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The Infinitesimal Generator of Markov Semigroup Associated with Multivalued Stochastic Differential Equation *

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Abstract: Under the condition that the coefficients are Lipschitz continuous, we study the infinitesimal generator of Markov semigroup corresponding to the multivalued stochastic equation. In order to provide a core of the infinitesimal generator, we investigate the associated multivalued elliptic equation and its viscosity solutions.

Keywords: multivalued stochastic differential equation; Markov semigroup; infinitesimal generator; viscosity solution

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§1. Introduction

In recent years, the multivalued stochastic differential equation (MSDE in short) has attracted much attention. In [1, 2], Cépa established the existence and uniqueness of a strong solution for the MSDE as following

$$\begin{cases} b(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}W_t\in \mathrm{d}X_t+A(X_t)\mathrm{d}t, \qquad t\geqslant 0,\\ X_0=x_0\in\overline{D(A)}, \end{cases} \tag{1}$$

where A is a maximal monotone operator on R^d, $b: R^d\to R^d$ and $\sigma: R^d\to R^d\otimes R^d$ are Lipschitz continuous, $(W_t)_{t\geqslant 0}$ is a d-dimensional Brownian motion defined in a filtered

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probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathsf{P})$, and $D(A) := \{x \in R^d : A(x) \neq \emptyset\}$. This has motivated a series of works on MSDEs. Rascanu [3] and Zhang [4] considered SDEs with maximal monotone operators in infinite dimensional spaces, Ren and Wu [5] and Maticiuc et al. [6] studied MSDEs driven by discontinuous processes, and Liu and Xu [7] investigated the average principle for MSDEs. For a systematic treatment of this subject we refer to the recent monograph [8]. For the study related to the Markovian property, Cépa [9], Ren, Wu and Zhang [5,10] proved that the solutions for MSDEs are strong Feller, ergodic, and the invariant measures have some regularity under certain conditions.

One of the significant problem in the theory of Markov processes is to characterize the generator of the processes. Barbu and Da Prato [11] investigated the generator of transition semigroup corresponding to the stochastic variational inequality (SVI in short)

$$\begin{cases} \mathrm{d}X(t) + F(X(t))\mathrm{d}t + \partial \mathbf{1}_K(X(t))\mathrm{d}t \ni \sqrt{Q}\mathrm{d}W(t), \qquad t \geqslant 0, \\ X(0) = x, \end{cases} \tag{2}$$

where $F : R^d \to R^d$ is a monotone operator, Q is a $d \times d$ strictly positive symmetric matrix, K is a closed, convex subset of R^d with non-empty interior $\mathring{K}$ and smooth boundary ∂K. As we known, $\partial \mathbf{1}_K$ is a specific maximal monotone operator on R^d. Denote by L its infinitesimal generator. Then [11] claims that a core of L is $D = \{\varphi \in C_b^2(R^d) : \partial\varphi/\partial n = 0$ on $\partial K\}$. Nevertheless, their method depends on some special properties of $\partial \mathbf{1}_K$. If we replace it by an arbitrary maximal monotone operator on R^d, the problem will become much more complicated.

Motivated by the above works, the main aim of this paper is to study the infinitesimal generator of the semigroup P_t associated to the solution of (1). $(P_t)_{t \geqslant 0}$ is a strong continuous contraction semigroup on $C_0(\overline{D(A)})$. Denote by L and $(R_\lambda)_{\lambda \geqslant 0}$ respectively its infinitesimal generator and resolvent. Thus, $(\lambda I - L)^{-1} = R_\lambda$ in $D(L)$. We aim at finding out a core of L. Taking into account of Proposition 3.1 in [12], a subset D of $D(L)$ is a core of L if and only if D is dense in $C_0(\overline{D(A)})$ and $\mathscr{R}(\lambda I - L|_D)$ is dense in $C_0(\overline{D(A)})$ for some $\lambda > 0$. Hence the key to our problem is to find out such D. We treat this problem by investigating the equation

$$-\lambda u + \frac{1}{2} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + g(x) \in \langle A(x), Du \rangle, \qquad x \in \overline{D(A)}, \tag{3}$$

where $a := \sigma\sigma*$ and the Einstein summation convention is used here and throughout the paper. However, this equation is different from the traditional partial differential equation because it involves a maximal monotone operator A and it can be hardly expected to admit a classic solution, so that we have to look for weak solutions to it. Moreover, since A is nonlinear and highly singular, it seems that the weak solution in viscosity sense rather than in distribution sense is more appropriate for our purpose.

