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A Note on Uniform Integrability of Random Variables in a Probability Space and Sublinear Expectation Space *

HU Zechun⋆

(*College of Mathematics, Sichuan University, Chengdu, 610064, China*)

ZHOU Qianqian

(*School of Mathematical Sciences, Nankai University, Tianjin, 300071, China*)

(*Department of Mathematics, Nanjing University, Nanjing, 210093, China*)

Abstract: In this note we discuss uniform integrability of random variables. In a probability space, we introduce two new notions on uniform integrability of random variables, and prove that they are equivalent to the classic one. In a sublinear expectation space, we give de La Vallée Poussin criterion for the uniform integrability of random variables and do some other discussions.

Keywords: uniform integrability; sublinear expectation

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§1. Introduction

It is well known that the uniform integrability of a family of random variables plays an important role in probability theory. As to the uniform integrability criterions, please refer to [1; P. 96], [2], [3; P. 94], [4], [5; P. 138] and [6].

In [7], the authors introduced the notion of a sequence of random variables being *uniformly nonintegrable* and gave some interesting characterizations of this uniform non-integrability. In [8], a weak notion of a sequence of random variables being uniformly nonintegrable was introduced and some equivalent characterizations were given. Motivated from [7] and [8], we will introduce two new notions of a sequence of random variables being uniformly integrable in a probability space, and prove that they are equivalent to the classic one.

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⋆Corresponding author, E-mail: zchu@scu.edu.cn.

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Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space. Suppose that all random variables under consideration are defined on this probability space. Let X be a random variable and $A \in \mathscr{F}$. We denote $\mathsf{E}(XI_A)$ by $\mathsf{E}(X : A)$.

Definition 1 A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be *uniformly integrable* (UI for short) if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \mathsf{E}(|X_n| : |X_n| \geqslant a) = 0. \tag{1}$$

Definition 2 [7] A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be *uniformly nonintegrable* (UNI for short) if

$$\lim_{a\to\infty} \inf_{n\geqslant 1} \mathsf{E}(|X_n| : |X_n| \leqslant a) = \infty.$$

Definition 3 [8] A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be *W-uniformly nonintegrable* (W-UNI for short) if

$$\lim_{a\to\infty} \inf_{n\geqslant 1} \mathsf{E}(|X_n| \wedge a) = \infty.$$

Definition 4 [8] A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be *W*-uniformly nonintegrable* (W*-UNI for short) if

$$\lim_{m\to\infty} \inf_{k\geqslant 1} \sum_{n=0}^{m} \mathsf{P}(|X_k| > n) = \infty.$$

For any random variable X, by the monotone convergence theorem, we have

$$\lim_{a\to\infty} \mathsf{E}(|X| \wedge a) = \mathsf{E}(|X|).$$

It follows that if X is integrable, then

$$\lim_{a\to\infty} [\mathsf{E}(|X|) - \mathsf{E}(|X| \wedge a)] = 0, \qquad \text{i.e.} \ \lim_{a\to\infty} \mathsf{E}(|X| - a : |X| \geqslant a) = 0. \tag{2}$$

In virture of (2), we introduce the following notion.

Definition 5 A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be *W-uniformly integrable* (W-UI for short) if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \mathsf{E}(|X_n| - a : |X_n| \geqslant a) = 0, \tag{3}$$

or equivalently,

$$\inf_{N\geqslant 1} \sup_{n\geqslant 1} \mathsf{E}(|X_n| - a : |X_n| \geqslant N) = 0.$$

For any random variable X, we have

$$\sum_{n=1}^{\infty} \mathsf{P}(|X| > n) \leqslant \mathsf{E}|X| \leqslant 1 + \sum_{n=1}^{\infty} \mathsf{P}(|X| > n). \tag{4}$$

In virtue of (4), we introduce the following notion.

Definition 6 A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be *W*-uniformly integrable* (W*-UI for short) if

$$\lim_{m\to\infty} \sup_{k\geqslant 1} \sum_{n=m}^{\infty} \mathsf{P}(|X_k| > n) = 0. \tag{5}$$

Remark 7 Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables. It is easy to know that it is UI if and only if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \mathsf{E}(|X_n| : |X_n| > a) = 0.$$

By $\mathsf{E}(|X_n| : |X_n| > a) = \mathsf{E}|X_n| - \mathsf{E}(|X_n| : |X_n| \leqslant a)$, we can say that UI corresponds to UNI in some sense. Similarly, we can say that W-UI corresponds to W-UNI and W*-UI corresponds to W*-UNI in some sense, respectively.

