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Smooth Density for a Class of Fractional SPDE with Fractional Noise *

CANG Yuquan LI Qinyi LIU Junfeng⋆

(*Department of Statistics, Nanjing Audit University, Nanjing, 211815, China*)

Abstract: In this paper we consider a class of fractional stochastic partial differential equation driven by fractional noise. We prove that the solution admits a smooth density at any fixed point $(t,x) \in [0,T] \times \mathbb{R}$ with $T > 0$ by using the techniques of Malliavin calculus.

Keywords: fractional stochastic partial differential equation; stable-like generator of variable order; fractional noise; Malliavin calculus; smooth density

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§1. Introduction and Main Result

Consider the following semilinear stochastic partial differential equations (abbr. SPDEs)

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \mathfrak{D}(x,D)u(t,x) + \dfrac{\partial f}{\partial x}(t,x,u(t,x)) + \dot{W}^H(t,x); \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R} \end{cases} \tag{1}$$

on the given domain $[0,T] \times \mathbb{R}$ with the initial condition $u_0(x) \in L^p(\mathbb{R})$, $p \geqslant 2$, where $\mathfrak{D}(x,D)$ denotes the Markovian generator of stable-like Feller process with variable order $\alpha(x)$. The coefficient $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable and $\dot{W}^H(t,x)$ is the fractional-colored noise. There have been a considerable body of literatures devoted to the study of stochastic partial differential equations (SPDEs), see [1–11] and etc.

In this paper, we are interested in equation (1) mainly because it involves a pseudo-differential operator $\mathfrak{D}(x,D)$ and an fractional-colored noise $\dot{W}^H(t,x)$. These kinds of

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⋆Corresponding author, E-mail: jordanjunfeng@163.com.

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equations have been used to model a variety of anomalous diffusion in continuum mechanics, particularly in connection with the investigation in turbulence. On one hand, the operator $\mathfrak{D}(x, D)$ contains the fractional Laplacian operator $(-\Delta)^{\alpha/2} = (-\partial^2/\partial x^2)^{\alpha/2}$ with a constant $\alpha \in (0, 2]$ as its special cases. The study of stable-like Markov generators $\mathfrak{D}(x, D)$ with variable order can be traced back to the seminal paper [12]. Further works on the transition densities associated with stable-like processes can be found in [13–16] and references therein. Thus it is natural to combine the SPDEs with the pseudo-differential $\mathfrak{D}(x, D)$. For example, Wu and Xie [11] studied a stochastic Burgers-type nonlinear equation with a pure jump Lévy time-space white noise in $\mathbb{R}^d$ which contained a d-dimensional pseudo-differential operator with a variable order $\alpha(x) : \mathbb{R}^d \to (0, 2)$. On the other hand, due to the various applications of the fractional Brownian motion, there are many works concerning SPDEs with fractional noise, see [1] and [2] where they studied the stochastic heat equation and stochastic wave equation with fractional noise respectively. Hu et al. [3] studied the parabolic Anderson model with fractional white noise by using Feynman-Kac formula and Mallaivin calculus. Liu and Yan [4] studied the existence and Hölder regularity of the mild solution of equation (1).

Once the existence and sample properties for the solution of SPDEs are proved, one usually would like to study the properties of the density for the solution. Especially, people would like to study the existence, smoothness (in the sense of Malliavin calculus) of density, Gaussian density estimates and etc. One can see [6] and [9] for the study of smoothness of the law for a stochastic wave equation in dimension 2 and 3, respectively. Nualart and Quer-Sardanyons [7] studied the smoothness of the density for spatially homogeneous SPDEs and [8] for the Gaussian density estimates for solutions to quasi-linear SPDEs.

Inspired by the above results, we would like to use the tools of Malliavin calculus to prove the smoothness of the density for the solution to equation (1) in this paper. That is, for any fixed $(t, x) \in [0, T] \times \mathbb{R}$, the law of the random variable $u(t, x)$ has an infinitely differentiable density with respect to the Lebesgue measure on $\mathbb{R}$. It was already proved by Liu and Yan [4] that the law of $u(t, x)$ is absolutely continuous with respect to Lebesgue measure, i.e. the law of $u(t, x)$ admits a density for any fixed $(t, x) \in [0, T] \times \mathbb{R}$. In addition, they also established lower and upper Gaussian bounds for the probability density of the mild solution.

According to [11], let us firstly make the following assumptions on some parameters which appeared in the subsequent discussions concerning some estimates with the Green function G introduced in Section 2:

Assumption 1 We follow the assumptions on the parameters α, α_L, α^U, β, γ appearing in Lemma 2.2 amd Lemma 2.3 in [11]

$$\alpha \in [\alpha_L, \alpha^U] \in (1,2), \qquad \beta \in \Big(0, \frac{1}{1+\alpha}\Big), \qquad \gamma \in (0, 1-\beta(\alpha+1)),$$

where

$$\alpha_L = \inf_{x\in\mathbb{R}} \alpha(x), \qquad \alpha^U = \sup_{x\in\mathbb{R}} \alpha(x),$$

and $\alpha(x)$ is a continuous function which will be introduced in Section 2.