Given $g \in C_0(\overline{D(A)})$, we will prove that $R_\lambda g$ is a viscosity solution of (3) in $\overline{D(A)}$ by Yosida approximation and some equivalent definitions of viscosity solution. More precisely, using a similar method to [13] we prove a comparison principle for viscosity solutions of (3), which implies that $R_\lambda g$ is the unique viscosity solution of (3) for all $\lambda > 0$. Denote by S_λ the viscosity solution space of (3). It follows that $S_\lambda = R_\lambda(C_0(\overline{D(A)}))$. Since $R_\lambda(G)$ is dense in $C_0(\overline{D(A)})$, it is a core of N for any $\lambda > 0$.

§2. Preliminaries

We present some notions and notations that will be used throughout the paper.

Given a multivalued operator $A : R^d \to 2^{R^d}$, we define

$$\mathrm{Gr}(A) := \{(x,y) \in R^{2d} : x \in D(A) \text{ and } y \in A(x)\}.$$

Definition 1 1) A monotone operator on R^d is a multivalued operator A satisfying

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0, \qquad \forall\, (x_1,y_1),(x_2,y_2) \in \mathrm{Gr}(A).$$

2) A maximum monotone operator on R^d is a monotone operator A satisfying

$$\langle x_1, y_1 \rangle \in \mathrm{Gr}(A) \Leftrightarrow \langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0, \qquad \forall\, (x_2,y_2) \in \mathrm{Gr}(A).$$

Below, we list some properties of a maximal monotone operator A which can be found in [14; Chapter 3].

(i) $\mathrm{Int}(D(A))$ and $\overline{D(A)}$ are convex subsets of R^d with $\mathrm{Int}(\overline{D(A)}) = \mathrm{Int}(D(A))$.

(ii) For any $n \in N^+$, $J_n = (I - A/n)^{-1}$ is a Lipschitz continuous function on R^d with the Lipschitz constant equal to 1. $A_n(x) := n[x - J_n(x)]$, $x \in R^d$ is called the Yosida approximation of A.

(iii) $A_n \in A(J_n(x))$.

(iv) $\lim\limits_{n\to\infty} A_n(x) = A^o(x)$, $\forall\, x \in D(A)$, where $A^o(x) := \mathrm{Proj}_{A(x)}(0)$.

We shall make the following **Hypotheses**:

(H1) b and σ are Lipschitz continuous: there exists an $C > 0$ such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leqslant C|x-y|.$$

(H2) $0 \in D(A)$ and $0 \in A(0)$.

Under the above hypotheses, Cépa [1] proved the existence and uniqueness of a strong solution to (1) by Yosida approximation. Denote by $(X_t(x_0), K_t)_{t\geqslant 0}$ the strong solution of (1) and $(X_t^n(x_0))_{t\geqslant 0}$ the strong solution of the equation

$$\begin{cases} b(X_t^n)\mathrm{d}t + \sigma(X_t^n)\mathrm{d}W_t = \mathrm{d}X_t^n + A_n(X_t^n)\mathrm{d}t, & t \geqslant 0, \\ X_0^n = x_0 \in K. \end{cases} \tag{4}$$

Theorem 2 ([13; Proposition.6]) The sequence $X_t^n(x)$ converges to $X_t(x)$ in $L^2(\Omega; C([0,T]; R^d))$ uniformly with respect to $x \in K$ for every bounded subset $K \subseteq \overline{D(A)}$.

For any $D \subset R^d$, denote by $C_0(D)$ the space of continuous functions on D vanishing at infinity with the norm $\|f\| := \sup\limits_{x\in D} |f(x)|$. Meanwhile, $\{X_t(x_0),\ x_0 \in \overline{D(A)}\}$ is a Markov family with transition probability $P_t(x_0, E) = \mathsf{P}(X_t(x_0) \in E)$, $E \in \mathscr{B}(R^d)$. By [10], the transition probability $P_t(\cdot,\cdot)$ is strong Feller. Hence it determines a Feller semigroup $(P_t)_{t\geqslant 0}$ in $C_0(\overline{D(A)})$, i.e.,

$$P_t\varphi(x) := \mathsf{E}[\varphi(X_t(x))], \qquad \varphi \in C_0(\overline{D(A)}),\ x \in \overline{D(A)},\ t \geqslant 0. \tag{5}$$

The resolvent of $\{P_t\}$ is given by

$$R_\lambda\varphi(x) := \int_0^\infty \mathrm{e}^{-\lambda t} P_t\varphi(x)\mathrm{d}t, \qquad \lambda > 0.$$

§3. Main Results

Denoted by L the infinitesimal generator of $(P_t)_{t\geqslant 0}$. In order to find out a core of L, we investigate the equation (3) where $g \in C_0(\overline{D(A)})$.

