

Characterizations on Almost Stochastic Dominance Revisited

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Abstract: Almost stochastic dominance has been receiving more attention in the financial and economic literature. In this short note, we characterize the almost first- and second-degree stochastic dominance by requiring one distribution to be “close to” a new distribution that dominates or is dominated by another distribution in the traditional sense of the first- and second-order stochastic dominance, respectively. We also investigate the concept of almost stochastic dominance for unbounded random variables.

Keywords: stochastic dominance; almost stochastic dominance; utility

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§1. Introduction

Stochastic dominance has been studied extensively in applied probability, particularly in the financial and economic literature. The concept of stochastic dominance is quite old, which is served as one of the main rules to rank risk prospects or distributions. Let X and Y be two random variables with respective distribution functions F and G . X is said to dominate Y in the sense of the *first-order stochastic dominance* (FSD), denoted by $X \succeq_1 Y$, if $F(x) \leq G(x)$ for all $x \in \mathfrak{R}$, and X is said to dominate Y in the sense of the *second-order stochastic dominance* (SSD), denoted by $X \succeq_2 Y$, if

$$F^{[2]}(x) \leq G^{[2]}(x) \quad \text{for all } x \in \mathfrak{R},$$

where $F^{[2]}$ and $G^{[2]}$ are the integrated distribution functions of F and G , respectively, defined by

$$F^{[2]}(x) = \int_{-\infty}^x F(t)dt, \quad G^{[2]}(x) = \int_{-\infty}^x G(t)dt.$$

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FSD and SSD can be characterized by utility functions. Suppose the risky prospects take values in the interval J . $X \succeq_1 Y$ if and only if $Eu(X) \geq Eu(Y)$ for $u \in \mathcal{U}_{1,J}$, and $X \succeq_2 Y$ if and only if $Eu(X) \geq Eu(Y)$ for $u \in \mathcal{U}_{2,J}$, where

$$\begin{aligned}\mathcal{U}_{1,J} &= \{u : u'(x) \geq 0, \forall x \in J\}, \\ \mathcal{U}_{2,J} &= \{u : u'(x) \geq 0, u''(x) \leq 0, \forall x \in J\}.\end{aligned}$$

We refer the interested reader to [1–4] for a survey on this topic.

Leshno and Levy^[5] extended the theory of stochastic dominance to the theory of almost stochastic dominance to reveal a preference for “most” decision makers but not for “all” of them. Tzeng et al.^[6] modified the definition of the *almost second-degree stochastic dominance* (ASSD) introduced by Leshno and Levy^[5]. We recall the definitions of the *almost first-degree stochastic dominance* (AFSD) and ASSD.

Definition 1 Let X and Y be two random variables, taking values in a bounded interval J , with respective distribution functions F and G , and let $0 < \epsilon < 1/2$.

- (i)^[5] X is said to dominate Y on interval J by ϵ -almost FSD, denoted by $X \succeq_{1,J}^{\text{almost}(\epsilon)} Y$ or $F \succeq_{1,J}^{\text{almost}(\epsilon)} G$, if and only if

$$\int_{S_{1,J}} [F(t) - G(t)] dt \leq \epsilon \|F - G\|_J, \quad (1)$$

where $S_{1,J} = \{t : F(t) > G(t), t \in J\}$, and

$$\|F - G\|_J = \int_J |F(t) - G(t)| dt.$$

- (ii)^[6] X is said to dominate Y on interval J by ϵ -almost SSD, denoted by $X \succeq_{2,J}^{\text{almost}(\epsilon)} Y$ or $F \succeq_{2,J}^{\text{almost}(\epsilon)} G$, if and only if $EX \geq EY$ and

$$\int_{S_{2,J}} [F^{[2]}(t) - G^{[2]}(t)] dt \leq \epsilon \|F^{[2]} - G^{[2]}\|_J, \quad (2)$$

where $S_{2,J} = \{t : F^{[2]}(t) > G^{[2]}(t), t \in J\}$, and

$$\|F^{[2]} - G^{[2]}\|_J = \int_J |F^{[2]}(t) - G^{[2]}(t)| dt.$$

It is easy to see that $S_{1,J} = S_{1,\mathbb{R}}$, $\|F - G\|_J = \|F - G\| := \|F - G\|_{\mathbb{R}}$ and, hence,

$$X \succeq_{1,J}^{\text{almost}(\epsilon)} Y \iff X \succeq_{1,\mathbb{R}}^{\text{almost}(\epsilon)} Y. \quad (3)$$