In this paper we shall prove the following result about the smoothness of the density of the solution to equation (1).

Theorem 2 Under Assumption 1 and assume that the coefficient f is a C^∞ function with bounded derivatives of any order greater than or equal to one with respect to the third variable (i.e. $f(s,y,\cdot) \in C_b^\infty([0,T]\times\mathbb{R}\times\mathbb{R})$). Then, for $H_1, H_2 > 1/2$ and for any fixed $(t,x) \in [0,T]\times\mathbb{R}$, the law of $u(t,x)$ to equation (1) admits a smooth density.

Remark 3 In [4], the authors also studied equation (1) and proved the existence and Hölder regularity of the solution. Moreover they proved the lower and upper Gaussian-type bounds for the density by using the techniques of Malliavin calculus. In this note, we continue studying the smoothness of the density of the solution to equation (1) which complements the results obtained about the density in [4].

The rest of this paper is organized as follows. In the next Section 2, we recall some known results on the operator $\mathfrak{D}(x,D)$ and present some preliminaries on Malliavin calculus for fractional noise. Section 3 is devoted to proving the smoothness of the density for solution to equation (1). We will firstly recall the main tools of the Malliavin calculus for iterated Malliavin derivative needed along the paper. Then we give the proof of Theorem 2 step by step.

§2. Preliminaries

We should firstly recall some known facts about the operator $\mathfrak{D}(x,D)$ (see [13] or [11] for more details). Let $\alpha(\cdot) : \mathbb{R} \mapsto (0,2)$ be a continuous function and $S_0 = \{-1,1\}$ and $\mathscr{B} = \mathscr{P}(S_0) = \{\phi, \{-1\}, \{1\}, S_0\}$. Given any finite symmetric Borelian measure $\widetilde{\mu}$ (the so-callled spectral measure) on $\mathscr{B}(S_0)$ depending smoothly on $x\in\mathbb{R}$, we define $s := z/|z|$ for $z \in \mathbb{R}\backslash\{0\}$. Then, $\mathfrak{D}(x,D)$ can be written as

$$(\mathfrak{D}(x,D)\varphi)(x) = \int_0^{+\infty}\int_{S_0}\Big[\varphi(x+z) - \varphi(x) - \frac{z\varphi'(x)}{1+|z|^2}\Big]\frac{\mathrm{d}|z|}{|z|^{1+\alpha(x)}}\widetilde{\mu}(x,\mathrm{d}s), \tag{2}$$

which is in the form of the Markovian generator of a stable-like Feller processes considered in [13], where φ is a Schwartz test function in $\mathscr{S}(\mathbb{R})$. In this paper, we are only in interested in dealing with the operator of a symmetric stable-like jump-diffusion process. We will assume that $\widetilde{\mu}$ is symmetric, that is $\widetilde{\mu}(x,A)=\widetilde{\mu}(x,-A)$ for any Borelian measurable subset $A\in\mathscr{B}(S_0)$. In particular, if further $\widetilde{\mu}$ is rotation invariant, then $\mathfrak{D}(x,D)=-(-\Delta)^{\alpha(x)/2}$.

Let $G(t;x,y)$ be the fundamental solution of the following parabolic equation

$$\begin{cases}\dfrac{\partial}{\partial t}G(t;x,y)=\mathfrak{D}(x,D)G(t;x,y), & (t,x)\in(0,+\infty)\times\mathbb{R};\\ \lim\limits_{t\searrow 0}G(t;x,y)=\delta_x(y), & x\in\mathbb{R}.\end{cases}\tag{3}$$

According to Theorem 5.1 in [13], $G(t;x,y)$ does exist under Assumption 1 and it is the transition probability density for a stable-like jump diffusion.

In this paper, we will use some useful estimates for Green function $G_\alpha(t,x)$ which is written for $G(t;x,y)$ with $\alpha(x)=\alpha$ proved in Lemma 2.2 and Lemma 2.3 in [11] (see also [13], [16] and etc). Here we omit the details.

Suppose that $W^H=\{W^H(t,x),t\in[0,T],x\in\mathbb{R}\}$ is a zero mean fractional Brownian sheet with the covariance function

$$\mathsf{E}[W^H(t,x)W^H(s,y)]=R^{H_1}(t,s)R^{H_2}(x,y),\qquad 1/2<H_1,H_2<1,$$

where we denote by $R^h(s,t)$, the covariance function of fractional Brownian motion with Hurst parameter h, that is,

$$R^h(t,s)=\frac{1}{2}(|t|^{2h}+|s|^{2h}-|t-s|^{2h}).$$

We will denote by $\mathscr{H}$ the canonical Hilbert space associated with the fractional Brownian sheet which is defined as the closure of indicator functions $\mathscr{E}=\{1_{[0,t]\times[0,x]},t\in[0,T],$ $x\in\mathbb{R}\}$ with respect to the inner product

$$\langle 1_{[0,t]\times[0,x]},1_{[0,s]\times[0,y]}\rangle_{\mathscr{H}}=R^{H_1}(t,s)R^{H_2}(x,y).$$

