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Moderate Deviations in $L_1(\mathbb{R}^d)$ for a Test of Symmetry Based on Kernel Density Estimator *

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Abstract: Let f_n be a non-parametric kernel density estimator based on a kernel function K and a sequence of independent and identically distributed random variables taking values in $\mathbb{R}^d$. In this paper we prove two moderate deviation theorems in $L_1(\mathbb{R}^d)$ for $\{f_n(x)-f_n(-x),\, n\geqslant 1\}$.

Keywords: symmetry test; kernel density estimator; moderate deviations

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§1. Introduction and Main Results

Let $\{X_i,\, i\geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in $\mathbb{R}^d$ on probability space $(\Omega,\mathscr{F},\mathsf{P})$ with unknown density function f. Let K be a measurable function such that

$$K(x)\geqslant 0,\qquad \int_{\mathbb{R}^d}K(x)\mathrm{d}x=1. \tag{1}$$

The kernel density estimator of f based on kernel function K is defined by

$$f_n(x)=\frac{1}{na_n^d}\sum_{i=1}^n K\Big(\frac{x-X_i}{a_n}\Big),\qquad x\in\mathbb{R}^d, \tag{2}$$

where $\{a_n,\, n\geqslant 1\}$ is a bandsequence, that is, a sequence of positive numbers sastifying

$$a_n\to 0,\quad na_n^d\to\infty,\qquad \text{as } n\to\infty. \tag{3}$$

As usual, we denote by $\|g\|_p=[\int_{\mathbb{R}^d}|g(x)|^p\mathrm{d}x]^{1/p}$, $p\geqslant 1$.

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He and Gao[1], Gao[2], Gińe and Guillou[3], Diallo and Louani[4], Louani[5] studied the limit properties for the kernel density estimator recently. The statistic $\sup_{x\in\mathbb{R}^d}|f_n(x)-f_n(-x)|$ was used to test the hypothesis that the density function $f(x)$ is symmetric about 0. He and Gao[1] studied moderate deviations and large deviations (cf. [6]) for $\sup_{x\in\mathbb{R}^d}|f_n(x)-f_n(-x)|$ of the density f in case $d=1$ by the empirical approach. Xu and Zhou[7,8] proved moderate deviations and large deviations for $\sup_{x\in\mathbb{R}^d}|f_n(x)-f_n(-x)|$ in case $d>1$. Gao[9] obtained moderate deviations and law of the iterated logarithm in $L_1(\mathbb{R}^d)$ for kernel density estimator. Motivated by Gao[9], we also try to use a measure transformation and Devroye partition method to deduce that moderate deviations hold for $\{f_n(x)-f_n(-x),\,n\geqslant 1\}$ in $L_1(\mathbb{R}^d)$ here. Moderate deviations in [1], Xu and Zhou[7] do not imply that in $L_1(\mathbb{R}^d)$ for $\{f_n(x)-f_n(-x),\,n\geqslant 1\}$, moderate deviations in $L_1(\mathbb{R}^d)$ for $\{f_n(x)-f_n(-x),\,n\geqslant 1\}$ also could not lead to that for $\sup_{x\in\mathbb{R}^d}|f_n(x)-f_n(-x)|$ in [1] and [7]. Moderate deviations in $L_1(\mathbb{R}^d)$ for $\{f_n(x)-f_n(-x),\,n\geqslant 1\}$ complement the results obtained by He and Gao[1], Xu and Zhou[7]. As in [9], we find the condition on the bandsequence such that $\{\|f_n(\cdot)-f_n(-\cdot)-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\|\}$ satisfies the moderate deviation principle.

Let b_n, $n\geqslant 1$ be a sequence of positive real numbers satisfying

$$\frac{n}{b_n}\to+\infty \quad \text{and} \quad \frac{n}{b_n^2}\to 0 \qquad \text{as } n\to+\infty. \tag{4}$$

We introduce the following conditions:

(A1) f is continuous and symmetric and $\lim\limits_{|x|\to\infty} f(x)=0$.

(A2) $\int_{\mathbb{R}^d}(1+|x|^{pd})K^2(x)\mathrm{d}x<\infty$, $\int_{\mathbb{R}^d}|x|^{pd}f(x)\mathrm{d}x<\infty$, for some $p>1$.