Set

$$A_-(x,q) := \liminf_{\substack{(x',q')\to(x,q)\\ x^*\in A(x')}} \langle x^*, q'\rangle, \qquad A_+(x,q) := \limsup_{\substack{(x',q')\to(x,q)\\ x^*\in A(x')}} \langle x^*, q'\rangle.$$

Then $A_+(x,q) = -A_-(x,-q)$. We have

Lemma 3 ([13; Lemma 2]) If $x \in \mathrm{Int}(D(A))$, $q \in R^d$, then $A_-(x;q) = \inf\limits_{x^*\in A(x)} \langle x^*, q\rangle$ (respectively, $A_+(x;q) = \sup\limits_{x^*\in A(x)} \langle x^*, q\rangle$). This equality still holds when $x \in bd(D(A))$, $q \in R^d$ and $\inf\limits_{n\in N_{\overline{D(A)}}(x)} \langle n, q\rangle > 0$. $N_{\overline{D(A)}}(x)$ is the set of unitarian normal vectors to $\overline{D(A)}$ in $x \in R^d$.

Now we introduce the notion of a viscosity solution for the equation (3).

Definition 4 We say that $u \in \mathrm{USC}(\overline{D(A)})$ is a viscosity sub-solution of equation (3) iff

$$-\lambda u + \frac{1}{2} a_{ij}(x) \frac{\partial^2 \psi}{\partial x_i \partial x_j} + b_i(x) \frac{\partial \psi}{\partial x_i} + g(x) \geqslant A_-(x; D\psi) \tag{6}$$

whenever $\psi \in C^2(\overline{D(A)})$ and x is the local maximum point of $u - \psi$.

Similarly, $v \in \mathrm{LSC}(\overline{D(A)})$ is a viscosity super-solution of equation (3) iff

$$-\lambda v + \frac{1}{2} a_{ij}(x) \frac{\partial^2 \psi}{\partial x_i \partial x_j} + b_i(x) \frac{\partial \psi}{\partial x_i} + g(x) \leqslant A_+(x; D\psi) \tag{7}$$

whenever $\psi \in C^2(\overline{D(A)})$ and x is the local minimum point of $v - \psi$.

A viscosity solution for (3) is both a viscosity super-solution and a sub-solution.

$\mathscr{T}(d)$ will stand for the set of symmetric matrices. For a subset K of R^d, a function $u : K \to R^d$ and $\widehat{x} \in K$, define $J_K^{2,+} u(\widehat{x})$ to be the set of $(p, X) \in R^d \times \mathscr{T}(d)$ satisfying

$$u(x) \leqslant u(\widehat{x}) + \langle p, x - \widehat{x} \rangle + \frac{1}{2} \langle X(x - \widehat{x}), x - \widehat{x} \rangle + o(|x - \widehat{x}|^2) \qquad \text{as } x \to \widehat{x}.$$

Similarly, $J_K^{2,-} u(\widehat{x})$ is the set of $(p, X) \in R^d \times \mathscr{T}(d)$ satisfying

$$u(x) \geqslant u(\widehat{x}) + \langle p, x - \widehat{x} \rangle + \frac{1}{2} \langle X(x - \widehat{x}), x - \widehat{x} \rangle + o(|x - \widehat{x}|^2) \qquad \text{as } x \to \widehat{x}.$$

Equivalently, $J_K^{2,-} u(x) := -J_K^{2,+}(-u)(x)$. Besides, we set

$$\begin{aligned}
\overline{J}_K^{2,+} u(x) := \{&(p, X) \in R^d \times \mathscr{T}(d) : \exists\, (x_n, p_n, X_n) \in K \times R^d \times \mathscr{T}(d) \text{ such that} \\
&(p_n, X_n) \in J_K^{2,+} u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X)\}, \quad \forall\, x \in K. \\
\overline{J}_K^{2,-} u(x) := \{&(p, X) \in R^d \times \mathscr{T}(d) : \exists\, (x_n, p_n, X_n) \in K \times R^d \times \mathscr{T}(d) \text{ such that} \\
&(p_n, X_n) \in J_K^{2,-} u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X)\}, \quad \forall\, x \in K.
\end{aligned}$$

Obviously, $\overline{J}_K^{2,+} u(x) = -\overline{J}_K^{2,-}(-u)(x)$.

The following proposition gives an equivalent definition of viscosity solution to (2).