Thus the mapping $1_{[0,t]\times[0,x]}\mapsto W^H(t,x)$ is an isometry between $\mathscr{E}$ and the linear space span of $\{W^H(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$. Moreover, the mapping can be extended to an isometry from $\mathscr{H}$ to a Gaussian space associated with W^H. This isometry will be denoted by $\varphi\mapsto W^H(\varphi)$ for any $\varphi\in\mathscr{H}$. Therefore, we can regard $W^H(\varphi)$ as the stochastic integral with respect to W^H. In general, we use the notation

$$W^H(\varphi)=\int_0^T\int_{\mathbb{R}}\varphi(s,y)W^H(\mathrm{d}s,\mathrm{d}y),\qquad \varphi\in\mathscr{H}.$$

Using the notation of stochastic integral with respect to W^H, one can introduce the following definition (i.e. [10]):

Definition 4 A real-valued stochastic process $u = \{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}\}$ is a mild solution of equation (1) if for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$\begin{aligned} u(t,x) = \int_{\mathbb{R}} G(t;x,y)u_0(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s;x,y)f(s,y,u(s,y))\mathrm{d}y\mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}} G(t-s;x,y)W^H(\mathrm{d}s,\mathrm{d}y). \end{aligned} \tag{4}$$

Now we recall the existence and uniqueness result of the solution to equation (1) which has already been proved by Liu and Yan [4].

Proposition 5 Suppose Assumption 1 is satisfied and f is Lipschitz continuous and satisfies the linear growth condition. Then there exists a unique mild solution $u(t,x)$ with $(t,x) \in [0,T] \times \mathbb{R}$, to equation (1) such that for all $T > 0$ and for some $p \geqslant 2$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathsf{E}|u(t,x)|^p < \infty.$$

§3. Proof of Theorem 2

The aim in this section is to prove Theorem 2. Firstly let us recall a useful result (see, for example, [17]):

Proposition 6 Let $F = (F^1, F^2, \ldots, F^m)$ be a random vector satisfying the following conditions:

(i) F^i belongs to $\mathbb{D}^\infty = \bigcap_{p\geqslant 1} \bigcap_{k\geqslant 1} \mathbb{D}^{k,p}$ for every $i = 1, 2, \ldots, m$.

(ii) The Malliavin matrix $\gamma_F = (\langle DF^i, DF^j\rangle_{\mathscr{H}})_{1\leqslant i,j\leqslant m}$ satisfies $(\det \gamma_F)^{-1} \in \bigcap_{p\geqslant 1} L^p(\Omega)$.

Then the random vector $F = (F^1, F^2, \ldots, F^m)$ has an infinitely differentiable density.

According to the above proposition, the proof of Theorem 2 will be achieved by showing that $u(t,x)$ belongs to the space $\mathbb{D}^\infty = \bigcap_{p\geqslant 1} \bigcap_{k\geqslant 1} \mathbb{D}^{k,p}$ and the inverse of the Malliavin matrix of $u(t,x)$ has moments of all order. The following proposition came from in [4].

Proposition 7 Under the assumptions of Theorem 2, if we further assume that $f \in C_b^1([0,T] \times \mathbb{R} \times \mathbb{R})$, then, at any fixed point $(t,x) \in [0,T] \times \mathbb{R}$, the random variable $u(t,x) \in \mathbb{D}^{1,2}$ and the Malliavin derivative $Du(t,x)$ satisfies

$$D_{v,z}u(t,x) = \int_v^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s;x,y)f'(s,y,u(s,y))D_{v,z}u(s,y)\mathrm{d}y\mathrm{d}s + G(t-v;x,z), \tag{5}$$

for all $v \leqslant t$ and $z \in \mathbb{R}$. Moreover, for any fixed $(t,x) \in [0,T] \times \mathbb{R}$, the law of the solution $u(t,x)$ is absolutely continuous with respect to the Lebesgue measure.

Before giving the proof of Theorem 2, we firstly need to extend Proposition 7 to any differentiability order. It is clear that a strengthening of the regularity of the coefficient f is needed. Recall that for any differentiable (in the Malliavin sense) random variable X and any $N \geqslant 1$, the iterated Malliavin derivative $D^N X$ defines an element of the Hilbert space $L^2(\Omega; \mathscr{H}^{\otimes N})$. We shall use the notation

$$D^N_{(\varphi_1,\varphi_2,\dots,\varphi_N)} X = \langle D^N X, \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_N \rangle_{\mathscr{H}^{\otimes N}},$$

for any $\varphi_i \in \mathscr{H}$, $i = 1, 2, \dots, N$. Thus, we have that

$$\|D^N X\|^2_{\mathscr{H}^{\otimes N}} = \sum_{j_1,j_2,\dots,j_N} \big|D^N_{(e_{j_1},e_{j_2},\dots,e_{j_N})} X\big|^2, \tag{6}$$

where $\{e_j\}_{j \geqslant 0}$ is a complete orthonormal system of $\mathscr{H}$.