As in Remark 1.1 in [9], if (A2) holds, then

$$\int_{\mathbb{R}^d}\sqrt{f(x)}\mathrm{d}x<\infty,$$

and

$$\sup_{n\geqslant 1}\int_{\mathbb{R}^d}\sqrt{\int_{\mathbb{R}^d}\frac{1}{a_n^d}K^2\Big(\frac{x-y}{a_n}\Big)f(y)\mathrm{d}y}\,\mathrm{d}x<\infty.$$

Write

$$I(g)=\begin{cases}\dfrac{1}{8}\displaystyle\int_{\mathbb{R}^d}\Big[\frac{g(x)}{f(x)}\Big]^2 f(x)\mathrm{d}x & \text{if } g\in L_1(\mathbb{R}^d) \text{ and } g(-x)=-g(x);\\ +\infty & \text{otherwise,}\end{cases} \tag{5}$$

and set

$$J(h)=\begin{cases}\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^d}\Big[\frac{h(x)}{f(x)}\Big]^2 f(x)\mathrm{d}x & \text{if } h\in L_1(\mathbb{R}^d) \text{ and } \int_{\mathbb{R}^d}h(x)\mathrm{d}x=0;\\ +\infty & \text{otherwise,}\end{cases} \tag{6}$$

where $0/0=0$.

Theorem 1 Suppose that (A1) and (A2) hold. If the width of windows $\{a_n,\, n\geqslant 1\}$ satisfies

(BC) $$\frac{n}{b_n^2 a_n^d}\to 0 \qquad \text{as } n\to+\infty,$$

then

(i) for any open subset G in $(L_1(\mathbb{R}^d),\,\|\cdot\|_1)$,

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}\in G\Big\}\geqslant-\inf_{g\in G}I(g), \tag{7}$$

(ii) for any open and convex subset G in $(L_1(\mathbb{R}^d),\,\|\cdot\|_1)$,

$$\lim_{n\to\infty}\frac{n}{b_n^2}\ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}\in G\Big\}=-\inf_{g\in G}I(g), \tag{8}$$

(iii) for any compact subset C in $(L_1(\mathbb{R}^d),\,\|\cdot\|_1)$, for any $\delta>0$, there exists an open subset $G_\delta\supset C$ such that

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}\in G_\delta\Big\}\leqslant-\inf_{g\in C}I(g)+\delta, \tag{9}$$

in particular,

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}\in C\Big\}\leqslant-\inf_{g\in C}I(g). \tag{10}$$

As in Remark 1.2 in [9], by Fatou's lemma, $I(\cdot)$ is lower-semicontinuous in $(L_1(\mathbb{R}^d),\,\|\cdot\|_1)$.

Remark 2 Gao[9] proved that if (A2) and (BC) hold, then for any open subset G in $(L_1(\mathbb{R}^d),\,\|\cdot\|_1)$, any compact subset C in $(L_1(\mathbb{R}^d),\,\|\cdot\|_1)$,

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\ln \mathsf{P}\Big\{\frac{n}{b_n}\{f_n(\cdot)-\mathsf{E}[f_n(\cdot)]\}\in G\Big\}\geqslant-\inf_{h\in G}J(h), \tag{11}$$

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\ln \mathsf{P}\Big\{\frac{n}{b_n}\{f_n(\cdot)-\mathsf{E}[f_n(\cdot)]\}\in C\Big\}\leqslant-\inf_{h\in C}J(h). \tag{12}$$

We define $F(h(\cdot))=h(\cdot)-h(-\cdot):(L_1(\mathbb{R}^d),\,\|\cdot\|_1)\mapsto(L_1(\mathbb{R}^d),\,\|\cdot\|_1)$. F is continuous. By contraction principle (cf. [6; p.126]), we could also obtain (7) and (10) from (11) and

(12). Indeed, we only need to prove $I(g)=\inf\limits_{F(h)=g} J(h)$. For $g\in L_1(\mathbb{R}^d)$ and $g(-x)=-g(x)$, choose $h=g/2$ and $h(-x)=-h(x)$, then $I(g)\geqslant \inf\limits_{F(h)=g} J(h)$. By Cauchy-Schwartz inequality and (A1), we see that

$$\inf_{F(h)=g} J(h)\geqslant \inf_{F(h)=g}\frac{1}{2}\int\Big[\frac{h(x)-h(-x)}{f}\Big]^2\frac{f}{4}\mathrm{d}x\geqslant \inf_{F(h)=g} I(g)=I(g).$$

Theorem 3 (i) Assume that (A1), (A2), and (BC) hold. Then for any open subset $G\subset[0,+\infty)$.

