpha}{x_i^\alpha} \lim\limits_{u\to\infty} \dfrac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\| > u)}, & n \geqslant 3; \\ 0, & n = 2, \end{cases} \tag{11}$$

and for $i \neq j$,

$$\mu\Bigg((x_i, \infty) \times (x_j, \infty) \times \bigcap_{\substack{k=1, \\ k\notin\{i,j\}}}^{n} \{x_k = 0\}\Bigg) = \tau_1(x_j \,|\, x_i) \frac{\gamma_i^\alpha}{x_i^\alpha} \lim_{u\to\infty} \frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\| > u)}. \tag{12}$$

Proof It's obvious that $\tau_1(0\,|\,x_i)=1$, $i=1,2,\ldots,n$. For $n=2$, by (1) and A3, we have

$$\begin{aligned}\mu((x_i,\infty)\times(x_j,\infty)) &= \lim_{u\to\infty}\frac{\mathsf{P}(X_i>ux_i,X_j>ux_j)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}\\ &= \lim_{u\to\infty}\frac{\mathsf{P}(X_j>ux_j\,|\,X_i>ux_i)\mathsf{P}(X_i>ux_i)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}\\ &= \tau_1(x_j\,|\,x_i)\frac{\gamma_i^\alpha}{x_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)},\end{aligned}$$

and

$$\begin{aligned}&\mu((x_i,\infty)\times\{x_j=0\})\\ &=\mu((x_i,\infty)\times\{x_j:x_j\in[0,\infty)\})-\lim_{\varepsilon_j\downarrow 0}\mu((x_i,\infty)\times\{x_j:x_j\in(\varepsilon_j,\infty)\})\\ &=\lim_{u\to\infty}\frac{\mathsf{P}(X_i>ux_i,X_j\geqslant 0)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}-\lim_{\varepsilon_j\downarrow 0}\tau_1(\varepsilon_j\,|\,x_i)\frac{\gamma_i^\alpha}{x_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}\\ &=\lim_{u\to\infty}\frac{\overline{F}_i(ux_i)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}-\tau_1(0\,|\,x_i)\frac{\gamma_i^\alpha}{x_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}\\ &=0.\end{aligned}$$

For $n\geqslant 3$, since $\mathsf{P}(\boldsymbol{X}\in u[\boldsymbol{x},\boldsymbol{\infty}])\leqslant\mathsf{P}(X_1>ux_1,X_2>ux_2,X_3>ux_3)$, by (1) and (9) we have

$$\begin{aligned}&0\leqslant\mu([\boldsymbol{x},\boldsymbol{\infty}])\\ &\quad=\lim_{u\to\infty}\frac{\mathsf{P}(\boldsymbol{X}\in u[\boldsymbol{x},\boldsymbol{\infty}])}{\mathsf{P}(\|\boldsymbol{X}\|>u)}\leqslant\lim_{u\to\infty}\frac{\mathsf{P}(X_1>ux_1,X_2>ux_2,X_3>ux_3)}{\mathsf{P}(X_1>u)}\\ &\quad=0,\end{aligned}$$

$$\begin{aligned}&\mu\Big((x_i,\infty)\times\bigcap_{j=1,j\neq i}^{n}\{x_j=0\}\Big)\\ &=\mu\Big((x_i,\infty)\times\bigcap_{j=1,j\neq i}^{n}\{x_j\in[0,\infty)\}\Big)-\lim_{\varepsilon_j\downarrow 0}\mu\Big((x_i,\infty)\times\bigcap_{j=1,j\neq i}^{n}\{x_j\in(\varepsilon_j,\infty)\}\Big)\\ &=\lim_{u\to\infty}\mathsf{P}\Big(X_i>ux_i,\bigcap_{j=1,j\neq i}^{n}\{X_j\geqslant 0\}\Big)\Big/\mathsf{P}(\|\boldsymbol{X}\|>u)-0\\ &=\lim_{u\to\infty}\frac{\overline{F}_i(ux_i)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}\\ &=\frac{\gamma_i^\alpha}{x_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)},\qquad i=1,2,\ldots,n,\end{aligned}$$