We denote by $\mathscr{D}^{N,p}$ the Sobolev-Watanabe space of random variable X such that

$$\|X\|_{N,p} = \Big[\mathsf{E}(|X|^p) + \sum_{j=1}^{N} \mathsf{E}(\|D^j X\|^p_{\mathscr{H}^{\otimes j}})\Big]^{1/p}.$$

Let $N \in \mathbb{N}$, fix a set $A_N = \{\varphi_i \in \mathscr{H}, i = 1, 2, \dots, N\}$ and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_N)$, $\widehat{\varphi}_i = (\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_N)$. Denote by $\mathscr{P}_m$ the set of partitions of A_N consisting of m disjoint subsets $p_1, p_2, \dots, p_m$, $m = 1, 2, \dots, N$, and by $|p_i|$ the cardinal of p_i. Let X be a random variable belonging to $\mathbb{D}^{N,2}$, $N \geqslant 1$, and g be a real $\mathscr{C}^N$-function with bounded derivatives up to order N. Leibniz's rule for Malliavin's derivatives yields

$$D^N_{\varphi}(g(X)) = \sum_{m=1}^{N} \sum_{\mathscr{P}_m} c_m g^{(m)}(X) \prod_{i=1}^{m} D^{|p_i|}_{p_i} X, \tag{7}$$

with positive coefficients c_m, $m = 2, 3, \dots, N$ and $c_1 = 1$. Let

$$\Delta^N_{\varphi}(g, X) := D^N_{\varphi} g(X) - g'(X) D^N_{\varphi} X.$$

Note that $\Delta^N_{\varphi}(g, X) = 0$ if $N = 1$ and it only depends on the Malliavin derivatives up to the order $N - 1$ if $N > 1$. Now we can show that the solution to equation (1) is infinitely differentiable in the Malliavin sense and also obtain the equation satisfied by the iterated Malliavin derivative.

Theorem 8 Assume the coefficient f is a C^{∞} function with bounded derivatives of any order greater than or equal to one. Then, for every $(t,x) \in [0,T] \times \mathbb{R}$, the random variable $u(t,x)$ belongs to the space $\mathbb{D}^\infty$. Moreover, for any $p \geqslant 1$ and $N \geqslant 1$, the N-iterated Malliavin derivative $D^N u(t,x)$ satisfies

$$\begin{aligned} D^N u(t,x) = &\int_0^t \mathrm{d}s \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s;x,y)\Delta^N(f,u(s,y))\mathrm{d}y \\ &+ \int_0^t \mathrm{d}s \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s;x,y) D^N u(s,y) f'(s,y,u(s,y))\mathrm{d}y, \end{aligned} \tag{8}$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathsf{E}\big[\|D^N u(t,x)\|^p_{L^p(\Omega;\mathscr{H}^{\otimes N})}\big] < +\infty.$$

We will prove this theorem by applying the next lemma, which follows from the fact that D^N is a closed operator defined on $L^p(\Omega)$ with values in $L^p(\Omega;\mathscr{H}^{\otimes N})$.

Lemma 9 Let $\{F_n\}_{n\geqslant 1}$ be a sequence of random variables belonging to $\mathbb{D}^{N,p}$. Assume that:

(i) there exists a random variable F such that F_n converges to F in $L^p(\Omega)$ as n tends to ∞;

(ii) the sequence $\{D^N F_n\}_{n\geqslant 1}$ converges in $L^p(\Omega;\mathscr{H}^{\otimes N})$.

Then $F \in \mathbb{D}^{N,p}$ and $D^N F = \lim\limits_{n\to\infty} D^N F_n$ in $L^p(\Omega;\mathscr{H}^{\otimes N})$.

Following the similar arguments in the proof of Theorem 1 in [4], we also consider the sequence of processes $\{u^{(n)}(t,x) : (t,x) \in [0,T]\times\mathbb{R}\}$ solving the equation (1)

$$\begin{cases} u^{(0)}(t,x) = \displaystyle\int_{\mathbb{R}} G(t;x,y)u_0(y)\mathrm{d}y; \\ u^{(n)}(t,x) = u^{(0)}(t,x) + \displaystyle\int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s;x,y) f(s,y,u^{(n-1)}(s,y))\mathrm{d}y\mathrm{d}s \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} G(t-s;x,y) W^H(\mathrm{d}y,\mathrm{d}s). \end{cases} \tag{9}$$

A standard argument (see, for instance, [6]) yields that $u^{(n)}(t,x) \in \mathbb{D}^\infty$, for all $n \geqslant 1$. Moreover, the derivative $D^N u^{(n)}(t,x)$ satisfies the following integral equation

$$\begin{aligned} D^N_\sigma u^{(n)}(t,x) = &\int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \frac{\partial G}{\partial y}(t-s;x,y)\Delta^N_\sigma(f,u^{(n-1)}(s,y)) \\ &+ \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \frac{\partial G}{\partial y}(t-s;x,y) D^N_\sigma u^{(n-1)}(s,y) f'(s,y,u^{(n-1)}(s,y)), \end{aligned} \tag{10}$$

where $\sigma = (\varphi_1,\varphi_2,\ldots,\varphi_N)$ and $\varphi_i \in \mathscr{H}$ with $i = 1,2,\ldots,N$.