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\ln\mathsf{P}\Big\{\frac{n}{b_n}\|[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\|_1\in G\Big\}\geqslant-\inf_{\lambda\in G}\frac{\lambda^2}{8},\tag{13}$$

and for any closed subset $F\subset[0,+\infty)$,

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\ln\mathsf{P}\Big\{\frac{n}{b_n}\|[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\|_1\in F\Big\}\leqslant-\inf_{\lambda\in F}\frac{\lambda^2}{8}.\tag{14}$$

In particular, for any $\lambda>0$,

$$\limsup_{n\to\infty}\frac{n}{b_n^2}\ln\mathsf{P}\Big\{\frac{n}{b_n}\|[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\|_1>\lambda\Big\}=-\frac{\lambda^2}{8}.\tag{15}$$

(ii) Let K be a bounded function with compact support, and let f also have compact support. Then (15) holds if and only if (BC) is valid.

§2. Proof of Theorem 1

The ideas of proof come from that of [9]. The lower bound is shown by a measure transformation. The upper bound for an open convex subset follows from the Hahn-Banach theorem and the Chebyshev inequality.

Proof of Theorem 1(i) Let G be an open subset in $(L_1(\mathbb{R}^d),\|\cdot\|_1)$. For any $g\in G$, choose $\delta>0$ such that $B(g,\delta):=\{\varphi\in L_1(\mathbb{R}^d);\ \|\varphi-g\|_1\leqslant\delta\}\subset G$. Then

$$\begin{aligned}&\liminf_{n\to\infty}\frac{n}{b_n^2}\ln\mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}\in G\Big\}\\ \geqslant&\liminf_{n\to\infty}\frac{n}{b_n^2}\ln\mathsf{P}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}-g\Big\|_1<\delta\Big\}.\end{aligned}$$

Therefore, the following lemma implies Theorem 1(i). □

Lemma 4 Assume that (A1), (A2), and (BC) hold. Then for any $g\in L_1(\mathbb{R}^d)$, and for any $\delta>0$,

$$\liminf_{n\to\infty}\frac{n}{b_n^2}\ln\mathsf{P}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}-g\Big\|_1<\delta\Big\}\geqslant-I(g).\tag{16}$$

Proof Without restriction of generality, we assume that $(\Omega, \mathscr{F}, \mathsf{P}) = (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mu)^{\mathbb{N}}$ where $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ and $\mathbb{N} = \{1, 2, \ldots\}$. Let $X_i(\omega) = \omega_i$, $i = 1, 2, \ldots$ be the coordinate variables on Ω. If $I(g) = \infty$, then (16) is trivial. Therefore, we need to prove (16) for g with $I(g) < \infty$. Moreover, if $I(g) < \infty$, write $\widetilde{g}_N(x) = g(x)I_{\{N^{-1}f(x)\leqslant|g(x)|\leqslant Nf(x)\}}$ and $g_N(x) = \widetilde{g}_N(x) - f(x)\int \widetilde{g}_N(y)\mathrm{d}y = \widetilde{g}_N(x)$, then $\int g_N(x)\mathrm{d}x = 0$, $\|g_N - g\|_1 \to 0$, and $I(g_N) \to I(g)$ as $N \to \infty$. Hence, we may assume that $g(x)/f(x)$ is a bounded function. Then for n large enough,

$$\nu_n(\mathrm{d}x) = \Big[f(x) + \frac{b_n}{2n}g(x)\Big]\mathrm{d}x$$

is a probability measure on $\mathbb{R}^d$, which is equivalent to μ. Let

$$\mathsf{Q}_n(\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n) = \nu_n(\mathrm{d}x_1)\nu_n(\mathrm{d}x_2)\cdots\nu_n(\mathrm{d}x_n).$$

Then for n large enough, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathsf{P}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} - g\Big\|_1 < \delta\Big\}\\
&= \int_{\{\|nb_n^{-1}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}-g\|_1<\delta\}} \prod_{i=1}^n \Big[1 + \frac{b_n g(x_i)}{2nf(x_i)}\Big]^{-1} \mathsf{Q}_n(\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n)\\
&= \int_{\{\|nb_n^{-1}\{[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\}-g\|_1<\delta\}} \exp\Big\{-\sum_{i=1}^n \ln\Big[1 + \frac{b_n g(x_i)}{2nf(x_i)}\Big]\Big\}\\
&\quad\times \mathsf{Q}_n(\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n)\\
&\geqslant \exp\Big\{-n\Big\{\mathsf{E}^{\nu_n}\ln\Big[1 + \frac{b_n g(X_1)}{2nf(X_1)}\Big] + \frac{b_n^2\varepsilon}{n^2}\Big\}\Big\}\\
&\quad\times \mathsf{Q}_n\Big\{A_{n,\varepsilon} \cap \Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} - g\Big\|_1 < \delta\Big\}\Big\},
\end{aligned}$$

where

$$A_{n,\varepsilon} := \Big\{\frac{n}{b_n^2}\sum_{i=1}^n \ln\Big[1 + \frac{b_n g(X_i)}{2nf(X_i)}\Big] \leqslant \frac{n^2}{b_n^2}\mathsf{E}^{\nu_n}\ln\Big[1 + \frac{b_n g(X_1)}{2nf(X_1)}\Big] + \varepsilon\Big\}.$$