In Section 2, we will prove that UI, W-UI and W*-UI are equivalent in a probability space.

Recently, motivated by the risk measures, superhedge pricing and modeling uncertain in finance, Peng[9–15] initiated the notion of independent and identically distributed (IID) random variables under sublinear expectations, proved the weak law of large numbers and the central limit theorems, defined the G-expectations, G-Brownian motions and built Itô's type stochastic calculus. In Section 3, we discuss uniform integrability of random variables in a sublinear expectation space, and present de La Vallée Poussin criterion for the uniform integrability of random variables and make some other discussions.

§2. Uniform Integrability in a Probability Space

In [8], we prove that

$$\text{UNI} \Rightarrow \text{W-UNI} \Leftrightarrow \text{W*-UNI},$$

and W-UNI is strictly weaker than UNI in general. While, as to UI, W-UI and W*-UI, we have the following result.

Theorem 8

$$\text{UI} \Leftrightarrow \text{W-UI} \Leftrightarrow \text{W*-UI}. \tag{6}$$

Proof Let $\{X_n,\, n \geqslant 1\}$ be a sequence of random variable in a probability space $(\Omega, \mathscr{F}, \mathsf{P})$.

UI $\Rightarrow$ W-UI: Suppose that $\{X_n,\, n \geqslant 1\}$ is UI. Then by Definition 1, Definition 5 and the inequality

$$\mathsf{E}(|X_n| - a : |X_n| > a) \leqslant \mathsf{E}(|X_n| : |X_n| > a),$$

we know that $\{X_n,\, n \geqslant 1\}$ is W-UI.

W-UI $\Rightarrow$ UI: Suppose that $\{X_n,\, n \geqslant 1\}$ is W-UI. For any set $A \in \mathscr{F}$, any positive constant C and any integer n, we have

$$\begin{aligned}\int_A |X_n| \mathrm{d}\mathsf{P} &= \int_{A\cap(|X_n|\geqslant C)} (|X_n| - C)\mathrm{d}\mathsf{P} + C\mathsf{P}[A \cap (|X_n| \geqslant C)] + \int_{A\cap(|X_n|<C)} |X_n|\mathrm{d}\mathsf{P} \\ &\leqslant \int_{|X_n|\geqslant C} (|X_n| - C)\mathrm{d}\mathsf{P} + 2C\mathsf{P}(A) \\ &\leqslant \sup_{k\geqslant 1} \int_{|X_k|\geqslant C} (|X_k| - C)\mathrm{d}\mathsf{P} + 2C\mathsf{P}(A). \end{aligned} \tag{7}$$

By the definition of W-UI, there exists a positive number C_0 such that

$$\sup_{k\geqslant 1} \int_{|X_k|\geqslant C_0} (|X_k| - C_0)\mathrm{d}\mathsf{P} < \frac{\epsilon}{2}. \tag{8}$$

Let $\delta = \epsilon/(4C_0)$. Then for any $A \in \mathscr{F}$ with $\mathsf{P}(A) < \delta$, by (7) and (8), we obtain that

$$\int_A |X_n|\mathrm{d}\mathsf{P} < \epsilon, \qquad \forall\, n \geqslant 1. \tag{9}$$

Setting $A = \Omega$ in (7) and using the definition of W-UI, we get that

$$\sup_{n\geqslant 1} \mathsf{E}(|X_n|) < \infty. \tag{10}$$

By (9) and (10), we obtain that $\{X_n,\, n \geqslant 1\}$ is UI.

W-UI $\Rightarrow$ W*-UI: For any random variable X and any positive integer m, by Fubini's theorem, we have

$$\begin{aligned}\sum_{n=m}^{\infty} \mathsf{P}(|X| > n) &= \sum_{n=m}^{\infty} \int_n^{n+1} \mathsf{P}(|X| > n)\mathrm{d}x \leqslant \sum_{n=m}^{\infty} \int_n^{n+1} \mathsf{P}(|X| > x - 1)\mathrm{d}x \\ &= \int_m^{\infty} \mathsf{P}(|X| > x - 1)\mathrm{d}x = \int_{m-1}^{\infty} \mathsf{P}(|X| > x)\mathrm{d}x \\ &= \int_\Omega \Big(\int_{m-1}^{\infty} I_{\{|X|>x\}}\mathrm{d}x \Big)\mathrm{d}\mathsf{P} \\ &= \mathsf{E}[|X| - (m-1) : |X| > m - 1] \\ &= \mathsf{E}[|X| - (m-1) : |X| \geqslant m - 1]. \end{aligned}$$

It follows that W-UI $\Rightarrow$ W*-UI.

