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A Comparison of Bounds on Three Sets of Copulas with Given Degree of Non-exchangeability

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Abstract: We establish best-possible supremum bounds of copulas with the degree of non-exchangeability $t = 3/4$, $t = 3/5$ and $t = 3/6 = 1/2$, and study the structures of these sets of copulas. The volumes between the upper and lower bounds are calculated to illustrate that the supremum bounds are specific practical and effective in narrowing the Fréchet-Hoeffding bounds.

Keywords: copulas; best-possible bounds; degree of non-exchangeability; narrowing effectiveness

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§1. Introduction

We encountered the Fréchet-Hoeffding bounds as universal bounds for copulas. The bounds can often be improved when we possess additional information about copula. Kaas et al. [1] studied the problem of finding best-possible upper bounds on the Value-at-Risk when the marginal distributions are known. Durante et al. [2] found best-possible bounds for copulas with given values in rectangles at corners of unit square.

The relationship between exchangeability and non-exchangeability is analogous to the relationship between independence and dependence. While there is but one copula for independent random variables (namely $C(u, v) = uv$), there is but one class of copulas for exchangeable random variables (the symmetric copulas). At the other extreme, there are several forms of complete or maximal dependence – examples include: X and Y are mutually completely dependent (counter monotone), if the copula of X and Y is $M(u, v) = \min(u, v)$ ($W(u, v) = \max(u + v - 1, 0)$). In this paper we shall establish best-possible supremum bounds of copulas with the degree of non-exchangeability $t = 3/4$, $t = 3/5$, and $t = 3/6 = 1/2$, and study the structures of these sets of copulas.

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Let X and Y be continuous random variables with joint distribution function H and margins F and G, respectively, and copula C, then X and Y are exchangeable if $H(x,y)=H(y,x)$ for all real x and y [3]. When X and Y are not exchangeable, $H(x,y)\neq H(y,x)$ for some x and y, and the supremum of $|H(x,y)-H(y,x)|$ can be used to measure the degree of non-exchangeability of X and Y. Since the set of values of $|H(x,y)-H(y,x)|$ for real x and y is the same as the set of values of $|C(u,v)-C(v,u)|$ for u and v in $\boldsymbol{I}=[0,1]$, the degree of non-exchangeability of H coincides with the the degree of non-exchangeability of the corresponding copula C, which can be written as:

$$\delta(C)=3\cdot\max_{(u,v)\in\boldsymbol{I}^2}|C(u,v)-C(v,u)|, \tag{1}$$

where 3 is the normalization factor [4].

Dong et al. [5] studied the nonexchangeable degrees of best-possible bounds for copulas specified at a single interior point. Beliakov et al. [6] established best-possible bound on the set of quasi-copulas with given degree of non-exchangeability. Fernández-Sánchez and Úbeda-Flores [7] provided lower and upper bounds on the degree of asymmetry of a copula with respect to a track. De Baets et al. [8] studied the degree of asymmetry of a quasi-copula with respect to a curve. Nelsen and other authors studied several types of best possible bounds in their papers (see [9–11]).

The best-possible bounds on the set of copulas with given degree of non-exchangeability was established in [6], where the best-possible lower bound $\underline{S}_t$ and upper bound $\overline{S}_t$ are the quasi-copula given by

$$\underline{S}_t(u,v)=\max(0,u+v-1,\min(u-1+2t/3,v-1+2t/3)); \tag{2}$$

$$\overline{S}_t(u,v)=\min(u,v,\max(1-t,u-t/3,v-t/3,u+v-t)), \tag{3}$$

$\underline{S}_t$ is a copula for any $t\in[0,1]$. For any $t\in[0,3/4]$, $\underline{S}_t=W=\max(0,u+v-1)$, and for any $t\in[0,3/4]$, $\underline{S}_t$ is a shuffle of M. The best-possible upper bound $\overline{S}_t(u,v)$ is not always a copula. For any $t\in[0,3/4]$, $\overline{S}_t(u,v)$ is a copula, while for any $t\in[0,3/4]$, it is a proper quasi-copula. For any $t\in[0,1/2]$, $\overline{S}_t=W=\min(u,v)$, and for any $t\in[1/2,3/4]$, it is a shuffle of M with support containing three segments for $t\in[1/2,3/5]$, five segments for $t\in[3/5,3/4]$, and four segments for $t=3/4$.