Proposition 5 $u \in \mathrm{USC}(\overline{D(A)})$ is a viscosity sub-solution of (3) if and only if

$$-\lambda u(x) + \frac{1}{2} \mathrm{tr}[a(x) X] + \langle b(x), p \rangle + g(x) \geqslant A_-(x, p), \qquad \forall\, (p, X) \in J^{2,+}_{\overline{D(A)}} u(x). \tag{8}$$

$v \in \mathrm{LSC}(\overline{D(A)})$ is a viscosity super-solution of (3) if and only if

$$-\lambda v(y) + \frac{1}{2} \mathrm{tr}[a(y) Y] + \langle b(y), q \rangle + g(y) \leqslant A_+(y, q), \qquad \forall\, (q, Y) \in J^{2,-}_{\overline{D(A)}} v(y). \tag{9}$$

Based on the continuity of $F(x,r,p,X) := -\lambda r + \mathrm{tr}[a(x)X]/2 + \langle b(x),p\rangle + g(x)$ on $R^d \times R \times R^d \times \mathscr{T}(d)$, we can replace $J^{2,+}_{D(A)}u(x)$ (respectively, $J^{2,+}_{D(A)}v(y)$) with $\overline{J}^{2,+}_{D(A)}u(x)$ (respectively, $\overline{J}^{2,-}_{D(A)}v(y)$).

The following theorem implies the uniqueness of viscosity solution for (3) in the space $C_0(\overline{D(A)})$.

Theorem 6 Let $\lambda > 0$ and $u \in C_0(\overline{D(A)})$ (respectively $v \in C_0(\overline{D(A)})$) be a viscosity sub-solution (respectively super-solution) of (3). Then $u \leqslant v$.

Proof Suppose that the contrary is valid, then there exists $\theta > 0$ and $x_0 \in \overline{D(A)}$ such that $\theta := u(x_0) - v(x_0) > 0$. As both u and v are bounded, there exists $N > 0$ such that $\|u\| + \|v\| < N$ on $\overline{D(A)}$. For $p \in [1,+\infty)$, $\epsilon < \theta/(2\|x_0\|^{2p})$ and arbitrary $\alpha > 0$, define

$$
\begin{aligned}
v_\epsilon(x) &:= v(x) + \frac{\epsilon}{2}\|x\|^{2p}, \qquad \forall\, x \in \overline{D(A)};\\
u_\epsilon(x) &:= u(x) - \frac{\epsilon}{2}\|x\|^{2p}, \qquad \forall\, x \in \overline{D(A)};\\
\psi_\alpha(x,y) &:= \frac{\alpha}{2}\|x-y\|^2, \qquad \forall\, x,y \in \overline{D(A)};\\
\phi_{\alpha,\epsilon}(x,y) &:= u_\epsilon(x) - v_\epsilon(y) - \psi_\alpha(x,y), \qquad \forall\, x,y \in \overline{D(A)}.
\end{aligned}
$$

We take $M_{\alpha,\epsilon} := \sup\limits_{(x,y)\in\overline{D(A)}^2} \phi_{\alpha,\epsilon}(x,y) < 2N$. As $\lim\limits_{\|x\|,\|y\|\to+\infty} \phi_{\alpha,\epsilon} = -\infty$, there exists $(x_{\alpha,\epsilon}, y_{\alpha,\epsilon}) \in \overline{D(A)} \times \overline{D(A)}$ such that $\phi(x_{\alpha,\epsilon}, y_{\alpha,\epsilon}) = M_{\alpha,\epsilon}$. Since $\epsilon < \theta/(2\|x_0\|^{2p})$, we have $M_{\alpha,\epsilon} \geqslant u(x_0) - v(x_0) - \epsilon\|x_0\|^{2p} \geqslant \theta/2$. Since u and v vanish at infinity, there exists $N' \in N^+$ such that $\|u(x)\| \vee \|v(x)\| \leqslant \theta/8$ whenever $\|x\| \geqslant N'$. Thus, we conclude that $\|x_{\alpha,\epsilon}\| \vee \|y_{\alpha,\epsilon}\| \leqslant N'$.

As $\alpha \to M_{\alpha,\epsilon}$ is decreasing, $\lim\limits_{\alpha\to+\infty} M_{\alpha,\epsilon}$ exists and is finite. Moreover, by [15; Lemma 3.1], we can conclude that $\lim\limits_{\alpha\to+\infty} \alpha\|x_{\alpha,\epsilon} - y_{\alpha,\epsilon}\| = 0$ and $\lim\limits_{\alpha\to+\infty} M_{\alpha,\epsilon} = \sup\limits_{x\in\overline{D(A)}}[u_\epsilon(x) - v_\epsilon(x)]$.