So, for simplicity, we sometimes suppress the subscript “ J ” in the notation $\succeq_{1,J}^{\text{almost}(\epsilon)}$. However, the implication (3) does not hold for the ASSD; see the discussion after Lemma 5. It is known that FSD implies SSD, while AFSD does not necessarily imply ASSD. For properties of AFSD and ASSD, see [7–9].

The purpose of this paper is to investigate further properties of AFSD and ASSD. First, we characterize AFSD and ASSD by requiring one distribution to be “close to” a new distribution that dominates or is dominated by another distribution in the traditional sense of FSD and SSD, respectively. Second, we reconsider the concepts of AFSD and ASSD for unbounded random variables. The main results for AFSD and ASSD are presented in Sections 2 and 3, respectively.

§2. Almost First-Degree Stochastic Dominance

An alternative characterization of AFSD is presented in the following theorem.

Theorem 2 (AFSD) Let X and Y be two general random variables taking values in J , where J is unnecessarily bounded.

- (i) $^{[5]} F \succeq_1^{\text{almost}(\epsilon)} G$ if and only if there exists a distribution function \tilde{F} such that $\tilde{F} \succeq_1 G$ and

$$\|F - \tilde{F}\| \leq \epsilon \|F - G\|.$$

- (ii) $F \succeq_1^{\text{almost}(\epsilon)} G$ if and only if there exists a distribution function \tilde{G} such that $F \succeq_1 \tilde{G}$ and

$$\|G - \tilde{G}\| \leq \epsilon \|F - G\|. \quad (4)$$

Proof (i) For the sufficiency, see the proof of Proposition 1 in [5]. However, there is a gap in the proof of the necessity of Proposition 1 in [5]. Their proof is complicated, and their Claim 1 ([5; p.1082]) is not enough for the proof of the necessity. We present an elegant proof of the necessity. First, define $\tilde{F}(t) = F(t) \wedge G(t)$ for $t \in \mathfrak{R}$. Then \tilde{F} is a distribution function with support contained in J , and $\tilde{F} \succeq_1 G$. Clearly, $\tilde{F}(t) - F(t) = 0$ for $t \notin S_1 := S_{1,J}$, and $\tilde{F}(t) - F(t) = G(t) - F(t)$ for $t \in S_1$. Then

$$\|\tilde{F}(t) - F(t)\| = \int_{S_1} [F(t) - G(t)] dt \leq \epsilon \|F - G\|.$$

(ii) *Sufficiency*: Assume that there exists a distribution function \tilde{G} such that $F \succeq_1 \tilde{G}$ and (4) holds. Then $\tilde{G}(t) \geq F(t)$ for $t \in \mathfrak{R}$. Note that for $t \in S_1$, $F(t) \geq G(t)$ and, hence,

$0 \leq F(t) - G(t) \leq \tilde{G}(t) - G(t)$. Therefore,

$$\int_{S_1} [F(t) - G(t)] dt \leq \|\tilde{G}(t) - G(t)\| \leq \epsilon \|F - G\|,$$

implying that $F \succeq_1^{\text{almost}(\epsilon)} G$.

Necessity: Define $\tilde{G}(t) = F(t) \vee G(t)$ for $t \in \mathfrak{R}$. Then $\tilde{G}(t)$ is also a distribution function with its support contained in J , and $F(t) \leq \tilde{G}(t)$, that is, $F \succeq_1 \tilde{G}$. Note that $\tilde{G}(t) = F(t)$ for $t \in S_1$, and $\tilde{G}(t) = G(t)$ for $t \notin S_1$. Thus,

$$\|\tilde{G} - G\| = \int_{S_1} |F(t) - G(t)| dt \leq \epsilon \|F - G\|.$$

This completes the proof of the theorem. \square

The next proposition states that, when all risk variables take values in the interval J , the common preferences of the decision makers with utility functions in $\mathcal{U}_{1,J}(\epsilon)$ exhibit the AFSD, where $0 < \epsilon < 1/2$, and

$$\mathcal{U}_{1,J}^*(\epsilon) = \left\{ u \in \mathcal{U}_{1,J} : \sup_{x \in J} u'(x) \leq \inf_{y \in J} u'(y) \cdot \frac{1 - \epsilon}{\epsilon} \right\}.$$