The three values of $t=3/4$, $t=3/5$ and $t=3/6=1/2$ were the dividing points in [6]. Extending the original study of Nelsen et al., we will discuss the comparison of bounds on three sets of copulas with given degree of non-exchangeability.

From Eq. (2) and (3), we have$\underline{S}_t(u,v)=\underline{S}_t(v,u)$ and $\overline{S}_t(u,v)=\overline{S}_t(v,u)$, then $\underline{S}_t$ and $\overline{S}_t$ are symmetric. The degree of non-exchangeability of them are 0, but not t. So $\underline{S}_t$ and

$\overline{S}_t$ are the general bounds of the set of copula with a given degree of non-exchangeability $t \in [0,1]$.

Our goal is to establish the best-possible supremum bounds or minimum (resp. maximum) of copulas with the degree of non-exchangeability $t = 3/4$, $t = 3/5$ and $t = 3/6 = 1/2$. These bounds are copulas of the shuffle of M. Our paper is organized as follows. In Section 2, we establish the best-possible bounds on the set of copulas with given degree of non-exchangeability $t = 3/4$, while in Section 3, we establish the best-possible bounds on the set of copulas with given degree of non-exchangeability $t = 3/5$ and $t = 1/2$. The latter implies the effective of our best-possible bounds in narrowing the Fréchet-Hoeffding bounds (see [11, 12] for narrowing effective).

§2. Best-Possible Bounds on the Set of Copulas with Non-exchangeability $t = 3/4$

We first tackle the problem of establishing best-possible bounds on the set of copulas with given degree of non-exchangeability $t = 3/4$. Let $\boldsymbol{C}_i$, $i = 1, 2, \ldots, 6$ denote the following sets of copulas

$$\begin{aligned}
\boldsymbol{C}_1 &= \{C \,|\, C(1/2, 1/4) = 0,\ C(1/4, 1/2) = 1/4\};\\
\boldsymbol{C}_2 &= \{C \,|\, C(1/4, 1/2) = 0,\ C(1/2, 1/4) = 1/4\};\\
\boldsymbol{C}_3 &= \{C \,|\, C(3/4, 1/4) = 0,\ C(1/4, 3/4) = 1/4\};\\
\boldsymbol{C}_4 &= \{C \,|\, C(3/4, 1/4) = 1/4,\ C(1/4, 3/4) = 0\};\\
\boldsymbol{C}_5 &= \{C \,|\, C(3/4, 1/2) = 1/4,\ C(1/2, 3/4) = 1/2\};\\
\boldsymbol{C}_6 &= \{C \,|\, C(1/2, 3/4) = 1/4,\ C(3/4, 1/2) = 1/2\}.
\end{aligned}$$

Note that $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are disjoint, as for $\boldsymbol{C}_3$ and $\boldsymbol{C}_4$, $\boldsymbol{C}_5$ and $\boldsymbol{C}_6$. A copula C is in $\boldsymbol{C}_1(\boldsymbol{C}_3, \boldsymbol{C}_5)$, if and only if C^{T} is in $\boldsymbol{C}_2(\boldsymbol{C}_4, \boldsymbol{C}_6)$, where C^{T} denotes the transpose of C, given by $C^{\mathsf{T}}(u, v) = C(v, u)$.

The following four copulas will play an important role in the sequel:

$$C_1(u, v) = \min(u, v, (u - 1/2)^+ + (v - 1/4)^+), \tag{4}$$

$$C_2(u, v) = \max(0, u + v - 1, 1/4 - (1/4 - u)^+ - (1/2 - v)^+), \tag{5}$$

$$C_3(u, v) = \min(u, v, 1/4 + (u - 3/4)^+ + (v - 1/2)^+), \tag{6}$$

$$C_4(u, v) = \max(0, u + v - 1, 1/2 - (1/2 - u)^+ - (3/4 - v)^+), \tag{7}$$

where $x^+ = \max(0, x)$. Both C_1, C_2, C_3 and C_4 are the shuffle of M with non-exchangeability $t = 3/4$, the support of them consists of three line segments in $\boldsymbol{I}^2$ (see Figure 1).