Using Theorem 3.2 in [15], we obtain that there exists $X, Y \in \mathscr{T}(d)$ such that

$$
(\alpha(x_{\alpha,\epsilon} - y_{\alpha,\epsilon}), X) \in \overline{J}^{2,+}_{D(A)} u_\epsilon(x_{\alpha,\epsilon}) \tag{10}
$$

and

$$
(-\alpha(x_{\alpha,\epsilon} - y_{\alpha,\epsilon}), -Y) \in \overline{J}^{2,+}_{D(A)} (-v_\epsilon)(y_{\alpha,\epsilon}). \tag{11}
$$

(11) is equivalent to $(\alpha(x_{\alpha,\epsilon} - y_{\alpha,\epsilon}), Y) \in \overline{J}^{2,-}_{D(A)} v_\epsilon(y_{\alpha,\epsilon})$. Meanwhile,

$$
-3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{12}
$$

From now on, we rewrite $(x_{\alpha,\epsilon}, y_{\alpha,\epsilon})$ as $(\widehat{x},\widehat{y})$ for simplicity. $(\alpha(\widehat{x}-\widehat{y}), X) \in \overline{J}^{2,+}_{\overline{D(A)}} u_\epsilon(\widehat{x})$ is equivalent to the fact that there exists $(x_n, p_n, X_n) \in \overline{D(A)} \times R^d \times \mathscr{T}(d)$ such that $(p_n, X_n) \in J^{2,+}_{\overline{D(A)}} u_\epsilon(x_n)$ and $\lim\limits_{n\to\infty}(x_n, u_\epsilon(x_n), p_n, X_n) = (\widehat{x}, u_\epsilon(\widehat{x}), \alpha(\widehat{x},\widehat{y}), X)$.

Therefore,

$$\begin{aligned} u(x) - \frac{\epsilon}{2}\|x\|^{2p} \leqslant{}& u(x_n) - \frac{\epsilon}{2}\|x_n\|^{2p} + \langle p_n, x - x_n\rangle \\ &+ \frac{1}{2}\langle X_n(x-x_n), x-x_n\rangle + o(\|x-x_n\|^2) \qquad \text{as } x \to x_n. \end{aligned} \tag{13}$$

Thus,

$$\begin{aligned} u(x) \leqslant{}& u(x_n) + \left(\frac{\epsilon}{2}\|x\|^{2p} - \frac{\epsilon}{2}\|x_n\|^{2p}\right) + \langle p_n, x - x_n\rangle \\ &+ \frac{1}{2}\langle X_n(x-x_n), x-x_n\rangle + o(\|x-x_n\|^2) \qquad \text{as } x_n \to x. \end{aligned} \tag{14}$$

Using the Taylor formula,

$$\begin{aligned} \frac{\epsilon}{2}\|x\|^{2p} ={}& \frac{\epsilon}{2}\|x_n\|^{2p} + p\epsilon\|x_n\|^{2p-2}\sum_{i=1}^d x_n^i(x^i - x_n^i) \\ &+ \epsilon p(p-1)\|x_n\|^{2p-4}\sum_{i,j=1}^d x_n^i x_n^j (x^i - x_n^i)(x^j - x_n^j) \\ &+ \frac{\epsilon}{2}p\|x_n\|^{2p-2}\sum_{i=1}^d (x^i - x_n^i)^2 + o\|x - x_n\|^2 \qquad \text{as } x_n \to x. \end{aligned}$$

Combining this with (14), we obtain that

$$\begin{aligned} u(x) \leqslant{}& u(x_n) + \langle p_n + \epsilon\beta(x_n), x - x_n\rangle \\ &+ \frac{1}{2}\langle [X_n + \epsilon B(x_n)](x-x_n), x-x_n\rangle + o(\|x-x_n\|^2), \end{aligned} \tag{15}$$

where $\beta(x) = p\|x\|^{2p-2}x$, $B(x) = p(2p-2)\|x\|^{2p-4}x\otimes x + p\|x\|^{2p-2}I$.

Since

$$(x_n, u(x_n), p_n + \epsilon\beta(x_n), X_n + \epsilon B(x_n)) \overset{n\to+\infty}{\longrightarrow} (\widehat{x}, u(\widehat{x}), \alpha(\widehat{x}-\widehat{y}) + \epsilon\beta(\widehat{x}), X + \epsilon B(\widehat{x})),$$

we have $(\alpha(\widehat{x}-\widehat{y}) + \epsilon\beta(\widehat{x}), X + \epsilon B(\widehat{x})) \in \overline{J}^{2,+}_{\overline{D(A)}} u(\widehat{x})$.

Similarly, $(\alpha(\widehat{x}-\widehat{y}) - \epsilon\beta(\widehat{y}), Y - \epsilon B(\widehat{y})) \in \overline{J}^{2,-}_{\overline{D(A)}} v(\widehat{y})$.