Proposition 3 [5] Let X and Y be two random variables taking values in the bounded interval $J = [\underline{x}, \bar{x}]$. Then, for $0 < \epsilon < 1/2$,

$$X \succeq_{1,J}^{\text{almost}(\epsilon)} Y \iff \mathbb{E}u(X) \geq \mathbb{E}u(Y), \quad u \in \mathcal{U}_{1,J}^*(\epsilon).$$

Remark 4 Proposition 3 also holds for $J = \mathfrak{R}$. The proof for the case of bounded J in [5] relies on the following equation:

$$\mathbb{E}u(X) - \mathbb{E}u(Y) = \int_J u'(x)[G(x) - F(x)]dx, \quad u \in \mathcal{U}_{1,J}. \quad (5)$$

The proof of (5) given by Leshno and Levy^[5] is not valid for general random variables. We outline the proof of the case $J = \mathfrak{R}$ as follows. Without loss of generality assume that $u \in \mathcal{U}_{1,\mathfrak{R}}$ such that $\mathbb{E}|u(X)| < \infty$ and $\mathbb{E}|u(Y)| < \infty$. Note that

$$\begin{aligned} \int_0^\infty u(x)dF(x) &= u(0)\bar{F}(0) + \int_0^\infty u'(x)\bar{F}(x)dx, \\ \int_{-\infty}^0 u(x)dF(x) &= u(0)F(0) - \int_{-\infty}^0 u'(x)F(x)dx. \end{aligned}$$

Then

$$\mathbb{E}u(X) = u(0) + \int_0^\infty u'(x)\bar{F}(x)dx - \int_{-\infty}^0 u'(x)F(x)dx. \quad (6)$$

Similarly, we have

$$Eu(Y) = u(0) + \int_0^\infty u'(x)\overline{G}(x)dx - \int_{-\infty}^0 u'(x)G(x)dx. \quad (7)$$

Therefore, (5) follows from (6) and (7) directly.

§3. Almost Second-Degree Stochastic Dominance

In order to investigate properties of ASSD for random variables, we need the next lemma.

Lemma 5 ^[10]

- (I) Let X be a random variable with distribution function F . Then the integrated distribution function $F^{[2]}$ satisfies that
- (i) $F^{[2]}(x)$ is increasing and convex on \mathfrak{R} ;
 - (ii) The right derivative $D^+F^{[2]}$ exists, and $0 \leq D^+F^{[2]} \leq 1$;
 - (iii) $\lim_{t \rightarrow -\infty} F^{[2]}(t) = 0$, and $\lim_{t \rightarrow \infty} [F^{[2]}(t) - t] = -EX$.
- (II) If a function $\Phi : \mathfrak{R} \rightarrow \mathfrak{R}$ satisfies the properties (i) and (iii) in Part (I), then there exists a random variable $X \sim F$ such that

$$F(x) = D^+\Phi(x), \quad F^{[2]}(x) = \Phi(x), \quad x \in \mathfrak{R}.$$

Now we come back to Definition 1 (ii), and assume that $J = \mathfrak{R}$. From Lemma 5 (I), it follows that

$$\lim_{t \rightarrow \infty} [F^{[2]}(t) - G^{[2]}(t)] = EY - EX.$$

Then, for $EX \neq EY$, we have $\|F^{[2]} - G^{[2]}\| = +\infty$ and, hence, (2) holds for any $\epsilon > 0$. Therefore, when $EX \neq EY$ and $J = \mathfrak{R}$, the definition of ASSD is meaningless. Otherwise, $EX > EY$ always implies that $X \succeq_{2,\mathfrak{R}}^{\text{almost}(\epsilon)} Y$. In order to avoid this trivial case, we give the definition of ASSD with $J = \mathfrak{R}$.