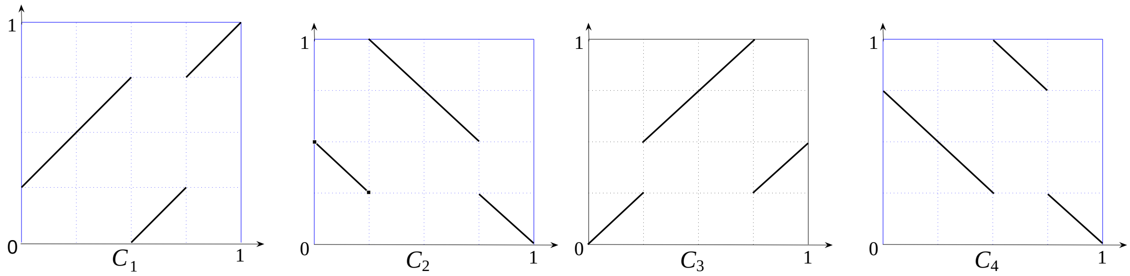

Figure 1 The supports of C_1, C_2, C_3 and C_4

Theorem 1 Let $\boldsymbol{C}$ denote the set of copulas with given degree of non-exchangeability $t = 3/4$, i.e., $\boldsymbol{C} = \{C \,|\, C$ is a copula and $\delta(C) = 3/4\}$. Then

(a) $\boldsymbol{C} = \boldsymbol{C}_1 \cup \boldsymbol{C}_2 \cup \boldsymbol{C}_3 \cup \boldsymbol{C}_4 \cup \boldsymbol{C}_5 \cup \boldsymbol{C}_6$, i.e., $C \in \boldsymbol{C}$ if and only if it compliances with any of the following conditions:

  (i) $C(1/2, 1/4) = 0$, $C(1/4, 1/2) = 1/4$ or $C(1/4, 1/2) = 0$, $C(1/2, 1/4) = 1/4$;

  (ii) $C(3/4, 1/4) = 0$, $C(1/4, 3/4) = 1/4$ or $C(1/4, 3/4) = 0$, $C(3/4, 1/4) = 1/4$;

  (iii) $C(3/4, 1/2) = 1/4$, $C(1/2, 3/4) = 1/2$ or $C(1/2, 3/4) = 1/4$, $C(3/4, 1/2) = 1/2$.

(b) For every $C \in \boldsymbol{C}, C \in \boldsymbol{C}_1 \cup \boldsymbol{C}_3$, if and only if $C_2 \prec C \prec C_1$; $C \in \boldsymbol{C}_2 \cup \boldsymbol{C}_4$, if and only if $C_2^{\mathsf{T}} \prec C \prec C_1^{\mathsf{T}}$; $C \in \boldsymbol{C}_5$, if and only if $C_4 \prec C \prec C_3$; $C \in \boldsymbol{C}_6$, if and only if $C_4^{\mathsf{T}} \prec C \prec C_3^{\mathsf{T}}$, where C_1, C_2, C_3 and C_4 are given in (4), (5), (6) and (7).

Proof (a) If $\delta(C) = 3/4$, then there exists a point $(u_0, v_0) \in \boldsymbol{I}^2$ with $u_0 < v_0$ such that either (i) $C(u_0, v_0) - C(v_0, u_0) = 1/4$ or (ii) $C(v_0, u_0) - C(u_0, v_0) = 1/4$. We show that in case (i), $C \in \boldsymbol{C}_1 \cup \boldsymbol{C}_3 \cup \boldsymbol{C}_5$, the proof that $C \in \boldsymbol{C}_2 \cup \boldsymbol{C}_4 \cup \boldsymbol{C}_6$ in case (ii) is similar and omitted.