and for all $(i,j)\in\mathbb{N}$,

$$\mu\bigg((x_i,\infty)\times(x_j,\infty)\times\bigcap_{\substack{l=1,\\ l\notin\{i,j\}}}^{n}\{x_l=0\}\bigg)$$

$$
\begin{aligned}
&= \mu\Bigg((x_i, \infty) \times (x_j, \infty) \times \bigcap_{\substack{l=1,\\ l\notin\{i,j\}}}^{n} \{x_l \in [0,\infty)\} \Bigg) \\
&\quad - \lim_{\varepsilon_l \downarrow 0} \mu\Bigg((x_i, \infty) \times (x_j, \infty) \times \bigcap_{\substack{l=1,\\ l\notin\{i,j\}}}^{n} \{x_l \in (\varepsilon_l,\infty)\} \Bigg) \\
&= \lim_{u\to\infty} \mathsf{P}\Bigg(X_i > ux_i, X_j > ux_j, \bigcap_{\substack{l=1,\\ l\notin\{i,j\}}}^{n} \{X_l \geqslant 0\} \Bigg) \Big/ \mathsf{P}(\|\boldsymbol{X}\| > u) - 0 \\
&= \lim_{u\to\infty} \frac{\mathsf{P}(X_j > ux_j \mid X_i > ux_i)\, \mathsf{P}(X_i > ux_i)}{\mathsf{P}(\|\boldsymbol{X}\| > u)} \\
&= \tau_1(x_j \mid x_i) \frac{\gamma_i^\alpha}{x_i^\alpha} \lim_{u\to\infty} \frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\| > u)}.
\end{aligned}
$$

Combining the above five equations directly yields (11) and (12). □

According to Proposition 8, under A1 and A2, the 0-th upper-orthant tail dependence (asymptotic independence) means that the probability of two or more components being simultaneously large when measured on suitable scales is negligible in comparison with the probability of one component being large. In this case, the measure μ spreads mass onto each axis but assigns no mass off the axes. Based on Proposition 9, under A1 and A3 with $n \geqslant 2$, the 1-th upper-orthant tail dependence tells that the probability of two components being simultaneously large when measured on suitable scales is significant. And as for $n \geqslant 3$, the measure μ spreads mass onto each axis and each upper-orthant of the 2-dimension subspace but assigns no mass on other parts of the space.

Let $\boldsymbol{a} - \Gamma_k$ be the subset of $\boldsymbol{x}$'s such that at least $n-k$ of all $a_i - x_i$'s are negative. Then, it can be expressed as

$$
\boldsymbol{a} - \Gamma_k = \Big\{ \boldsymbol{x} : \sum_{i=1}^{n} I(a_i - x_i) \leqslant k \Big\} = \Delta_0 + \Delta_1 + \cdots + \Delta_k,
$$

where, for $j = 0, 1, \ldots, k$,

$$
\Delta_j = \Big\{ \boldsymbol{x} : \sum_{i=1}^{n} I(a_i - x_i) = j \Big\}
$$

is the subset of $\boldsymbol{x}$ such that exactly j elements of $\boldsymbol{a} - \boldsymbol{x}$ are nonnegative and $n-j$ of them are negative. The following two lemmas evaluate the Radon measure of $\boldsymbol{a} - \Gamma_k$ in our context.

Lemma 10 Under A1 and A2, suppose that $\boldsymbol{X} \geqslant \boldsymbol{0}$. For the risk process $\boldsymbol{R}_t$ in (3),

$$
\frac{\mu(\boldsymbol{a} - \Gamma_k)}{\lim\limits_{u\to\infty} L(u)/\mathsf{P}(\|\boldsymbol{X}\| > u)} = \begin{cases} \sum\limits_{i=1}^{n} \dfrac{\gamma_i^\alpha}{x_i^\alpha}, & k = n-1; \\ 0, & 0 \leqslant k \leqslant n-2, \end{cases} \tag{13}
$$

where $a_1 + a_2 + \cdots + a_n = 1$ with $a_i \in (0,1)$ for $i = 1, 2, \ldots, n$.