Lemma 10 Under the same hypothesis in Theorem 2, for $p \geqslant 1$ and every $N \geqslant 1$, we have

$$\sup_{n\geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathsf{E}\big[\|D^N u^{(n)}(t,x)\|^p_{\mathscr{H}^{\otimes N}}\big] < +\infty. \tag{11}$$

Proof We will use an induction argument with respect to N with $p \geqslant 2$. For $N = 1$, (11) has been proved in [4]. Next let us assume that

$$\sup_{n \geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathsf{E}\big[\|D^k u^{(n)}(t,x)\|^p_{\mathscr{H}^{\otimes k}}\big] < +\infty,$$

for any $k = 1, 2, \ldots, N-1$. Let $\sigma = (e_{j_1}, e_{j_2}, \ldots, e_{j_N})$. Then, by (6), we have that

$$\mathsf{E}\big[\|D^N u^{(n)}(t,x)\|^p_{\mathscr{H}^{\otimes N}}\big] = \mathsf{E}\Big[\sum_{j_1,j_2,\cdots,j_N} |D^N_\sigma u^{(n)}(t,x)|^2\Big]^{p/2} \leqslant C \sum_{i=1}^{2} N_i,$$

where

$$N_1 = \mathsf{E}\Big[\sum_{j_1,j_2,\cdots,j_N} \Big|\int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \frac{\partial G}{\partial y}(t-s;x,y)\Delta^N_\sigma(f, u^{(n-1)}(s,y))\Big|^2\Big]^{p/2},$$
$$N_2 = \mathsf{E}\Big[\sum_{j_1,j_2,\cdots,j_N} \Big|\int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \frac{\partial G}{\partial y}(t-s;x,y) D^N_\sigma u^{(n-1)}(s,y) f'(s,y,u^{(n-1)}(s,y))\Big|^2\Big]^{p/2}.$$

Using similar arguments (this time for deterministic integration), Hölder inequality and the estimates on $\partial G/\partial y$, we obtain

$$\begin{aligned} N_1 &\leqslant C \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \frac{\partial G}{\partial y}(t-s;x,y)\mathsf{E}\big[\|\Delta^N(f, u^{(n-1)}(s,y))\|^p_{\mathscr{H}^{\otimes N}}\big] \\ &\leqslant C \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathsf{E}\big[\|\Delta^N(f, u^{(n-1)}(s,y))\|^p_{\mathscr{H}^{\otimes N}}\big], \end{aligned}$$

which again by the induction hypothesis, then N_1 is uniformly bounded in n, t and x.

Finally, as for N_2,

$$N_2 \leqslant C \int_0^t \mathrm{d}s \sup_{(r,y)\in[0,s]\times\mathbb{R}} \mathsf{E}\big[\|D^N u^{(n-1)}(r,y)\|^p_{\mathscr{H}^{\otimes N}}\big].$$

Summarizing the above estimates obtained for N_1 and N_2, we obtain

$$\begin{aligned} &\sup_{(s,y)\in[0,t]\times\mathbb{R}} \mathsf{E}\big[\|D^N u^{(n)}(s,y)\|^p_{\mathscr{H}^{\otimes N}}\big] \\ &\leqslant C\Big\{1 + \int_0^t \mathrm{d}s \sup_{(r,y)\in[0,s]\times\mathbb{R}} \mathsf{E}\big[\|D^N u^{(n-1)}(r,y)\|^p_{\mathscr{H}^{\otimes N}}\big]\Big\}. \end{aligned}$$

We can complete the proof of this lemma by using the Gronwall's lemma. □

For $N \geqslant 1$, we introduce the assumption that the sequence $\{D^j u^{(n)}(t,x), n \geqslant 1\}$ converges in $L^p(\Omega;\mathscr{H}^{\otimes j})$, $j = 1, 2, \ldots, N-1$, with the convention that $L^p(\Omega;\mathscr{H}^{\otimes 0}) = L^p(\Omega)$. We denote this assumption by $(\mathbf{H}_{N-1})$. For $N > 1$, $(\mathbf{H}_{N-1})$ implies that $u(t,x) \in$

$\mathbb{D}^{j,p}$ and the sequences $D^j u^{(n)}(t,x)$, $n \geqslant 1$ converge in $L^p(\Omega;\mathscr{H}^{\otimes j})$ to $D^j u(t,x)$ with $j = 1, 2, \ldots, N-1$. In addition, by Lemma 10,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathsf{E}\big[\|D^j X\|^p_{L^p(\Omega;\mathscr{H}^{\otimes j})}\big] < +\infty, \tag{12}$$

for all $j = 1, 2, \ldots, N-1$.