Assume that $C(u_0, v_0) - C(v_0, u_0) = 1/4$. From $C(u, v) - C(v, u) \leqslant \min(u, 1-v, v-u)$ (see Eq. (3) in (4)), we have $\min(u_0, 1-v_0, v_0-u_0) = 1/4$, and hence $(u_0, v_0) = (1/4, 1/2)$, $(1/4, 3/4), (1/2, 3/4)$. So

$$C(1/4, 1/2) - C(1/2, 1/4) = 1/4,$$
$$C(1/4, 3/4) - C(3/4, 1/4) = 1/4,$$
$$C(1/2, 3/4) - C(3/4, 1/2) = 1/4.$$

As for any copula $C(u, v)$ satisfing the inequality:

$$W = \max(0, u + v - 1) \leqslant C(u, v) \leqslant \min(u, v) = M,$$

where W and M are Fréchet-Hoeffding lower and upper bounds.

Because of (i) $C(1/4,1/2) \leqslant 1/4$ and $C(1/2,1/4) \geqslant 0$; (ii) $C(1/4,3/4) \leqslant 1/4$ and $C(3/4,1/4) \geqslant 0$; (iii) $C(1/2,3/4) \leqslant 1/2$ and $C(3/4,1/2) \geqslant 1/4$, it follows that (i) $C(1/4,1/2) = 1/4$ and $C(1/2,1/4) = 0$, (ii) $C(1/4,3/4) = 1/4$ and $C(3/4,1/4) = 0$, (iii) $C(1/2,3/4) = 1/2$ and $C(3/4,1/2) = 1/4$. Hence $C \in \boldsymbol{C}_1 \cup \boldsymbol{C}_3 \cup \boldsymbol{C}_5$.

(b) Let $C \in \boldsymbol{C}$, assume $C_2 \prec C \prec C_1$. Hence $C(1/4,1/2) = 1/4$, $C(1/2,1/4) = 0$, or $C(1/4,3/4) = 1/4$ and $C(3/4,1/4) = 0$, thus $C \in \boldsymbol{C}_1 \cup \boldsymbol{C}_3$. Now let $C \in \boldsymbol{C}_1 \cup \boldsymbol{C}_3$. Theorem 3.2.3 of [3] state that any copula C satisfying $C(a,b) = \theta$ for given values $(a,b) \in [0,1]^2$ and $\theta \in [0,1]$ must satisfy

$$\max(u, v, \theta - (a-u)^+ - (b-v)^+) \leqslant C(u,v) \leqslant \min(u, v, \theta + (u-a)^+ + (v-b)^+).$$

So an upper bound for any copula with $C(1/4,1/2) = 1/4$ is a shuffle of M:

$$\overline{C}_1(u,v) = \min(u, v, 1/4 + (u-1/4)^+ + (v-1/2)^+),$$

the upper bound with $C(1/2,1/4) = 0$ is

$$C_1(u,v) = \min(u, v, (u-1/2)^+ + (v-1/4)^+)$$

(see Eq. (4)); the upper bound with $C(1/4,3/4) = 1/4$ is

$$\widetilde{C}_1(u,v) = \min(u, v, 1/4 + (u-1/4)^+ + (v-3/4)^+),$$

and the upper bound with $C(3/4,1/4) = 0$ is

$$\widehat{C}_1(u,v) = \min(u, v, (u-3/4)^+ + (v-1/4)^+).$$

Since

$$\min(C_1(u,v), \overline{C}_1(u,v), \widetilde{C}_1(u,v), \widehat{C}_1(u,v)) = C_1(u,v),$$

the least upper bound for any copula with $C(1/4,1/2) = 1/4$, $C(1/2,1/4) = 0$ or $C(1/4, 3/4) = 1/4$, $C(3/4,1/4) = 0$ is $C_1(u,v)$. $C_1(u,v)$ is an element of $\boldsymbol{C}$ for its non-exchangeability being $t = 3/4$.