Definition 6 Let X and Y be two random variables with respective distribution functions F and G , and let $0 < \epsilon < 1/2$. Then $X \succeq_{2,\mathfrak{R}}^{\text{almost}(\epsilon)} Y$ or $F \succeq_{2,\mathfrak{R}}^{\text{almost}(\epsilon)} G$, if $EX = EY$ and

$$\int_{S_2} [F^{[2]}(t) - G^{[2]}(t)] dt \leq \epsilon \|F^{[2]} - G^{[2]}\|,$$

where $S_2 = \{t : F^{[2]}(t) > G^{[2]}(t), t \in \mathfrak{R}\}$.

Similar to Theorem 2 for AFSD, Theorems 7, 8 and 9 characterize ASSD, and give us an insight on understanding the notion of ASSD. For any distribution F , denote by μ_F the mean of F .

Theorem 7 (ASSD) Let F and G be two distribution functions with their supports contained in a bounded interval J . Then, for $\epsilon \in (0, 1/2)$, $F \succeq_{2,J}^{\text{almost}(\epsilon)} G$ if and only if there exists a distribution function \tilde{G} such that $F \succeq_2 \tilde{G}$, $\mu_G = \mu_{\tilde{G}}$, and

$$\|G^{[2]} - \tilde{G}^{[2]}\|_J \leq \epsilon \|F^{[2]} - G^{[2]}\|_J. \quad (8)$$

Proof *Sufficiency:* Assume that $F \succeq_2 \tilde{G}$, $\mu_G = \mu_{\tilde{G}}$, and (8) holds. Then $\tilde{G}^{[2]}(t) \geq F^{[2]}(t)$ for $t \in \mathfrak{R}$. Also, $F^{[2]}(t) \geq G^{[2]}(t)$ for $t \in S_{2,J}$. Then, $0 \leq F^{[2]}(t) - G^{[2]}(t) \leq \tilde{G}^{[2]}(t) - G^{[2]}(t)$ for $t \in S_{2,J}$. Thus,

$$\begin{aligned} \int_{S_{2,J}} [F^{[2]}(t) - G^{[2]}(t)] dt &\leq \int_{S_{2,J}} [\tilde{G}^{[2]}(t) - G^{[2]}(t)] dt \\ &\leq \|\tilde{G}^{[2]}(t) - G^{[2]}\|_J \\ &\leq \epsilon \|F^{[2]} - G^{[2]}\|_J, \end{aligned}$$

that is, $F \succeq_{2,J}^{\text{almost}(\epsilon)} G$.

Necessity: Define $\Phi(t) = F^{[2]}(t) \vee G^{[2]}(t)$ for $t \in \mathfrak{R}$. Then $\Phi(t)$ is increasing and convex. By Lemma 5 (I), we have

$$\lim_{t \rightarrow -\infty} \Phi(t) = 0, \quad \lim_{t \rightarrow \infty} [\Phi(t) - t] = -\mu_G,$$

where the second equality follows from $\mu_F \geq \mu_G$ and

$$\lim_{t \rightarrow \infty} [F^{[2]}(t) - t] = -\mu_F, \quad \lim_{t \rightarrow \infty} [G^{[2]}(t) - t] = -\mu_G.$$

Again by Lemma 5 (II), there exists a distribution function \tilde{G} such that $\tilde{G}^{[2]}(t) = \Phi(t)$. Moreover, $\mu_G = \mu_{\tilde{G}}$. Since $\tilde{G}^{[2]}(t) \geq F^{[2]}(t)$ for $t \in \mathfrak{R}$, it follows that $F \succeq_2 \tilde{G}$. On the other hand, $\tilde{G}^{[2]}(t) = F^{[2]}(t)$ for $t \in S_{2,J}$, and $\tilde{G}^{[2]}(t) = G^{[2]}(t)$ for $t \notin S_{2,J}$. Therefore,

$$\|\tilde{G}^{[2]} - G^{[2]}\|_J = \int_{S_{2,J}} [F^{[2]}(t) - G^{[2]}(t)] dt \leq \epsilon \|F^{[2]} - G^{[2]}\|_J.$$

This completes the proof of the theorem. \square

From the proof of Theorem 7, we can obtain the next result.