Proof of Theorem 8 Fix $(t,x) \in [0,T] \times \mathbb{R}$, $p \geqslant 2$. We apply Lemma 9 to $F_n = u^{(n)}(t,x)$ and $F = u(t,x)$. We know that the assumption (i) of Lemma 9 is satisfied. One can see the proof of Theorem 1 in [4].

Let us check the sequence $\{D^N u^{(n)}(t,x)\}_{n\geqslant 1}$ converges in the space $L^p(\Omega;\mathscr{H}^{\otimes N})$, for every $N \geqslant 1$ and $p \geqslant 2$, which implies that the random variables $D^N u(t,x)$ exists, belongs to $L^p(\Omega;\mathscr{H}^{\otimes N})$ and by Lemma 10 satisfies

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathsf{E}\big[\|D^N u(s,y)\|^p_{\mathscr{H}^{\otimes N}}\big] < +\infty.$$

Owing to Lemma 10, it suffices to check the assertion $p = 2$. We will use an induction argument N. For $N = 1$, the proof is given in [4]. Assume the induction hypothesis $(\mathbf{H}_{N-1})$ holds. Let $\mathbb{B}_{p,N}$ be the class of $\mathscr{H}^{\otimes N}$-valued processes $\{\Gamma(t,x); (t,x) \in [0,T]\times\mathbb{R}\}$ and satisfy

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathsf{E}\big[\|\Gamma(s,y)\|^p_{\mathscr{H}^{\otimes N}}\big] < +\infty.$$

We consider the stochastic integral equation in $\mathbb{B}_{p,N}$,

$$U(t,x) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}z \frac{\partial G}{\partial z}(t-s;x,z)[\Delta(f, u(s,z)) + U(s,z) f'(s,z,u(s,z))].$$

There exists a unique solution to the equation. Then under the Hypothesis $(\mathbf{H}_{N-1})$ one obtains that

$$U(t,x) = \lim_{n\to\infty} D^N u^{(n)}(t,x), \qquad \text{in } \ L^2(\Omega;\mathscr{H}^{\otimes N}),$$

by following the similar arguments in the proof of Theorem 2 in [9]. Moreover the limit is uniform in (t,x). Then by uniqueness of the solution $U = D^N u$ and the process $D^N u(t,x)$ satisfies equation (8). $\square$

Next let us prove the L^p-integrability of the inverse of of the Malliavin derivative of $u(t,x)$ for any fixed $(t,x) \in [0,T]\times\mathbb{R}$.

Theorem 11 Assume that the coefficient f is C^1-function with bounded Lipschitz continuous derivatives. Then, for any $q \geqslant 2$

$$\mathsf{E}\big[\|Du(t,x)\|^{-q}_{\mathscr{H}}\big] < +\infty. \tag{13}$$

This result, together with Theorem 8 applied to equation (1), yields the main result Theorem 2 in this paper.

Proof of Theorem 11 We need to show that the inverse of the Malliavin derivative of $u(t,x)$ has moments of order $q \geqslant 2$, that is, for all $q \geqslant 2$

$$\mathsf{E}\big[\|Du(t,x)\|_{\mathscr{H}}^{-q}\big] < +\infty.$$

It turns out (see, for instance, Lemma 2.3.1 in [17]) that it suffices to check that for any $q \geqslant 2$, there exists an $\varepsilon_0(q) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$\mathsf{P}\big[\|Du(t,x)\|_{\mathscr{H}}^{2} < \varepsilon\big] \leqslant C\varepsilon^q. \tag{14}$$

Let $\mathscr{H}([0,t]\times\mathbb{R})$ denote the Hilbert space over the rectangle $[0,t]\times\mathbb{R}$. Note that the positivity of the Green function G guarantees that the solution of equation (4) is nonnegative. Then proceeding as in the proof of Theorem 5.1 in [4], fix $\delta \in (0,1]$, denote by $\Psi_H(r_1,r_2;z_1,z_2) = 4H_1H_2(2H_1-1)(2H_2-1)|r_1-r_2|^{2H_1-2}|z_1-z_2|^{2H_2-2}$, then we obtain that