Since

$$\frac{n}{b_n^2}\sum_{i=1}^n \ln\Big[1 + \frac{b_n g(X_i)}{2nf(X_i)}\Big] = \frac{1}{b_n}\sum_{i=1}^n \frac{g(X_i)}{2f(X_i)} - \frac{1}{8n}\sum_{i=1}^n \frac{g^2(X_i)}{f^2(X_i)} + O\Big(\frac{b_n}{n}\Big),$$

$$\frac{n}{b_n}\mathsf{E}^{\nu_n}\Big[\frac{g(X_1)}{2f(X_1)}\Big] = 2I(g) + O\Big(\frac{b_n}{n}\Big), \qquad \frac{1}{8}\mathsf{E}^{\nu_n}\Big[\frac{g^2(X_1)}{f^2(X_1)}\Big] = I(g) + O\Big(\frac{b_n}{n}\Big),$$

and

$$\frac{n^2}{b_n^2}\mathsf{E}^{\nu_n}\ln\Big[1 + \frac{b_n g(X_1)}{2nf(X_1)}\Big] = I(g) + O\Big(\frac{b_n}{n}\Big),$$

by the Chebyshev inequality, for each $\eta > 0$, for n large enough, we get

$$\mathsf{Q}_n\Big[\Big|\frac{1}{8n}\sum_{i=1}^n \frac{g^2(X_i)}{f^2(X_i)} - I(g)\Big| > \eta\Big] \leqslant \mathsf{Q}_n\Big\{\Big|\frac{1}{8n}\sum_{i=1}^n \frac{g^2(X_i)}{f^2(X_i)} - \frac{1}{8}\mathsf{E}^{\nu_n}\Big[\frac{g^2(X_1)}{f^2(X_1)}\Big]\Big| > \frac{\eta}{2}\Big\}$$

$$\leqslant \frac{16}{n\eta^2}\mathsf{E}^{\nu_n}\Big\{\frac{g^2(X_1)}{f^2(X_1)} - \mathsf{E}^{\nu_n}\Big[\frac{g^2(X_1)}{f^2(X_1)}\Big]\Big\}^2$$

and

$$\mathsf{Q}_n\Big[\Big|\frac{1}{2b_n}\sum_{i=1}^{n}\frac{g(X_i)}{f(X_i)} - 2I(g)\Big| > \eta\Big] \leqslant \mathsf{Q}_n\Big\{\Big|\frac{1}{2b_n}\sum_{i=1}^{n}\Big\{\frac{g(X_i)}{f(X_i)} - \mathsf{E}^{\nu_n}\Big[\frac{g(X_1)}{f(X_1)}\Big]\Big\}\Big| > \frac{\eta}{2}\Big\}$$
$$\leqslant \frac{n}{b_n^2\eta^2}\mathsf{E}^{\nu_n}\Big\{\frac{g(X_1)}{f(X_1)} - \mathsf{E}^{\nu_n}\Big[\frac{g(X_1)}{f(X_1)}\Big]\Big\}^2.$$

Therefore

$$\mathsf{Q}_n(A_{n,\varepsilon}) \to 1.$$

On the other hand, since

$$\mathsf{E}^{\mathsf{Q}_n}[f_n(\cdot) - f_n(-\cdot)] = \mathsf{E}[f_n(\cdot) - f_n(-\cdot)] + \frac{b_n}{2n}\int \frac{1}{a_n^d}\Big[K\Big(\frac{x-y}{a_n}\Big) - K\Big(\frac{-x-y}{a_n}\Big)\Big]g(y)\mathrm{d}y$$

and

$$\int \Big|\frac{1}{2}\int \frac{1}{a_n^d}\Big[K\Big(\frac{x-y}{a_n}\Big) - K\Big(\frac{-x-y}{a_n}\Big)\Big]g(y)\mathrm{d}y - g(x)\Big|\mathrm{d}x \to 0 \qquad \text{as } n\to\infty,$$

we obtain

$$\limsup_{n\to\infty}\mathsf{E}^{\mathsf{Q}_n}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} - g\Big\|_1\Big\}$$
$$\leqslant \limsup_{n\to\infty}\mathsf{E}^{\mathsf{Q}_n}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}^{\mathsf{Q}_n}[f_n(\cdot) - f_n(-\cdot)]\}\Big\|_1\Big\}.$$