Proof For $0 \leqslant k \leqslant n-2$,

$$\boldsymbol{a} - \Gamma_k = \Big\{\boldsymbol{x} : \sum_{i=1}^{n} I(a_i - x_i) \leqslant k\Big\}.$$

This means that at least two or more components are simultaneously large. By (10), we have $\mu(\boldsymbol{a} - \Gamma_k) = 0$.

When $k = n-1$, it holds that

$$\boldsymbol{a} - \Gamma_{n-1} = \Big\{\boldsymbol{x} : \sum_{i=1}^{n} I(a_i - x_i) \leqslant n-1\Big\} = \Delta_0 + \Delta_1 + \cdots + \Delta_{n-1},$$

where

$$\Delta_{n-1} = \Big\{\boldsymbol{x} : \bigcup_{i=1}^{n} \Big[\bigcap_{j \neq i} \{x_j : 0 \leqslant x_j \leqslant a_j\} \cap \{x_i : x_i > a_i\}\Big]\Big\}.$$

By (10), we also have $\mu(\Delta_i) = 0$ for $i = 0, 1, \ldots, n-2$ and

$$\mu(\Delta_{n-1}) = \sum_{i=1}^{n} \frac{\gamma_i^\alpha}{x_i^\alpha} \lim_{u\to\infty} \frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\| > u)}.$$

This invokes

$$\mu(\boldsymbol{a} - \Gamma_k) = \sum_{i=1}^{n} \frac{\gamma_i^\alpha}{x_i^\alpha} \lim_{u\to\infty} \frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\| > u)}. \qquad \square$$

Lemma 11 Under A1 and A3 with nonnegative $\boldsymbol{X}$ and for the risk process $\boldsymbol{R}_t$ in (3),

$$\frac{\mu(\boldsymbol{a} - \Gamma_k)}{\lim\limits_{u\to\infty} L(u)/\mathsf{P}(\|\boldsymbol{X}\| > u)} = \begin{cases} \tau_1(a_2 \,|\, a_1)\dfrac{\gamma_1^\alpha}{a_1^\alpha}, & n = 2,\ k = 0; \\ \dfrac{\gamma_1^\alpha}{a_1^\alpha} + [1 - \tau_1(a_1 \,|\, a_2)]\dfrac{\gamma_2^\alpha}{a_2^\alpha}, & n = 2,\ k = 1; \\ \sum\limits_{1\leqslant i<j\leqslant n} \tau_1(a_j \,|\, a_i)\dfrac{\gamma_i^\alpha}{a_i^\alpha}, & n \geqslant 3,\ k = n-2; \\ \sum\limits_{i=1}^{n} \dfrac{\gamma_i^\alpha}{a_i^\alpha} + \sum\limits_{1\leqslant i<j\leqslant n} \tau_1(a_j \,|\, a_i)\dfrac{\gamma_i^\alpha}{a_i^\alpha}, & n \geqslant 3,\ k = n-1; \\ 0, & n \geqslant 3,\ 0 \leqslant k \leqslant n-3, \end{cases} \tag{14}$$

where $a_1 + a_2 + \cdots + a_n = 1$ with $a_i \in (0, 1)$ for $i = 1, 2, \ldots, n$.

Proof For $n = 2$, note that

$$\begin{aligned} \boldsymbol{a} - \Gamma_0 &= \{(x_1, x_2) : I(a_1 - x_1) + I(a_2 - x_2)) \leqslant 0\} \\ &= \{(x_1, x_2) : x_1 > a_1, x_2 > a_2\}, \\ \boldsymbol{a} - \Gamma_1 &= \{(x_1, x_2) : I(a_1 - x_1) + I(a_2 - x_2)) \leqslant 1\} \\ &= \{(x_1, x_2) : x_1 > a_1, x_2 > a_2\} \cup \{(x_1, x_2) : x_1 > a_1, 0 \leqslant x_2 \leqslant a_2\} \\ &\quad \cup \{(x_1, x_2) : 0 \leqslant x_1 \leqslant a_1, x_2 > a_2\}. \end{aligned}$$