W*-UI $\Rightarrow$ W-UI: For any random variable X and any positive integer m, by Fubini's theorem, we have

$$\begin{aligned}\sum_{n=m}^{\infty}\mathsf{P}(|X|>n) &\geqslant \sum_{n=m}^{\infty}\int_n^{n+1}\mathsf{P}(|X|>x)\mathrm{d}x=\int_m^{\infty}\mathsf{P}(|X|>x)\mathrm{d}x\\ &=\int_m^{\infty}\Big(\int_{\Omega}I_{\{|X|>x\}}\mathrm{d}\mathsf{P}\Big)\mathrm{d}x=\int_{\Omega}\Big(\int_m^{\infty}I_{\{|X|>x\}}\mathrm{d}x\Big)\mathrm{d}\mathsf{P}\\ &=\mathsf{E}(|X|-m:|X|>m)\\ &=\mathsf{E}(|X|-m:|X|\geqslant m).\end{aligned}$$

It follows that W*-UI $\Rightarrow$ W-UI.

Hence (6) holds, and the proof is complete. □

§3. Uniform Integrability in a Sublinear Expectation Space

In this section, we discuss the uniform integrability of random variables in a sublinear expectation space. At first, we present some basic settings about sublinear expectations. Please refer to [9–16] for more details.

Let $(\Omega,\mathscr{F})$ be a given measurable space and $\mathscr{H}$ be a linear space of $\mathscr{F}$-measurable real functions defined on Ω such that for any constant number c, $c\in\mathscr{H}$; if $X\in\mathscr{H}$, then $|X|\in\mathscr{H}$ and $XI_A\in\mathscr{H}$ for any $A\in\mathscr{F}$.

Definition 9 A sublinear expectation $\mathscr{E}$ on $\mathscr{H}$ is a functional $\mathscr{E}:\mathscr{H}\to\mathbb{R}$ satisfying the following properties:

(a) Monotonicity: $\mathscr{E}(X)\geqslant\mathscr{E}(Y)$, if $X\geqslant Y$.

(b) Constant preserving: $\mathscr{E}(c)=c$, $\forall\, c\in\mathbb{R}$.

(c) Sub-additivity: $\mathscr{E}(X+Y)\leqslant\mathscr{E}(X)+\mathscr{E}(Y)$.

(d) Positive homogeneity: $\mathscr{E}(\lambda X)=\lambda\mathscr{E}(X)$, $\forall\,\lambda\geqslant 0$.

The triple $(\Omega,\mathscr{H},\mathscr{E})$ is called a sublinear expectation space.

Definition 10 ([16; Definition 3.1]) For $p\in[1,\infty)$, the map

$$\|\cdot\|_p : X\mapsto[\mathscr{E}(|X|^p)]^{1/p}$$

forms a seminorm on $\mathscr{H}$. Define the space $\mathscr{L}^p(\mathscr{F})$ as the completion under $\|\cdot\|_p$ of the set $\{X\in\mathscr{H}:\|X\|_p<\infty\}$ and then $L^p(\mathscr{F})$ as the equivalence classes of $\mathscr{L}^p$ modulo equality in $\|\cdot\|_p$.

Definition 11 ([16; Definition 3.2]) Consider $K \subset L^1$. K is said to be uniformly integrable if $\lim\limits_{c\to\infty}\sup\limits_{X\in K}\mathscr{E}(I_{\{|X|\geqslant c\}}|X|)=0$.

Theorem 12 ([16; Theorem 3.1]) Suppose K is a subset of L^1. Then K is uniformly integrable if and only if the following two conditions hold.

(i) $\{\mathscr{E}(|X|)\}_{X\in K}$ is bounded.

(ii) For any $\epsilon<0$ there is a $\delta>0$ such that for all $A\in\mathscr{F}$ with $\mathscr{E}(I_A)\leqslant\delta$, we have $\mathscr{E}(I_A|X|)<\epsilon$ for all $X\in K$.

Now we present the following de La Vallée Poussin criterion for the uniform integrability.