Now, applying the Cauchy-Schwartz inequality to

$$\mathsf{E}^{\mathsf{Q}_n}\Big\{\Big|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}^{\mathsf{Q}_n}[f_n(\cdot) - f_n(-\cdot)]\}\Big|\Big\},$$

we see that for any $\eta > 0$,

$$\mathsf{E}^{\mathsf{Q}_n}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}^{\mathsf{Q}_n}[f_n(\cdot) - f_n(-\cdot)]\}\Big\|_1\Big\}$$
$$= \int_{\mathbb{R}^d}\mathsf{E}^{\mathsf{Q}_n}\Big\{\Big|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}^{\mathsf{Q}_n}[f_n(\cdot) - f_n(-\cdot)]\}\Big|\Big\}\mathrm{d}x$$
$$\leqslant \sqrt{\frac{n}{b_n^2a_n^d}}\int_{\mathbb{R}^d}\sqrt{\mathsf{E}^{\nu_n}\Big\{\Big[K\Big(\frac{x-X_1}{a_n}\Big) - K\Big(\frac{-x-X_1}{a_n}\Big)\Big]^2\Big\}/a_n^d}\,\mathrm{d}x.$$

And so

$$\mathsf{E}^{\mathsf{Q}_n}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}^{\mathsf{Q}_n}[f_n(\cdot) - f_n(-\cdot)]\}\Big\|_1\Big\} = 0,$$

and

$$\mathsf{Q}_n\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} - g\Big\|_1 < \delta\Big\} \to 1.$$

Therefore, for any $\varepsilon > 0$,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\Big\|\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} - g\Big\|_1 < \delta\Big\} \\
\geqslant\ & -\limsup_{n\to\infty} \frac{n^2}{b_n^2}\Big\{\mathsf{E}^{\nu_n} \ln\Big[1 + \frac{b_n g(X_1)}{2nf(X_1)}\Big] + \frac{b_n^2 \varepsilon}{n^2}\Big\} = -I(g) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we have (16). □

Proof of Theorem 1(ii) It is sufficient for (8) to prove that for any open convex subset G,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} \in G\Big\} \leqslant -\inf_{g\in G} I(g). \tag{17}$$

Now let G be an open convex subset. Since (17) is trivial if $\inf\limits_{g\in G} I(g) = 0$, we can assume $\inf\limits_{g\in G} I(g) > 0$. For any $N > 0$ and $0 < \varepsilon < \inf\limits_{g\in G} I(g)$, put $U = \{\varphi \in L_1(\mathbb{R}^d); I(\varphi) \leqslant t_N\}$ where $t_N = \min\big\{N, \inf\limits_{g\in G} I(g) - \varepsilon\big\}$. Then $U \cap G = \emptyset$, and thus by the Habh-Banach theorem, there exist $h \in L_\infty(\mathbb{R}^d)$ and $c \in \mathbb{R}$ such that $H \cap U = \emptyset$ and $G \subset H$, where $H = \{\varphi \in L_1(\mathbb{R}^d); \int h(x)\varphi(x)\mathrm{d}x > c\}$. Hence, by the Chebyshev inequality, we deduce that for any $\alpha > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} \in G\Big\} \\
\leqslant\ & \limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\int h(x)\{[f_n(x) - f_n(-x)] - \mathsf{E}[f_n(x) - f_n(-x)]\}\mathrm{d}x \geqslant c\Big\} \\
\leqslant\ & -\alpha c + \limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{E}\Big\{\exp\Big\{\alpha b_n \int h(x)\{[f_n(x) - f_n(-x)] - \mathsf{E}[f_n(x) - f_n(-x)]\}\mathrm{d}x\Big\}\Big\}.
\end{aligned}$$

By a Taylor series expansion, we get

$$\begin{aligned}
\Lambda(h) &\equiv \lim_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{E}\Big\{\exp\Big\{b_n \int h(x)\{[f_n(x) - f_n(-x)] - \mathsf{E}[f_n(x) - f_n(-x)]\}\mathrm{d}x\Big\}\Big\} \\
&= \frac{1}{2}\Big\{\int [h(x) - h(-x)]^2 f(x)\mathrm{d}x - \Big\{\int [h(x) - h(-x)]f(x)\mathrm{d}x\Big\}^2\Big\}.
\end{aligned}$$