By (12), we have

$$\mu(\boldsymbol{a}-\Gamma_0)=\tau_1(a_2\,|\,a_1)\frac{\gamma_1^\alpha}{a_1^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)},$$
$$\mu(\boldsymbol{a}-\Gamma_1)=\Big[\frac{\gamma_1^\alpha}{a_1^\alpha}+(1-\tau_1(a_1\,|\,a_2))\frac{\gamma_2^\alpha}{a_2^\alpha}\Big]\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}.$$

For $n\geqslant 3$ and $k=n-2$, it holds that

$$\boldsymbol{a}-\Gamma_{n-2}=\Big\{\boldsymbol{x}:\sum_{i=1}^n I(a_i-x_i)\leqslant n-2\Big\}=\Delta_0+\Delta_1+\cdots+\Delta_{n-2},$$

where

$$\Delta_{n-2}=\Big\{\boldsymbol{x}:\bigcup_{i\neq j}\Big[\bigcap_{l\notin\{i,j\}}\{x_l:0\leqslant x_l\leqslant a_l\}\cap\{x_i:x_i>a_i\}\cap\{x_j:x_j>a_j\}\Big]\Big\}.$$

Also due to (11) and (12), we have

$$\mu(\Delta_i)=0,\qquad i=0,1,\dots,n-3,$$

and

$$\mu(\Delta_{n-2})=\sum_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\frac{\gamma_i^\alpha}{a_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}.$$

Therefore, we conclude that

$$\mu(\boldsymbol{a}-\Gamma_{n-2})=\sum_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\frac{\gamma_i^\alpha}{a_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}.$$

For $n\geqslant 3$ and $k=n-1$,

$$\boldsymbol{a}-\Gamma_{n-1}=\Big\{\boldsymbol{x}:\sum_{i=1}^n I(a_i-x_i)\leqslant n-1\Big\}=\Delta_0+\Delta_1+\cdots+\Delta_{n-2}+\Delta_{n-1},$$

where

$$\Delta_{n-1}=\Big\{\boldsymbol{x}:\bigcup_{i=1}^n\Big[\bigcap_{j\neq i}\{x_j:0\leqslant x_j\leqslant a_j\}\cap\{x_i:x_i>a_i\}\Big]\Big\}.$$

Similarly, from (11) and (12) it follows that

$$\mu(\Delta_i)=0,\qquad i=0,1,\dots,n-3,$$
$$\mu(\Delta_{n-2})=\sum_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\frac{\gamma_i^\alpha}{a_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)},$$

and

$$\mu(\Delta_{n-1})=\sum_{i=1}^n\frac{\gamma_i^\alpha}{a_i^\alpha}\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}.$$

Thus, we have

$$\mu(\boldsymbol{a}-\Gamma_{n-1}) = \Big[\sum_{i=1}^{n}\frac{\gamma_i^\alpha}{a_i^\alpha} + \sum_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\frac{\gamma_i^\alpha}{a_i^\alpha}\Big]\lim_{u\to\infty}\frac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}.$$

As for $n\geqslant 3$ and $k=0,1,\ldots,n-3$, it holds that $\mu(\boldsymbol{a}-\Gamma_k)=0$. $\square$

Now, we are ready to present the two main results.

Proposition 12 Under A1 and A2, for the risk process $\boldsymbol{R}_t$ in (3) with $\boldsymbol{X}\geqslant\boldsymbol{0}$, $t^*>0$ and large u, it holds that

$$\lim_{u\to\infty}\frac{\phi_k(u,t^*)}{\lambda t^* L(u)} = \begin{cases}\sum\limits_{i=1}^{n}\dfrac{\gamma_i^\alpha}{a_i^\alpha}, & k=n-1;\\ 0, & k=0,1,\ldots,n-2,\end{cases} \tag{15}$$

where $a_i\in(0,1)$ for $i=1,2,\ldots,n$ and $a_1+a_2+\cdots+a_n=1$.