An lower bound for any copula with $C(1/4,1/2) = 1/4$ is

$$C_2(u,v) = \max(0, u+v-1, 1/4 - (1/4-u)^+ - (1/2-v)^+),$$

the lower bound with $C(1/2,1/4) = 0$ is

$$\widetilde{C}_2(u,v) = \max(u, v, -(1/2-u)^+ - (1/4-v)^+),$$

the lower bound with $C(1/4,3/4)=1/4$ is

$$\widehat{C}_2(u,v)=\max(u,v,1/4-(1/4-u)^+-(3/4-v)^+),$$

the lower bound with $C(3/4,1/4)=0$ is

$$\overline{C}_2(u,v)=\max(u,v,-(3/4-u)^+-(1/4-v)^+).$$

Since

$$\min(C_2(u,v),\overline{C}_2(u,v),\widetilde{C}_2(u,v),\widehat{C}_2(u,v))=C_2(u,v),$$

the greatest lower bound for any copula with $C(1/4,1/2)=1/4$, $C(1/2,1/4)=0$ or $C(1/4,3/4)=1/4$, $C(3/4,1/4)=0$ is $C_2(u,v)$. $C_2(u,v)$ is also an element of C for its non-exchangeability being $t=3/4$. The case $C\in\boldsymbol{C}_2\cup\boldsymbol{C}_4$ is similar and omitted.

Assume $C_6\prec C\prec C_5$. Hence $C(1/2,3/4)=1/2$ and $C(3/4,1/2)=1/4$, thus $C\in\boldsymbol{C}_5$. Now let $C\in\boldsymbol{C}_5$. An upper bound for any copula with $C(1/2,3/4)=1/2$ is a shuffle of M:

$$\overline{C}_3(u,v)=\min(u,v,1/2+(u-1/2)^++(v-3/4)^+),$$

another upper bound for any copula with $C(3/4,1/2)=1/4$ is

$$C_3(u,v)=\min(u,v,1/4+(u-3/4)^++(v-1/2)^+).$$

Since

$$\min(C_3(u,v),\overline{C}_3(u,v))=C_3(u,v),$$

the least upper bound for any copula with $C(1/2,3/4)=1/2$, $C(3/4,1/2)=1/4$ is $C_3(u,v)$. $C_3(u,v)$ is an element of C for its non-exchangeability being $t=3/4$.

An lower bound for any copula with $C(1/2,3/4)=1/2$ is

$$C_4(u,v)=\max(0,u+v-1,1/2-(1/2-u)^+-(3/4-v)^+),$$

another lower for any copula with $C(3/4,1/2)=1/4$ is

$$\overline{C}_4(u,v)=\max(u,v,1/4-(3/4-u)^+-(1/2-v)^+).$$

Since

$$\max(C_4(u,v),\overline{C}_4(u,v))=C_4(u,v),$$

the greatest lower bound for any copula with $C(1/2,3/4)=1/2$, $C(3/4,1/2)=1/4$ is $C_4(u,v)$. $C_4(u,v)$ is also an element of C for its non-exchangeability being $t=3/4$. The case $C\in\boldsymbol{C}_6$ is similar and omitted. $\square$

Theorem 1 means that there are four best possible upper and lower bounds for the set of copulas with given degree of non-exchangeability $t = 3/4$, denoted by $\boldsymbol{C}_U = \{C_1, C_3, C_1^{\mathsf{T}}, C_3^{\mathsf{T}}\}$ as the set of upper bounds, and $\boldsymbol{C}_L = \{C_2, C_4, C_2^{\mathsf{T}}, C_4^{\mathsf{T}}\}$ the lower bounds set.

§3. Best-Possible Bounds of Copulas with Non-exchangeability $t = 3/5$ and $1/2$

Similar to the theory of Section 2, the following eight copulas play an important role in the sequel of best-possible bounds of copulas with given degree of non-exchangeability $t = 3/5$:

$$
\begin{aligned}
C_1^5(u,v) &= \min(u, v, (u-2/5)^+ + (v-1/5)^+);\\
C_2^5(u,v) &= \max(0, u+v-1, 1/5 - (1/5-u)^+ - (2/5-v)^+);\\
C_3^5(u,v) &= \min(u, v, (u-3/5)^+ + (v-2/5)^+);\\
C_4^5(u,v) &= \max(0, u+v-1, 1/5 - (2/5-u)^+ - (3/5-v)^+);\\
C_5^5(u,v) &= \min(u, v, 1/5 + (u-3/5)^+ + (v-2/5)^+);\\
C_6^5(u,v) &= \max(0, u+v-1, 2/5 - (2/5-u)^+ - (3/5-v)^+);\\
C_7^5(u,v) &= \min(u, v, 2/5 + (u-4/5)^+ + (v-3/5)^+);\\
C_8^5(u,v) &= \max(0, u+v-1, 3/5 - (3/5-u)^+ - (4/5-v)^+).
\end{aligned}
$$