Theorem 8 (ASSD) $F \succeq_{2,\mathfrak{R}}^{\text{almost}(\epsilon)} G$ if and only if there exists a distribution function \tilde{G} such that $F \succeq_2 \tilde{G}$, $\mu_F = \mu_G = \mu_{\tilde{G}}$, and

$$\|G^{[2]} - \tilde{G}^{[2]}\| \leq \epsilon \|F^{[2]} - G^{[2]}\|.$$

By a similar proof to that of Theorem 7, we get the next theorem. It is still unknown whether the conditions in Theorem 9 is necessity.

Theorem 9 (ASSD) Let F and G be two distributed distributions with their supports contained in a bounded interval J . If there exists a distribution function \tilde{F} such that $\tilde{F} \succeq_2 G$, and

$$\|F^{[2]} - \tilde{F}^{[2]}\|_J \leq \epsilon \|F^{[2]} - G^{[2]}\|_J, \quad (9)$$

then $F \succeq_{2,J}^{\text{almost}(\epsilon)} G$.

ASSD on a bounded interval J can be characterized by utility functions in $\mathcal{U}_{2,J}^*(\epsilon)$, as stated in the following proposition, where $0 < \epsilon < 1/2$, and

$$\mathcal{U}_{2,J}^*(\epsilon) = \left\{ u \in \mathcal{U}_{2,J} : \sup_{x \in J} [-u''(x)] \leq \inf_{y \in J} [-u''(y)] \cdot \frac{1-\epsilon}{\epsilon} \right\}.$$

Proposition 10 ^[6] Let X and Y be two random variables taking values in a bounded interval J , and let $0 < \epsilon < 1/2$. Then

$$X \succeq_{2,J}^{\text{almost}(\epsilon)} Y \iff \mathbb{E}u(X) \geq \mathbb{E}u(Y), \quad u \in \mathcal{U}_{2,J}^*(\epsilon).$$

However, for $J = \mathbb{R}$, $X \succeq_{2,\mathbb{R}}^{\text{almost}(\epsilon)} Y$ does not have a characterization in terms of expected-utility maximization like Proposition 10 since $\mathcal{U}_{2,\mathbb{R}}^*(\epsilon) = \emptyset$. To see it, suppose that a utility function $u \in \mathcal{U}_{2,J}^*(\epsilon)$. Then there exists two constants $0 < \alpha < \beta < \infty$ such that $2\alpha \leq -u''(x) \leq 2\beta < \infty$ for $x \in \mathbb{R}$. Integrating with respect to x yields that

$$-2\beta x + c_1 \leq u'(x) \leq -2\alpha x + c_2, \quad \forall x \in \mathbb{R},$$

where $c_1, c_2 \in \mathbb{R}$ are constants. Then, for some constants $d_1, d_2 \in \mathbb{R}$,

$$-\beta x^2 + c_1 x + d_1 \leq u(x) \leq -\alpha x^2 + c_2 x + d_2, \quad \forall x \in \mathbb{R}.$$

It is easy to see that $u'(x) < 0$ for x large enough. This means that $u \notin \mathcal{U}_{2,J}^*(\epsilon)$. Therefore, $\mathcal{U}_{2,\mathbb{R}}^*(\epsilon) = \emptyset$.

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几乎随机占优刻画再讨论

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摘 要: 几乎随机占优在经济和金融的研究中正受到人们越来越多的关注. 在本文中, 我们给出了一阶和二阶几乎随机占优的一个刻画, 即两个分布之间满足一阶或二阶几乎随机占优关系, 当且仅当存在一个新的分布, 使得该新分布充分贴近其中一个分布, 且该新分布在通常一阶或二阶随机占优意义下占优或被占优于另外一个分布函数. 我们也研究了无界随机变量几乎随机占优的概念.

关键词: 随机占优; 几乎随机占优; 效用

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