$$\begin{aligned}
&\|Du(t,x)\|_{\mathscr{H}}^2\\
=&\int_0^t\int_0^t\int_{\mathbb{R}^2} D_{r_1,z_1}u(t,x)D_{r_2,z_2}u(t,x)\Psi_H(r_1,r_2;z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\\
\geqslant&\int_{[(1-\delta)t,t]}^t\int_{[(1-\delta)t,t]}^t\int_{\mathbb{R}^2} D_{r_1,z_1}u(t,x)D_{r_2,z_2}u(t,x)\Psi_H(r_1,r_2;z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\\
=&\|G(t-\cdot;x,*)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}^2\\
&+2\Big\langle G(t-\cdot;x,*),\int_{(1-\delta)t}^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(t-s;x,y)f'(s,y,u(s,y))Du(s,y)\mathrm{d}y\mathrm{d}s\Big\rangle_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\\
&+\Big\|\int_{(1-\delta)t}^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(t-s;x,y)f'(s,y,u(s,y))Du(s,y)\mathrm{d}y\mathrm{d}s\Big\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}^2\\
\geqslant&\|G(t-\cdot;x,*)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}^2-2\|G(t-\cdot;x,*)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\\
&\cdot\int_{(1-\delta)t}^t\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s;x,y)f'(s,y,u(s,y))\Big|\|Du(s,y)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\mathrm{d}y\mathrm{d}s\\
&-\int_{(1-\delta)t}^t\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s;x,y)f'(s,y,u(s,y))\Big|\|Du(s,y)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}^2\mathrm{d}y\mathrm{d}s\\
=&\frac{1}{2}\|G(t-\cdot;x,*)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}^2-I(t,x,\delta),
\end{aligned}\tag{15}$$

where we denote by

$$I(t,x,\delta) = 2\|G(t-\cdot;x,*)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}$$

$$\begin{aligned}
&\cdot \int_{(1-\delta)t}^{t}\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s;x,y)f'(s,y,u(s,y))\Big|\|Du(s,y)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\mathrm{d}y\mathrm{d}s\\
&+\int_{(1-\delta)t}^{t}\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s;x,y)f'(s,y,u(s,y))\Big|\|Du(s,y)\|^2_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\mathrm{d}y\mathrm{d}s\\
:=\ & I_1(t,x,\delta)+I_2(t,x,\delta). \qquad (16)
\end{aligned}$$

The lower bound for $\|G(t-\cdot;x,*)\|^2_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}$ has been proved by Liu and Yan [4] as follows

$$\|G(t-\cdot;x,*)\|^2_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\geqslant C(\delta t)^{2H_1+4\gamma}. \qquad (17)$$

We have already decomposed the term $I(t,x,\delta)$ into two terms, so that we need to find the upper bounds for $\mathsf{E}|I_i(t,x,\delta)|^p$, $i=1,2$. On one hand, owing to Hölder inequality and (4.11) in [4], we get

$$\begin{aligned}
&\mathsf{E}|I_1(t,x,\delta)|^p\\
\leqslant\ & 2^p\mathsf{E}\Big(\|G(t-\cdot;x,*)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\\
&\cdot\int_0^t\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s;x,y)\Big||f'(s,y,u(s,y))|\|Du(s,y)\|_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\mathrm{d}y\mathrm{d}s\Big)^p\\
\leqslant\ & C(\delta t)^{p(\alpha H_1+H_2-1)/\alpha}\cdot(\delta t)^{p(\alpha H_1+H_2-1)/\alpha+p-p/\alpha}\\
=\ & C(\delta t)^{p[2(\alpha H_1+H_2-1)/\alpha+1-1/\alpha]}.
\end{aligned}$$

On the other hand, using the similar arguments, one proves that $\mathsf{E}|I_2(t,x,\delta)|^p$ may be bounded, up to some positive constant, by $(\delta t)^{p[2(\alpha H_1+H_2-1)/\alpha+1-1/\alpha]}$. So

$$\begin{aligned}
&\mathsf{P}\big[\|Du(t,x)\|^2_{\mathscr{H}}<\varepsilon\big]\\
\leqslant\ & \mathsf{P}\Big[I(t,x,\delta)\geqslant\frac{1}{2}\|G(t-\cdot;x,*)\|^2_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}-\varepsilon\Big]\\
\leqslant\ & \Big[\frac{1}{2}\|G(t-\cdot;x,*)\|^2_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}-\varepsilon\Big]^{-p}\mathsf{E}[|I(t,x,\delta)|^p],
\end{aligned}$$

for any $p\geqslant 2$, where $I(t,x,\delta)$ is defined by (16). Thus, we have proved that

$$\begin{aligned}
&\mathsf{P}\big[\|Du(t,x)\|^2_{\mathscr{H}}<\varepsilon\big]\\
\leqslant\ & \Big[\frac{1}{2}\|G(t-\cdot;x,*)\|^2_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}-\varepsilon\Big]^{-p}(\delta t)^{p[2(\alpha H_1+H_2-1)/\alpha+1-1/\alpha]}.
\end{aligned}$$

At this point, we choose $\delta=\delta(\varepsilon,t)$ in such a way that

$$\begin{aligned}
&\|G(t-\cdot;x,*)\|^2_{\mathscr{H}([(1-\delta)t,t]\times\mathbb{R})}\\
=\ & 4\varepsilon\geqslant C(\delta t)^{2H_1+4\gamma}=C(\delta t)^{[2(\alpha H_1+H_2-1)/\alpha+4\gamma+(2-2H_2)/\alpha]}
\end{aligned}$$

$$= C(\delta t)^{2(\alpha H_1 + H_2 - 1)/\alpha},$$

that implies $(\delta t) \leqslant C\varepsilon^{\alpha/[2(\alpha H_1 + H_2 - 1)]}$. Hence, one gets

$$\mathsf{P}\left[\|Du(t,x)\|^2_{\mathscr{H}} < \varepsilon\right] \leqslant C\varepsilon^{p(\alpha-1)/[2(\alpha H_1 + H_2 - 1)]}.$$