Theorem 13 Let K be a subset of L^1. Then K is uniformly integrable if and only if there is a nonnegative function φ defined on $[0,\infty)$ such that $\lim\limits_{t\to\infty}\varphi(t)/t=\infty$ and $\sup\limits_{X\in K}\mathscr{E}(\varphi\circ|X|)<\infty$.

Proof As to the sufficiency, refer to [16; Corollary 3.1.1]. In the following, we give the proof of the necessity. The idea comes from the corresponding proof in a probability space (see e.g. [17; Theorem 7.4.5]).

Suppose that K is uniformly integrable. For any constant $a>0$, we have $\mathscr{E}[(|X|-a)^+]\leqslant\mathscr{E}(|X|I_{\{|X|\geqslant a\}})$. It follows that there exists a sequence $\{n_k\}$ of integers such that $n_k\uparrow\infty$ and

$$\sup_{X\in K}\mathscr{E}[(|X|-n_k)^+]<2^{-k},\qquad k\geqslant 1. \tag{11}$$

Define a function

$$\varphi(t)=\sum_{k\geqslant 1}(n-n_k)^+,\qquad n\leqslant t<n+1,\ n=0,1,2,\ldots.$$

Then φ is a nonnegative, nondecreasing and right continuous function. What's more, we have

$$\lim_{n\to\infty}\frac{\varphi(n)}{n}=\lim_{n\to\infty}\sum_{k\geqslant 1}\Big(1-\frac{n_k}{n}\Big)^+=\infty,$$

which implies that $\lim\limits_{t\to\infty}\varphi(t)/t=\infty$.

By Fubini's theorem, the monotone convergence theorem ([16; Theorem 2.2]), the sublinear property of $\mathscr{E}$ and (11), we obtain that for any $X\in K$,

$$\begin{aligned}\mathscr{E}(\varphi\circ|X|)&=\mathscr{E}\Big[\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(n-n_k)^+I_{\{n\leqslant|X|<n+1\}}\Big]\\&=\mathscr{E}\Big[\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(n-n_k)^+I_{\{n\leqslant|X|<n+1\}}\Big]\end{aligned}$$

$$
\begin{aligned}
&= \mathscr{E}\Big[\lim_{m\to\infty}\sum_{k=1}^{m}\sum_{n=0}^{\infty}(n-n_k)^+ I_{\{n\leqslant |X|<n+1\}}\Big] \\
&= \lim_{m\to\infty}\mathscr{E}\Big[\sum_{k=1}^{m}\sum_{n=0}^{\infty}(n-n_k)^+ I_{\{n\leqslant |X|<n+1\}}\Big] \\
&\leqslant \lim_{m\to\infty}\sum_{k=1}^{m}\mathscr{E}\Big[\sum_{n=0}^{\infty}(n-n_k)^+ I_{\{n\leqslant |X|<n+1\}}\Big] \\
&= \sum_{k=1}^{\infty}\mathscr{E}[(|X|-n_k)^+] < 1. \qquad \square
\end{aligned}
$$

With respect to Definitions 5 and 6, we introduce the following two notions.

Definition 14 Consider $K \subset L^1$. K is said to be W-uniformly integrable (W-UI for short) if

$$\lim_{a\to\infty}\sup_{X\in K}\mathscr{E}[I_{\{|X|\geqslant a\}}(|X|-a)] = 0. \tag{12}$$

Definition 15 Consider $K \subset L^1$. K is said to be S-uniformly integrable (S-UI for short) if

$$\lim_{m\to\infty}\sup_{X\in K}\sum_{n=m}^{\infty}\mathscr{E}(I_{\{|X|>n\}}) = 0. \tag{13}$$

Proposition 16 Suppose that K is a family of random variables in a sublinear expectation $(\Omega, \mathscr{H}, \mathscr{E})$. Then we have

$$\text{UI} \Leftrightarrow \text{W-UI} \Leftarrow \text{S-UI}. \tag{14}$$

Proof UI $\Rightarrow$ W-UI: Suppose that K is UI. Then by Definition 11, Definition 14 and the inequality

$$\mathscr{E}[(|X|-a)I_{\{|X|>a\}}] \leqslant \mathsf{E}[|X|I_{\{|X|>a\}}],$$

we know that K is W-UI.