Due to the definition of viscosity sub-solution (respectively, super-solution) for equation (3), we obtain that

$$\begin{cases} & -\lambda u(\widehat{x}) + \dfrac{1}{2}\mathrm{tr}\{a(\widehat{x})[\widehat{X} + \epsilon B(\widehat{x})]\} + \langle b(\widehat{x}), \alpha(\widehat{x}-\widehat{y}) + \epsilon\beta(\widehat{x})\rangle + g(\widehat{x}) \\ \geqslant & A_-(\widehat{x}; \alpha(\widehat{x}-\widehat{y}) + \epsilon\beta(\widehat{x})), \\ & -\lambda v(\widehat{y}) + \dfrac{1}{2}\mathrm{tr}\{a(\widehat{y})[\widehat{Y} - \epsilon B(\widehat{y})]\} + \langle b(\widehat{y}), \alpha(\widehat{x}-\widehat{y}) - \epsilon\beta(\widehat{y})\rangle + g(\widehat{y}) \\ \leqslant & A_+(\widehat{y}; \alpha(\widehat{x}-\widehat{y}) - \epsilon\beta(\widehat{y})). \end{cases} \tag{16}$$

For any $x^* \in A(\widehat{x})$ and $y^* \in A(\widehat{y})$, $\langle x, x^*\rangle \geqslant 0$ and $\langle y, y^*\rangle \geqslant 0$. Therefore,

$$\begin{aligned}\langle x^*, \alpha(\widehat{x}-\widehat{y}) + \epsilon p\|\widehat{x}\|^{2p-2}\widehat{x}\rangle &\geqslant \langle x^*, \alpha(\widehat{x}-\widehat{y})\rangle \geqslant \langle y^*, \alpha(\widehat{x}-\widehat{y})\rangle \\ &\geqslant \langle y^*, \alpha(\widehat{x}-\widehat{y}) - \epsilon p\|\widehat{y}\|^{2p-2}\widehat{y}\rangle. \end{aligned} \tag{17}$$

Lemma 3 implies that

$$\begin{aligned} A_-(\widehat{x}; \alpha(\widehat{x}-\widehat{y}) + \epsilon\beta(\widehat{x})) &= \inf_{x^*\in A(\widehat{x})} \langle x^*, \alpha(\widehat{x}-\widehat{y}) + \epsilon p\|\widehat{x}\|^{2p-2}\widehat{x}\rangle, \\ A_+(\widehat{y}; \alpha(\widehat{x}-\widehat{y}) - \epsilon\beta(\widehat{y})) &= \sup_{y^*\in A(\widehat{y})} \langle y^*, \alpha(\widehat{x}-\widehat{y}) - \epsilon p\|\widehat{y}\|^{2p-2}\widehat{y}\rangle. \end{aligned}$$

Hence, we have

$$\begin{aligned} &-\lambda u(\widehat{x}) + \frac{1}{2}\mathrm{tr}\{a(\widehat{x})[\widehat{X} + \epsilon B(\widehat{x})]\} + \langle b(\widehat{x}), \alpha(\widehat{x}-\widehat{y}) + \epsilon\beta(\widehat{x})\rangle - g(\widehat{x}) \\ \geqslant &-\lambda v(\widehat{y}) + \frac{1}{2}\mathrm{tr}\{a(\widehat{y})[\widehat{Y} - \epsilon B(\widehat{y})]\} + \langle b(\widehat{y}), \alpha(\widehat{x}-\widehat{y}) - \epsilon\beta(\widehat{y})\rangle - g(\widehat{y}). \end{aligned}$$

Equivalently,

$$\begin{aligned} &\lambda[u(\widehat{x}) - v(\widehat{y})] - \frac{1}{2}\mathrm{tr}[a(\widehat{x})\widehat{X} - a(\widehat{y})\widehat{Y}] - \langle b(\widehat{x}) - b(\widehat{y}), \alpha(\widehat{x}-\widehat{y})\rangle + g(\widehat{x}) - g(\widehat{y}) \\ \leqslant &\frac{\epsilon}{2}\mathrm{tr}[a(\widehat{x})B(\widehat{x}) + a(\widehat{y})B(\widehat{y})] + \epsilon[\langle b(\widehat{x}), \beta(\widehat{x})\rangle + \langle b(\widehat{y}), \beta(\widehat{y})\rangle]. \end{aligned} \tag{18}$$

Since $M_{\alpha,\epsilon} \geqslant \theta/2$, we have

$$u(\widehat{x}) - u(\widehat{y}) \geqslant \frac{\theta}{2} + \epsilon(\|x\|^{2p} + \|y\|^{2p}). \tag{19}$$

By (12) we obtain

$$\mathrm{tr}[a(\widehat{x})\widehat{X} - a(\widehat{y})\widehat{Y}] \leqslant 3\alpha\,\mathrm{tr}[a(\widehat{x}) - a(\widehat{y})] \leqslant 3\alpha C^2\|\widehat{x}-\widehat{y}\|^2. \tag{20}$$