Proof By Proposition 3, we have, for any $t^*>0$ and $u\geqslant 0$,

$$\lim_{u\to\infty}\frac{\phi_k(u,t^*)}{\mathsf{P}(\|\boldsymbol{X}\|>u)} = \lambda t^*\mu(\boldsymbol{a}-\Gamma_k).$$

Also from Lemma 10 it follows that

$$\mu(\boldsymbol{a}-\Gamma_k) = \begin{cases}\sum\limits_{i=1}^{n}\dfrac{\gamma_i^\alpha}{a_i^\alpha}\lim\limits_{u\to\infty}\dfrac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}, & k=n-1;\\ 0, & 0\leqslant k\leqslant n-2.\end{cases}$$

Consequently, we have (15) and this completes the proof. $\square$

Proposition 13 Under A1 and A3, for the risk process $\boldsymbol{R}_t$ in (3) with $\boldsymbol{X}\geqslant\boldsymbol{0}$, $t^*>0$ and large u, we have

$$\lim_{u\to\infty}\frac{\phi_k(u,t^*)}{\lambda t^* L(u)} = \begin{cases}\tau_1(a_2\,|\,a_1)\dfrac{\gamma_1^\alpha}{a_1^\alpha}, & n=2,\ k=0;\\ \dfrac{\gamma_1^\alpha}{a_1^\alpha}+[1-\tau_1(a_1\,|\,a_2)]\dfrac{\gamma_2^\alpha}{a_2^\alpha}, & n=2,\ k=1;\\ \sum\limits_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\dfrac{\gamma_i^\alpha}{a_i^\alpha}, & n\geqslant 3,\ k=n-2;\\ \sum\limits_{i=1}^{n}\dfrac{\gamma_i^\alpha}{a_i^\alpha}+\sum\limits_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\dfrac{\gamma_i^\alpha}{a_i^\alpha}, & n\geqslant 3,\ k=n-1;\\ 0, & n\geqslant 3,\ k=0,1,\ldots,n-3,\end{cases} \tag{16}$$

where $a_i\in(0,1)$ for $i=1,2,\ldots,n$ and $a_1+a_2+\cdots+a_n=1$.

Proof From Lemma 11, it follows that

$$\mu(\boldsymbol{a}-\Gamma_k)$$
$$=\begin{cases} \tau_1(a_2\,|\,a_1)\dfrac{\gamma_1^\alpha}{a_1^\alpha}\lim\limits_{u\to\infty}\dfrac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}, & n=2,\ k=0;\\ \Big[\dfrac{\gamma_1^\alpha}{a_1^\alpha}+(1-\tau_1(a_1\,|\,a_2))\dfrac{\gamma_2^\alpha}{a_2^\alpha}\Big]\lim_{u\to\infty}\dfrac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}, & n=2,\ k=1;\\ \sum\limits_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\dfrac{\gamma_i^\alpha}{a_i^\alpha}\lim_{u\to\infty}\dfrac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}, & n\geqslant 3,\ k=n-2;\\ \Big[\sum\limits_{i=1}^n\dfrac{\gamma_i^\alpha}{a_i^\alpha}+\sum\limits_{1\leqslant i<j\leqslant n}\tau_1(a_j\,|\,a_i)\dfrac{\gamma_i^\alpha}{a_i^\alpha}\Big]\lim\limits_{u\to\infty}\dfrac{L(u)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}, & n\geqslant 3,\ k=n-1;\\ 0, & n\geqslant 3,\ k=0,1,\ldots,n-3. \end{cases}$$

On the other hand, by Proposition 3 we also have, for any $t^*>0$ and $u\geqslant 0$,

$$\lim_{u\to\infty}\frac{\phi_k(u,t^*)}{\mathsf{P}(\|\boldsymbol{X}\|>u)}=\lambda t^*\mu(\boldsymbol{a}-\Gamma_k).$$

This invokes (16) and hence completes the proof. □

§5. One Numerical Example

To close this note, we present one example as an illustration of the asymptotic ruin probability we developed in previous section.