W-UI $\Rightarrow$ UI: Suppose that K is W-UI. For any set $A \in \mathscr{F}$, any positive constant C and any $X \in K$, we have

$$
\begin{aligned}
\mathscr{E}(|X|I_A) &= \mathscr{E}[(|X|-C)I_{A\cap\{|X|\geqslant C\}} + CI_{A\cap\{|X|\geqslant C\}} + |X|I_{A\cap\{|X|<C\}}] \\
&\leqslant \mathscr{E}[(|X|-C)I_{A\cap\{|X|\geqslant C\}}] + \mathscr{E}(CI_{A\cap\{|X|\geqslant C\}}) + \mathscr{E}(|X|I_{A\cap\{|X|<C\}}) \\
&\leqslant \mathscr{E}[(|X|-C)I_{\{|X|\geqslant C\}}] + 2C\mathscr{E}(I_A).
\end{aligned} \tag{15}
$$

By Definition 14 there exists a positive number C_0 such that

$$\sup_{X\in K}\mathscr{E}[(|X|-C_0)I_{\{|X|\geqslant C_0\}}] < \frac{\epsilon}{2}. \tag{16}$$

Let $\delta = \epsilon/(4C_0)$. Then for any $A \in \mathscr{F}$ with $\mathscr{E}(I_A) < \delta$, by (15) and (16), we obtain that

$$\mathscr{E}(|X|I_A) < \epsilon, \qquad \forall\, X \in K. \tag{17}$$

Setting $A=\Omega$ in (15) and using Definition 14, we get that

$$\sup_{X\in K}\mathscr{E}(|X|)<\infty. \tag{18}$$

By (17), (18) and Theorem 12, we obtain that K is UI.

S-UI $\Rightarrow$ W-UI: Suppose that K is S-UI. For any $X\in K$ and any integer m, by the monotone convergence theorem ([16; Theorem 2.2]) and the sublinear property of $\mathscr{E}$, we get

$$\begin{aligned}
\mathscr{E}[(|X|-m)I_{\{|X|\geqslant m\}}]&=\mathscr{E}[(|X|-m)I_{\{|X|>m\}}]\\
&=\mathscr{E}\Big(\int_m^\infty I_{\{|X|>x\}}\mathrm{d}x\Big)=\mathscr{E}\Big(\sum_{n=m}^\infty\int_n^{n+1}I_{\{|X|>x\}}\mathrm{d}x\Big)\\
&=\mathscr{E}\Big(\lim_{l\to\infty}\sum_{n=m}^{l}\int_n^{n+1}I_{\{|X|>x\}}\mathrm{d}x\Big)=\lim_{l\to\infty}\mathscr{E}\Big(\sum_{n=m}^{l}\int_n^{n+1}I_{\{|X|>x\}}\mathrm{d}x\Big)\\
&\leqslant\lim_{l\to\infty}\sum_{n=m}^{l}\mathscr{E}\Big(\int_n^{n+1}I_{\{|X|>x\}}\mathrm{d}x\Big)\\
&\leqslant\sum_{n=m}^{\infty}\mathscr{E}(I_{\{|X|>n\}}),
\end{aligned}$$

which together with Definitions 14 and 15 implies that K is W-UI. □

Remark 17 The part "W-UI $\Rightarrow$ W*-UI" of the proof of Theorem 8 tell us that in general we don't have that W-UI $\Rightarrow$ S-UI in a sublinear expectation space. In the following, we will give a counterexample.

Let $\Omega=\{0,1,2,\ldots\}$. For any $n=2,3,\ldots$, define a probability measure P_n on Ω as follows:

$$\mathsf{P}_n(n)=\frac{1}{n\ln n},\qquad \mathsf{P}_n(0)=1-\frac{1}{n\ln n}.$$

Denote by E_n the expectation with respect to the probability measure P_n. Define the sublinear expectation $\mathscr{E}$ by

$$\mathscr{E}[\,\cdot\,]:=\sup_{n\geqslant 2}\mathsf{E}_n[\,\cdot\,].$$

Let X be a random variable defined on Ω by

$$X(n)=n,\qquad n=0,1,2,\ldots.$$

We have

$$\begin{aligned}
\lim_{m\to\infty}\mathscr{E}(|X|I_{\{|X|\geqslant m\}})&=\lim_{m\to\infty}\sup_{n\geqslant 2}\mathsf{E}_n(|X|I_{\{|X|\geqslant m\}})=\lim_{m\to\infty}\sup_{n\geqslant m}n\times\frac{1}{n\ln n}\\
&=\lim_{m\to\infty}\frac{1}{\ln m}=0,
\end{aligned}$$

which implies that $\{X\}$ is UI and thus $\{X\}$ is W-UI by Proposition 16.