Considering the definition of $B(x)$ and $\beta(x)$, we conclude that there exists a constant C' such that

$$\begin{aligned} &\lambda\frac{\theta}{2} + \lambda\epsilon(\|\widehat{x}\|^{2p} + \|\widehat{y}\|^{2p}) \\ \leqslant &C'\epsilon(1 + \|\widehat{x}\|^{2p} + \|\widehat{y}\|^{2p}) + \frac{3}{2}\alpha C^2\|\widehat{x}-\widehat{y}\|^2 + C\alpha\|\widehat{x}-\widehat{y}\|^2 + |g(\widehat{x}) - g(\widehat{y})|. \end{aligned} \tag{21}$$

Recalling that $\lim\limits_{\alpha\to+\infty} \alpha\|\widehat{x}-\widehat{y}\| = 0$ and $\|\widehat{x}\| \vee \|\widehat{y}\| < N'$, there exists a constant C'' such that

$$\lambda\frac{\theta}{2} \leqslant C''\epsilon, \qquad \forall\, \epsilon > 0.$$

Therefore, we get a contradiction and the theorem has been proved. $\square$

Let $g \in C_0(\overline{D(A)})$, we are able to show that the unique viscosity solution of (3) is $u(x) = R_\lambda g(x)$ by Yosida approximation. For any $n \in N^+$, denote by $X_t^n(x)$ the strong solution of equation (4).

Lemma 7 For fixed $\lambda > 0$, $u_n(x) := \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathsf{E}[g(X_t^n(x))]\mathrm{d}t$ converges to $u(x) := \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathsf{E}[g(X_t(x))]\mathrm{d}t$ uniformly in $\mathbb{M}$ as $n \to +\infty$, whenever $\mathbb{M}$ is a bounded subset of $\overline{D(A)}$.

Proof Set

$$F_T^n(x) := \int_0^T \mathrm{e}^{-\lambda t}\mathsf{E}|g(X_t^n(x)) - g(X_t(x))|\mathrm{d}t,$$
$$F^n(x) := \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathsf{E}|g(X_t^n(x)) - g(X_t(x))|\mathrm{d}t.$$

Obviously,

$$|u_n(x) - u(x)| \leqslant F^n(x). \tag{22}$$

Since g is bounded in $\overline{D(A)}$, it yields that for any $\epsilon > 0$, there exists $T' > 0$, such that for every $T > T'$,

$$|F_T^n(x) - F^n(x)| \leqslant \frac{\epsilon}{4}, \qquad \forall\, x \in \mathbb{M} \text{ and } n \in N^+. \tag{23}$$

There exists $r > 0$ such that $\|x\| \leqslant r$ for $x \in \mathbb{M}$. We take ξ satisfying $|g(x)| \leqslant \xi$, $\forall\, x \in \overline{D(A)}$. Let

$$\omega(\delta, K) := \sup_{\|x\|,\|y\|\leqslant K,\|x-y\|\leqslant\delta} |g(x) - g(y)|.$$

By [16; Proposition 2.2], for fixed $\widehat{T} := T' + 1$ and every $p \geqslant 1$, there exists $C_p > 0$ such that

$$\mathsf{E}\Big[\sup_{t\in[0,\widehat{T}]} |X_t^n(x)|^p\Big] \leqslant C_p(1+\|x\|^p),\ \forall\, n \in N^+ \text{ and } \mathsf{E}\Big[\sup_{t\in[0,\widehat{T}]} |X_t(x)|^p\Big] \leqslant C_p(1+\|x\|^p). \tag{24}$$

Meanwhile, there exists $K > 0$ such that

$$C_2\Big(\frac{1 - \mathrm{e}^{-\lambda\widehat{T}}}{\lambda}\Big)(1 + r^2)\frac{\xi}{K^2} \leqslant \frac{\epsilon}{16}.$$

What's more, there exists $\delta > 0$ such that

$$\frac{1 - \mathrm{e}^{-\lambda\widehat{T}}}{\lambda}\omega(\delta, K) \leqslant \frac{\epsilon}{4}.$$

Applying Theorem 2, there exists there exists $N > 0$ such that for $n > N$,

$$\sup_{x\in\mathbb{M}} \mathsf{E}\Big[\sup_{0\leqslant t\leqslant\widehat{T}} |X_t^n(x) - X_t(x)|^2\Big] \leqslant \frac{\epsilon\delta^2}{4}.$$