For the tail equivalent claim vector (X_1,X_2) with the Gumbel copula discussed in Example 6, there is some $L\in\mathscr{R}_\alpha$ with $\alpha>1$ such that X_i's survival function $\overline{F}_i$ satisfies

$$\lim_{t\to\infty}\frac{\overline{F}_i(t)}{L(t)}=\gamma_i^\alpha,\qquad \gamma_i>0,\ i=1,2.$$

In this example, we suppose that $L(t)$ is only depend on t. Also, $a_1+a_2+\cdots+a_n=1$ and $a_i>0$ is the element of the initial capital allocation vector, $i=1,2,\ldots,n$.

By standard calculus, we have, for $x_1,x_2>0$,

$$\lim_{t\to\infty}\frac{\overline{F}_2(tx_2)}{\overline{F}_1(tx_1)}=\frac{x_1^\alpha\gamma_2^\alpha}{x_2^\alpha\gamma_1^\alpha},$$

and

$$0<\tau_1(x_2\,|\,x_1)=1+\frac{x_1^\alpha\gamma_2^\alpha}{x_2^\alpha\gamma_1^\alpha}-\Big[1+\Big(\frac{x_1^\alpha\gamma_2^\alpha}{x_2^\alpha\gamma_1^\alpha}\Big)^{1/\beta}\Big]^\beta<1.$$

That is, (X_1,X_2) is 1-st upper-orthant tail dependent.

By (16) in Proposition 13, we have, for $\boldsymbol{R}_t$ in (3) with $(X_1, X_2) \geqslant \mathbf{0}$ and $t^* > 0$,

$$\begin{aligned} g(a_1, a_2, \alpha, \beta, \gamma_1, \gamma_2) &= \lim_{u\to\infty} \frac{\phi_k(u, t^*)}{\lambda t^* L(u)} \\ &= \begin{cases} \tau_1(a_2 \,|\, a_1)\dfrac{\gamma_1^\alpha}{a_1^\alpha}, & k = 0; \\ \dfrac{\gamma_1^\alpha}{a_1^\alpha} + [1 - \tau_1(a_1 \,|\, a_2)]\dfrac{\gamma_2^\alpha}{a_2^\alpha}, & k = 1 \end{cases} \\ &= \begin{cases} \dfrac{\gamma_1^\alpha}{a_1^\alpha} + \dfrac{\gamma_2^\alpha}{(1-a_1)^\alpha} - \Big[\Big(\dfrac{\gamma_1}{a_1}\Big)^{\alpha/\beta} + \Big(\dfrac{\gamma_2}{1-a_1}\Big)^{\alpha/\beta}\Big]^\beta, & k = 0; \\ \Big[\Big(\dfrac{\gamma_1}{a_1}\Big)^{\alpha/\beta} + \Big(\dfrac{\gamma_2}{1-a_1}\Big)^{\alpha/\beta}\Big]^\beta, & k = 1. \end{cases} \end{aligned}$$

Obviously $[(\gamma_1/a_1)^{\alpha/\beta} + (\gamma_2/(1-a_1))^{\alpha/\beta}]^\beta$ is monotonically increasing in $\beta \in (0, 1)$ and it holds that

$$\Big[\Big(\frac{\gamma_1}{a_1}\Big)^{\alpha/\beta} + \Big(\frac{\gamma_2}{1-a_1}\Big)^{\alpha/\beta}\Big]^\beta \longrightarrow \begin{cases} \dfrac{\gamma_1^\alpha}{a_1^\alpha} + \dfrac{\gamma_2^\alpha}{(1-a_1)^\alpha}, & \beta \to 1-; \\ \max\Big\{\dfrac{\gamma_1^\alpha}{a_1^\alpha}, \dfrac{\gamma_2^\alpha}{(1-a_1)^\alpha}\Big\}, & \beta \to 0+. \end{cases}$$

Note that the dependence is weakened as β increases from $\beta = 0+$ (comonotonicity) to $\beta = 1-$ (independence), we come up with the following facts: as the initial capital u is large,