We also have

$$\lim_{m\to\infty}\sum_{n=m}^{\infty}\mathscr{E}(I_{\{|X|>n\}})=\lim_{m\to\infty}\sum_{n=m}^{\infty}\sup_{k\geqslant 2}\mathsf{E}_k(I_{\{|X|>n\}})=\lim_{m\to\infty}\sum_{n=m}^{\infty}\sup_{k\geqslant n+1}\frac{1}{k\ln k}$$
$$=\lim_{m\to\infty}\sum_{n=m}^{\infty}\frac{1}{(n+1)\ln(n+1)}=+\infty,$$

which implies that $\{X\}$ is not S-UI.

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References

[1] CHUNG K L. *A Course in Probability Theory* [M]. 2nd ed. New York: Academic Press, 1974.

[2] CHONG K M. On a theorem concerning uniform integrability [J]. *Publ Inst Math* (*Beograd*) (*NS*). 1979, **25(39)**: 8–10.

[3] CHOW Y S, TEICHER H. *Probability Theory: Independence, Interchangeability, Martingales* [M]. 3rd ed. New York: Springer-Verlag, 1997.

[4] HU T C, ROSALSKY A. A note on the de La Vallée Poussin criterion for uniform integrability [J]. *Statist Probab Lett*, 2011, **81(1)**: 169–174.

[5] KLENKE A. *Probability Theory: A Comprehensive Course* [M]. 2nd ed. London: Springer-Verlag, 2014.

[6] CHANDRA T K. de La Vallée Poussin's theorem, uniform integrability, tightness and moments [J]. *Statist Probab Lett*, 2015, **107**: 136–141.

[7] CHANDRA T K, HU T C, ROSALSKY A. On uniform nonintegrability for a sequence of random variables [J]. *Statist Probab Lett*, 2016, **116**: 27–37.

[8] HU Z C, PENG X. Uniform nonintegrability of random variables [J]. *Front Math China*, 2018, **13(1)**: 41–53.

[9] PENG S G. Filtration consistent nonlinear expectations and evaluations of contingent claims [J]. *Acta Math Appl Sin Engl Ser*, 2004, **20(2)**: 191–214.

[10] PENG S G. Nonlinear expectations and nonlinear Markov chains [J]. *Chinese Ann Math Ser B*, 2005, **26(2)**: 159–184.

[11] PENG S G. G-expectation, G-Brownian motion and related stochastic calculus of Itô type [M] // BENTH F E, DI NUNNO G, LINDSTRØM T, et al. (eds) *Stochastic Analysis and Applications* (*Abel Symposia, Vol. 2*). Berlin: Springer, 2007: 541–567.

[12] PENG S G. G-Brownian motion and dynamic risk measure under volatility uncertainty [J]. arXiv preprint arXiv:0711.2834v1, 2007.

[13] PENG S G. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation [J]. *Stochastic Process Appl*, 2008, **118(12)**: 2223–2253.

[14] PENG S G. A new central limit theorem under sublinear expectations [J]. arXiv preprint arXiv:0803.2656v1, 2008.

[15] PENG S G. Nonlinear expectations and stochastic calculus under uncertainty [J]. arXiv preprint arXiv:1002.4546v1, 2010.

[16] COHEN S N, JI S L, PENG S G. Sublinear expectations and martingales in discrete time [J]. arXiv preprint arXiv:1104.5390v1, 2011.

[17] YAN J A. *Lectures on Measure Theory* [M]. 2nd ed. Beijing: Science Press, 2004. (in Chinese)

[18] HU T C, ROSALSKY A. A note on random variables with an infinite absolute first moment [J]. *Statist Probab Lett*, 2015, **97**: 212–215.

关于概率空间与次线性期望空间中的随机变量一致可积性的一个注记

胡泽春

(四川大学数学学院, 成都, 610064)

周倩倩

(南开大学数学科学学院, 天津, 300071; 南京大学数学系, 南京, 210093)

摘　要: 在这篇注记中我们讨论随机变量的一致可积性. 在概率空间中, 我们引进了随机变量一致可积性的两个新的定义, 并证明了他们与经典定义等价. 在次线性期望空间中, 我们给出了随机变量一致可积性的德拉瓦利普桑准则, 并作了一些其它讨论.

关键词: 一致可积性; 次线性期望

中图分类号: O211.4; O211.5