Therefore for any $\alpha > 0$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} \in G\Big\} \leqslant -\alpha c + \alpha^2\Lambda(h),$$

and so

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} \in G\Big\} \leqslant -\frac{c^2}{4\Lambda(h)}.$$

Noting that $\varphi \in U$ implies that $\varphi(-\cdot) = -\varphi(\cdot) \in U$, we have $U \subset \{\varphi; |\int h(x)\varphi(x)\mathrm{d}x| \leqslant |c|\}$. Therefore,

$$\begin{aligned}2\Lambda(h) &= \int \Big\{h(x) - h(-x) - \int [h(y) - h(-y)] f(y)\mathrm{d}y\Big\}^2 f(x)\mathrm{d}x \\ &= \sup_{8I(\varphi)\leqslant 1} \Big|\int [h(x) - h(-x)]\varphi(x)\mathrm{d}x\Big|^2 \\ &= \frac{1}{8t_N} \sup_{\varphi\in U} \Big|\int [h(x) - h(-x)]\varphi(x)\mathrm{d}x\Big|^2 \leqslant \frac{c^2}{2t_N}.\end{aligned}$$

Hence,

$$\begin{aligned}&\limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} \in G\Big\} \\ \leqslant & -t_N = -\min\Big\{N, \inf_{g\in G} I(g) - \varepsilon\Big\}.\end{aligned}$$

Now, first letting $N \to \infty$, and then letting $\varepsilon \to 0$, we get (17). $\square$

Proof of Theorem 1(iii) Let C be a compact subset. For any $\delta > 0$, and for any $g \in C$, there exists an open ball $U_g \ni g$ such that $\inf\limits_{\varphi\in U_g} I(\varphi) \geqslant I(g) - \delta$, since $I(\cdot)$ is lower-semicontinuous. Choose finite $g_1, g_2, \ldots, g_m$ such that $C \subset \bigcup\limits_{i=1}^m U_{g_i}$ and write $G_\delta = \bigcup\limits_{i=1}^m U_{g_i}$. Then

$$\begin{aligned}&\limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} \in G_\delta\Big\} \\ \leqslant & \max_{1\leqslant i\leqslant m} \Big\{-\inf_{\varphi\in U_{g_i}} I(\varphi)\Big\} \leqslant -\inf_{g\in C} I(g) + \delta. \qquad \square\end{aligned}$$

§3. Proof of Theorem 3

The ideas of the proof come also from that in [9]. The lower bound is a consequence of Theorem 1. Here are two basic steps in proving the upper bound. As in [9], the upper bound follows by Devroye's proof in [10]. The Devroye partition plays an important role in proof of the upper bound, here it requires precise estimates to get the MDP.

Proof of Theorem 3(i) Denote $\Psi : (L_1(\mathbb{R}^d), \|\cdot\|_1) \mapsto [0, \infty)$ by $\Psi(\varphi) = \|\varphi\|_1$. Then Ψ is continuous from $\Psi : (L_1(\mathbb{R}^d), \|\cdot\|_1)$ to $[0, \infty)$ and

$$\inf_{\Psi(g)=\lambda} I(g) = \frac{\lambda^2}{8}. \tag{18}$$

By Cauchy-Schwarz inequality,

$$\inf_{\Psi(g)=\lambda} I(g) = \inf_{\Psi(g)=\lambda} \frac{1}{8} \int \Big(\frac{g}{\sqrt{f}}\Big)^2 \mathrm{d}x \int (\sqrt{f}\,)^2 \mathrm{d}x \geqslant \inf_{\Psi(g)=\lambda} \frac{1}{8} \Psi(g)^2 \geqslant \frac{\lambda^2}{8};$$

on the other hand, if one takes $g(x) = \lambda[I_A(x) - I_B(x)]f(x)$ where $A \cap B = \emptyset$, $A = -B$, $A \cup B = \mathbb{R}^d$ and $\int_A f(x)\mathrm{d}x = 1/2$, then $\|g\|_1 = \lambda$ and $I(g) = \lambda^2/8$. Therefore (18) holds.