Finally, and it suffices to take p sufficiently large such that $p(\alpha - 1)/[2(\alpha H_1 + H_2 - 1)] \geqslant q$. □

References

[1] BALAN R M, TUDOR C A. Stochastic heat equation with multiplicative fractional-colored noise [J]. *J Theoret Probab*, 2010, **23(3)**: 834–870.

[2] BALAN R M. The stochastic wave equation with multiplicative fractional noise: a malliavin calculus approach [J]. *Potential Anal*, 2012, **36(1)**: 1–34.

[3] HU Y Z, NUALART D, SONG J. Feynman-Kac formula for heat equation driven by fractional white noise [J]. *Ann Probab*, 2011, **39(1)**: 291–326.

[4] LIU J F, YAN L T. Solving a nonlinear fractional stochastic partial differential equation with fractional noise [J]. *J Theoret Probab*, 2016, **29(1)**: 307–347.

[5] LIU J F, TUDOR C A. Analysis of the density of the solution to a semilinear SPDE with fractional noise [J]. *Stochastics*, 2016, **88(7)**: 959–979.

[6] MILLET A, SANZ-SOLÉ M. A stochastic wave equation in two space dimension: smoothness of the law [J]. *Ann Probab*, 1999, **27(2)**: 803–844.

[7] NUALART D, QUER-SARDANYONS L. Existence and smoothness of the density for spatially homogeneous SPDEs [J]. *Potential Anal*, 2007, **27(3)**: 281–299.

[8] NUALART D, QUER-SARDANYONS L. Gaussian density estimates for solutions to quasi-linear stochastic partial differential equations [J]. *Stochastic Process Appl*, 2009, **119(11)**: 3914–3938.

[9] QUER-SARDANYONS L, SANZ-SOLÉ M. A stochastic wave equation in dimension 3: smoothness of the law [J]. *Bernoulli*, 2004, **10(1)**: 165–186.

[10] WALSH J B. An introduction to stochastic partial differential equations [M] // CARMONA R, KESTEN H, WALSH J B. *Ecole d'Été de Probabilités de Saint-Flour XIV – 1984*, Berlin: Springer-Verlag, 1986: 265–439.

[11] WU J L, XIE B. On a Burgers type nonlinear equation perturbed by a pure jump Lévy noise in $\mathbb{R}^d$ [J]. *Bull Sci Math*, 2012, **136(5)**: 484–506.

[12] BASS R F. Uniqueness in law for pure jump Markov processes [J]. *Probab Theory Related Fields*, 1988, **79(2)**: 271–287.

[13] KOLOKOLTSOV V. Symmetric stable laws and stable-like jump-diffusions [J]. *Proc London Math Soc*, 2000, **80(3)**: 725–768.

[14] KOMATSU T. Markov processes associated with certain integro-differential operators [J]. *Osaka J Math*, 1973, **10(2)**: 271–303.

[15] KOMATSU T. Continuity estimates for solutions of parabolic equations associated with jump type Dirichlet forms [J]. *Osaka J Math*, 1988, **25(3)**: 697–728.

[16] KOMATSU T. Uniform estimates for fundamental solutions associated with non-local Dirichlet forms [J]. *Osaka J Math*, 1995, **32(4)**: 833–860.

[17] NUALART D. *The Malliavin Calculus and Related Topics* [M]. 2nd ed. Berlin: Springer-Verlag, 2006.

[18] HOH W. Pseudo differential operators with negative definite symbols of variable order [J]. *Rev Mat Iberoam*, 2000, **16(2)**: 219–241.

[19] JACOB N, LEOPOLD H G. Pseudo differential operators with variable order of differentiation generating feller semigroups [J]. *Integral Equations Operator Theory*, 1993, **17(4)**: 544–553.

[20] KIKUCHI K, NEGORO A. On Markov process generated by pseudodifferential operator of variable order [J]. *Osaka J Math*, 1997, **34(2)**: 319–335.

由分数噪声驱动的一类分数阶随机偏微分方程的光滑密度研究

苍玉权　李沁怡　刘俊峰

(南京审计大学统计学系, 南京, 211815)

摘　要: 本文中, 我们研究了由分数噪声驱动的一类分数阶随机偏微分方程, 利用 Malliavin 分析技巧, 证明了该类方程的适度解在任意固定的点 $(t,x)\in[0,T]\times\mathbb{R}$ 具有光滑密度.

关键词: 分数阶随机偏微分方程; 变系数的稳定类过程生成元; 分数噪声; Malliavin 分析; 光滑密度

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