Therefore, for arbitrary $n > N$ and $x \in \mathbb{M}$,

$$F_{\widehat{T}}^n(x) = \int_0^{\widehat{T}} \mathrm{e}^{-\lambda t}\mathsf{E}|g(X_t^n(x)) - g(X_t(x))|\mathrm{d}t$$

$$
\begin{aligned}
&\leqslant \int_0^{\widehat{T}} \mathrm{e}^{-\lambda t}\omega(\delta, K)\mathsf{P}(\|X_t^n(x) - X_t(x)\| \leqslant \delta)\mathrm{d}t \\
&\quad + \int_0^{\widehat{T}} \mathrm{e}^{-\lambda t} 2\xi \mathsf{P}(\|X_t^n(x) - X_t(x)\| \geqslant \delta)\mathrm{d}t \\
&\quad + \int_0^{\widehat{T}} \mathrm{e}^{-\lambda t} 2\xi \mathsf{P}(\|X_t^n(x)\| \vee \|X_t(x)\| \geqslant K)\mathrm{d}t \\
&\leqslant \Big(\frac{1-\mathrm{e}^{-\lambda\widehat{T}}}{\lambda}\Big)\Big\{\omega(\delta, K) + \mathsf{E}\Big[\sup_{0\leqslant t\leqslant\widehat{T}} |X_t^n(x) - X_t(x)|^2\Big]\Big/\delta^2 \\
&\quad + 2\xi\Big\{\mathsf{E}\Big[\sup_{0\leqslant t\leqslant\widehat{T}}|X_t^n(x)|^2\Big] + \mathsf{E}\Big[\sup_{0\leqslant t\leqslant\widehat{T}}|X_t(x)|^2\Big]\Big\}\Big/K^2\Big\} \\
&\leqslant \frac{\epsilon}{2} + 4C_2(1+r^2)\Big(\frac{1-\mathrm{e}^{-\lambda\widehat{T}}}{\lambda}\Big)\frac{\xi}{K^2} \\
&\leqslant \frac{3\epsilon}{4}.
\end{aligned}
$$

Combining the above inequality with (22) and (23), the lemma is proved. □

Theorem 8 Under the hypotheses, $u(x) = R_\lambda g(x)$ is the viscosity solution of (3).

Proof Let $\psi \in C^2(\overline{D(A)})$ such that $u - \psi$ reaches local maximum in $x \in \overline{D(A)}$. Let $\mathbb{M}$ be a bounded neighborhood of x in $\overline{D(A)}$, and hence $\{u_n(x)\}_{n=1}^{+\infty}$ converges to u uniformly in $\mathbb{M}$ by Lemma 7. Thus, there exists $x_n \to x$ such that $u_n - \psi$ attains local maximum in x_n. Meanwhile, u_n is the viscosity solution of

$$
-\lambda u_n + \frac{1}{2}a_{ij}(x)\frac{\partial^2 u_n}{\partial x_i \partial x_j} + b_i(x)\frac{\partial u_n}{\partial x_i} + g(x) = \langle A_n(x), Du_n\rangle. \tag{25}
$$

It yields that

$$
-\lambda u_n(x_n) + \frac{1}{2}a_{ij}(x_n)\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x_n) + b_i(x_n)\frac{\partial \psi}{\partial x_i}(x_n) + g(x_n) \geqslant \langle A_n(x_n), D\psi(x_n)\rangle.
$$

Notice that $J_n(x_n) \to x$ and $A_n(x_n) \in A(J_n x_n)$. Taking the limit as $n \to +\infty$ in above, we obtain that

$$
-\lambda u(x) + \frac{1}{2}a_{ij}(x)\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x) + b_i(x)\frac{\partial \psi}{\partial x_i}(x) + g(x) \geqslant A_-(x; D\psi(x)).
$$

We conclude that u is the viscosity sub-solution of (3). In the same way, u is the viscosity super-solution of (3) as well. □

Theorem 9 Denote by

$S_\lambda := \Big\{u \in C_0(\overline{D(A)}) \,|\, \exists\, g \in C_0(\overline{D(A)})$ such that u is the viscosity solution of

$$-\lambda u+\frac{1}{2}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+b_i(x)\frac{\partial u}{\partial x_i}+g(x)\in\langle A(x),Du\rangle,\ \text{in}\ \overline{D(A)}\Big\}.$$

For any $\lambda>0$, S_λ is a core of L.

Proof Actually, $S_\lambda = R_\lambda(C_0(\overline{D(A)}))$. Meanwhile, for any $f\in D(L)$, We take $g:=(\lambda I-L)f\in C_0(\overline{D(A)})$. Thus, $f=R_\lambda g$ and $D(L)=S_\lambda$. Noticing that $D(L)$ is dense in $C_0(\overline{D(A)})$, we conclude that S_λ is dense in $C_0(\overline{D(A)})$. Therefore, S_λ is a core of L for any $\lambda>0$. □

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与多值随机微分方程相关马氏半群的无穷小生成元

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摘 要： 当多值随机微分方程的扩散及漂移系数满足利普希兹连续性条件时, 我们考虑其解的无穷小生成元问题. 为了找出该无穷小生成元的核, 我们研究了对应的多值椭圆方程及其粘性解.

关键词： 多值随机微分方程; 马氏半群; 无穷小生成元; 粘性解

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