Lower bound: Let G be an open subset in $[0, \infty)$. Then $\Psi^{-1}(G)$ is an open subset in $(L_1(\mathbb{R}^d), \|\cdot\|_1)$; hence by Theorem 1,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\|[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\|_1 \in G\Big\} \\
= &\liminf_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\{[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\} \in \Psi^{-1}(G)\Big\} \\
\geqslant &- \inf_{\Psi(g)\in G} I(g) = - \inf_{\lambda \in G} \frac{\lambda^2}{8}.
\end{aligned}$$

Upper bound: Let F be a closed subset in $[0, \infty)$, and let $\lambda = \inf\{x;\, x \in F\}$. Without loss of generality, we can assume $\lambda > 0$. Then for any $0 < \varepsilon < \lambda$,

$$\begin{aligned}
&\mathsf{P}\Big\{\frac{n}{b_n}\|[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\|_1 \in F\Big\} \\
\leqslant &\mathsf{P}\Big\{\frac{n}{b_n}\|[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\|_1 > \lambda - \varepsilon\Big\}.
\end{aligned}$$

Thus, it is sufficient for the upper bound to prove for any $\lambda > 0$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \ln \mathsf{P}\Big\{\frac{n}{b_n}\|[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\|_1 > \lambda\Big\} \leqslant -\frac{\lambda^2}{8}.$$

By (A2) and (BC) and Lemma A.1 in [9], without loss of restricition we can assume that there exists a constant $1 < L < \infty$ such that $\{K \neq 0\} \cup \{f \neq 0\} \subset [-L+1, L-1]^d$, and

$$K(x) = \sum_{j=1}^m c_j I_{A_j}(x), \qquad \sum_{j=1}^m c_j |A_j| = 1, \qquad |A_j| > 0,$$

where $0 < c_j < \infty$, $j = 1, 2, \ldots, m$ are constants, and $A_j \subset [-L, L]^d$, $j = 1, 2, \ldots, m$ are disjoint rectangles, and $|A| = \int_A \mathrm{d}x$ (see Lemmas A.5 – A.7 in Appendix in [9] with $K[(x - X_i)/a_n] - K[(-x - X_i)/a_n]$ in place of $K[(x - X_i)/a_n]$). Write $K_j(x) = (|A_j|)^{-1} I_{A_j}(x)$ and

$$f_{n;j} = \frac{1}{na_n^d} \sum_{i=1}^n K_j\Big(\frac{x - X_i}{a_n}\Big).$$

Then by $f_n = \sum_{j=1}^m c_j |A_j| f_{n;j}$ and $\sum_{j=1}^m c_j |A_j| = 1$, we deduce that for any $\lambda > 0$,

$$\begin{aligned}
&\Big\{\frac{n}{b_n}\|[f_n(\cdot) - f_n(-\cdot)] - \mathsf{E}[f_n(\cdot) - f_n(-\cdot)]\|_1 > \lambda\Big\} \\
\subset &\bigcup_{j=1}^m \Big\{\frac{n}{b_n}\|[f_{n;j}(\cdot) - f_{n;j}(-\cdot)] - \mathsf{E}[f_{n;j}(\cdot) - f_{n;j}(-\cdot)]\|_1 > \lambda\Big\}.
\end{aligned}$$

Therefore, we only prove the upper bound for $K(x)=(|A|)^{-1}I_A(x)$, where A is a rectangle. There is no loss of generality in assuming that $A=[0,1]^d$, i.e., $K(x)=I_{[0,1]^d}(x)$. In this case,

$$
\begin{aligned}
&\|[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\|_1\\
&=\frac{1}{a_n^d}\int_{\mathbb{R}^d}|\mu_n(x+a_nA)-\mu_n(-x+a_nA)-[\mu(x+a_nA)-\mu(-x+a_nA)]|\mathrm{d}x,
\end{aligned}
$$

where $\mu(B)=\int_B f(x)\mathrm{d}x$ and $\mu_n(B)=n^{-1}\sum\limits_{i=1}^{n}\delta_{X_i}(B)$ is the empirical measure for X_i, $i=1,2,\ldots,n$. Define the partition Ψ of $\mathbb{R}^d$ as follows (see [9] and [10]):

$$
\Psi:=\Big\{\prod_{j=1}^{d}\Big[\frac{(i_j-1)a_n}{N},\frac{i_ja_n}{N}\Big);\ i_j\in\mathbb{Z},\ j=1,2,\ldots,d\Big\},
$$

where N is a constant to be chosen as in [9]. Set

$$
D_x=(x+a_nA)-\bigcup_{B\in\Psi,\,B\subset x+a_nA,\,B\cap[-L,L]^d\neq\emptyset}B.
$$