- the probability for both lines' reserves to be negative $\phi_0(u, t^*)$ decreases as the interdependence between the two lines is weakened. This means that comparing to 0-th upper-orthant tail dependent, 1-th upper-orthant tail dependent increases the ruin probability $\phi_0(u, t^*)$.
- the probability for at least one line's reserve to be negative $\phi_1(u, t^*)$ decreases as the interdependence between the two lines is intensified. This means that comparing to 0-th upper-orthant tail dependent, 1-th upper-orthant tail dependent decreases the ruin probability $\phi_1(u, t^*)$.
- for fixed $\gamma_1 > 0$, $\gamma_2 > 0$, $\alpha > 1$ and $0 < \beta < 1$, it holds that

$$g\Big(\frac{1}{2}, \frac{1}{2}, \alpha, \beta, \gamma_1, \gamma_2\Big) = \begin{cases} 2^\alpha\big[\gamma_1^\alpha + \gamma_2^\alpha - \big(\gamma_1^{\alpha/\beta} + \gamma_2^{\alpha/\beta}\big)^\beta\big], & k = 0; \\ 2^\alpha\big(\gamma_1^{\alpha/\beta} + \gamma_2^{\alpha/\beta}\big)^\beta, & k = 1, \end{cases}$$

$$g\Big(\frac{\gamma_1}{\gamma_1+\gamma_2}, \frac{\gamma_2}{\gamma_1+\gamma_2}, \alpha, \beta, \gamma_1, \gamma_2\Big) = \begin{cases} (2 - 2^\beta)(\gamma_1 + \gamma_2)^\alpha, & k = 0; \\ 2^\beta(\gamma_1 + \gamma_2)^\alpha, & k = 1. \end{cases}$$

By routine algebras, we have

$$g\Big(\frac{1}{2},\frac{1}{2},\alpha,\beta,\gamma_1,\gamma_2\Big) \geqslant g\Big(\frac{\gamma_1}{\gamma_1+\gamma_2},\frac{\gamma_2}{\gamma_1+\gamma_2},\alpha,\beta,\gamma_1,\gamma_2\Big).$$

That is, the uniform allocation of initial capital is not a better way to reduce the ruin probability because the proportional allocation based on their tails performs much better.

Appendix

Proof of Proposition 3 Since our ruin set Γ_k is a special case in [6], we only to prove that $\mu(\partial(\boldsymbol{a}-\Gamma_k))=0$. Similarly to the proof of Lemma 7 in [6], notice that

$$\boldsymbol{a}-\Gamma_k=\Big\{\boldsymbol{x}:\sum_{i=1}^{n} I(a_i-x_i)\leqslant k\Big\},$$

then

$$\partial(\boldsymbol{a}-\Gamma_k)\subset\bigcup_i\{\boldsymbol{x}:x_i=a_i\}=\bigcup_i H_i.$$

Let $W_i=\{\boldsymbol{x}:x_i>a_i\}$, we find that

$$\mu(W_i)\geqslant\mu\Big(\bigcup_{q\in\mathbb{Q}\cap(1,+\infty)} qH_i\Big)=\sum_{q\in\mathbb{Q}\cap(1,+\infty)}\mu(qH_i)=\mu(H_i)\sum_{q\in\mathbb{Q}\cap(1,+\infty)} q^{-\alpha}.$$

Since $\mu(W_i)\in(0,+\infty)$ and $\sum\limits_{q\in\mathbb{Q}\cap(1,+\infty)} q^{-\alpha}=\infty$, we must have $\mu(H_i)=0$. Hence,

$$\partial(\boldsymbol{a}-\Gamma_k)\leqslant\sum_i\mu(H_i)=0. \qquad \square$$

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多元重尾索赔下的渐近有限时间破产概率

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摘　要: 本文研究进行多线生意的保险公司. 在同时面临 1 阶上象限尾相依重尾索赔的情形下, 基于所谓的 k-out-of-n 破产集, 我们得到 Radon 测度的显示表达并推导出有限时间内部分支线公司破产时的渐近破产概率. 我们给出了具体阐释主要结论的数值例子.

关键词: 多元正则变化; k-out-of-n 破产集; 上象限尾相依

中图分类号: O211.6; O211.9