Then

$$
\begin{aligned}
&\frac{1}{a_n^d}\int_{\mathbb{R}^d}|\mu_n(x+a_nA)-\mu_n(-x+a_nA)-[\mu(x+a_nA)-\mu(-x+a_nA)]|\mathrm{d}x\\
&\leqslant 2\frac{1}{a_n^d}\int_{\mathbb{R}^d}|\mu_n(x+a_nA)-\mu(x+a_nA)|\mathrm{d}x\\
&\leqslant 2\sum_{B\in\Psi,\,B\cap[-L,L]^d\neq\emptyset}|\mu_n(B)-\mu(B)|+2\frac{1}{a_n^d}\int_{\mathbb{R}^d}|\mu_n(D_x)-\mu(D_x)|\mathrm{d}x,
\end{aligned}
$$

where the last inequality is due to $a_n^{-d}\int_{B\subset x+a_nA}\mathrm{d}x\leqslant 1$. And the rest of the proof is the same as that of upper bound of Theorem 1.2(1) in [9] with $\lambda/2$ and $\delta/2$ in place of λ and δ respectively, so we omitted the proof. □

Proof of Theorem 3(ii) Similarly as in [9], we only need to prove necessity. Let K be a bounded function with compact support, and let f have also compact support. If (15) holds, then

$$
\frac{n}{b_n}\|[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\|_1\xrightarrow{\mathsf{P}}0. \tag{19}
$$

Now we take $\mathbb{B}=L_1(\mathbb{R}^d)$, and

$$
\xi_{i,n}=\frac{1}{a_n^d}\Big\{K\Big(\frac{x-X_i}{a_n}\Big)-K\Big(\frac{-x-X_i}{a_n}\Big)-\mathsf{E}\Big[K\Big(\frac{x-X_i}{a_n}\Big)-K\Big(\frac{-x-X_i}{a_n}\Big)\Big]\Big\}
$$

in Lemma A.1 in [9], then by (19) and Lemma A.1 in [9], we have

$$
\frac{n}{b_n}\mathsf{E}\{\|[f_n(\cdot)-f_n(-\cdot)]-\mathsf{E}[f_n(\cdot)-f_n(-\cdot)]\|_1\}\to 0. \tag{20}
$$

Since K is bounded, K and f have compact support, hence we have

$$\limsup_{n\to\infty}\int \frac{a_n^d\mathsf{E}(|\xi_{1,n}|^3)}{\mathsf{E}(|\xi_{1,n}|^2)}\mathrm{d}x<\infty,$$

where $0/0=0$. By Lemma A.2 in [9], we have

$$\Big|\sqrt{n}\mathsf{E}|[f_n(x)-f_n(-x)]-\mathsf{E}[f_n(x)-f_n(-x)]|-\sqrt{\frac{2}{\pi}}\sqrt{\mathsf{E}(|\xi_{1,n}|^2)}\Big|\leqslant\frac{A\mathsf{E}(|\xi_{1,n}|^3)}{\sqrt{n}\mathsf{E}(|\xi_{1,n}|^2)}.$$

Hence

$$\begin{aligned}&\Big|\frac{n}{b_n}\mathsf{E}\|[f_n(\cdot)-f_n(-\cdot)]-E[f_n(\cdot)-f_n(-\cdot)]\|_1-\sqrt{\frac{2}{\pi}}\frac{\sqrt{n}}{b_n\sqrt{a_n^d}}\int\sqrt{a_n^d\mathsf{E}(|\xi_{1,n}|^2)}\,\mathrm{d}x\Big|\\&\leqslant\int\frac{Aa_n^d\mathsf{E}(|\xi_{1,n}|^3)}{b_na_n^d\mathsf{E}(|\xi_{1,n}|^2)}\mathrm{d}x.\end{aligned}\tag{21}$$

Finally, by (20) and $\lim\limits_{n\to\infty}\int\sqrt{a_n^d\mathsf{E}(|\xi_{1,n}|^2)}\,\mathrm{d}x=c_1\int\sqrt{f(x)}\,\mathrm{d}x$, we see that (21) implies (BC). □

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核密度估计的对称检验在 $L_1(\mathbb{R}^d)$ 下的中偏差

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摘　要: 设 f_n 是基于一个核函数 K 和取值于 $\mathbb{R}^d$ 的独立同分布随机变量列的一个非参数核密度估计. 本文证明了 $\{f_n(x) - f_n(-x), n \geqslant 1\}$ 在 $L_1(\mathbb{R}^d)$ 空间下的两个中偏差定理.

关键词: 对称检验; 核密度估计; 中偏差

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