Theorem 2 Let $\boldsymbol{C}_5 = \{C \mid C \text{ is a copula and } \delta(C) = 3/5\}$. Then

(a) $C \in \boldsymbol{C}_5$ if and only if C meets any of the following conditions:

(i) $C(2/5, 1/5) = 0$, $C(1/5, 2/5) = 1/5$ or $C(2/5, 1/5) = 1/5$, $C(1/5, 2/5) = 0$;

(ii) $C(3/5, 2/5) = 0$, $C(2/5, 3/5) = 1/5$ or $C(3/5, 2/5) = 1/5$, $C(2/5, 3/5) = 0$;

(iii) $C(3/5, 2/5) = 1/5$, $C(2/5, 3/5) = 2/5$ or $C(3/5, 2/5) = 2/5$, $C(2/5, 3/5) = 1/5$;

(iv) $C(4/5, 2/5) = 1/5$, $C(2/5, 4/5) = 2/5$ or $C(4/5, 2/5) = 2/5$, $C(2/5, 4/5) = 1/5$;

(v) $C(4/5, 3/5) = 2/5$, $C(3/5, 4/5) = 3/5$ or $C(4/5, 3/5) = 3/5$, $C(3/5, 4/5) = 2/5$.

(b) For every $C \in \boldsymbol{C}_5$, C is satisfies the conditions (i) of (a) if and only if $C_2^5 \prec C \prec C_1^5$ or $C_2^{5\mathsf{T}} \prec C \prec C_1^{5\mathsf{T}}$; C satisfies (ii) if and only if $C_4^5 \prec C \prec C_3^5$, or $C_4^{5\mathsf{T}} \prec C \prec C_3^{5\mathsf{T}}$; C satisfies (iii) or (iv) if and only if $C_6^5 \prec C \prec C_5^5$, or $C_6^{5\mathsf{T}} \prec C \prec C_5^{5\mathsf{T}}$; C satisfies (v) if and only if $C_8^5 \prec C \prec C_7^5$ or $C_8^{5\mathsf{T}} \prec C \prec C_7^{5\mathsf{T}}$.

The proof of Theorem 2 is similar to Theorem 1, slightly here. There are eight pairs of best possible upper and lower bounds for the set of copulas with given degree of non-exchangeability $t=3/5$, denoted by respectively

$$\begin{aligned}\boldsymbol{C}_U^5&=\{C_1^5,C_3^5,C_5^5,C_7^5;C_1^{5^\top},C_3^{5^\top},C_5^{5^\top},C_7^{5^\top}\},\\ \boldsymbol{C}_L^5&=\{C_2^5,C_4^5,C_6^5,C_8^5;C_2^{5^\top},C_4^{5^\top},C_6^{5^\top},C_8^{5^\top}\}.\end{aligned}$$

The following twelve copulas play an important role in the sequel of best-possible bounds of copulas with given degree of non-exchangeability $t=1/2$:

$$\begin{aligned}
C_1^6(u,v)&=\min(u,v,(u-1/3)^++(v-1/6)^+);\\
C_2^6(u,v)&=\max(0,u+v-1,1/6-(1/6-u)^+-(1/3-v)^+);\\
C_3^6(u,v)&=\min(u,v,1/6+(u-1/2)^++(v-1/3)^+);\\
C_4^6(u,v)&=\max(0,u+v-1,1/3-(1/3-u)^+-(1/2-v)^+);\\
C_5^6(u,v)&=\min(u,v,1/6+(u-2/3)^++(v-1/2)^+);\\
C_6^6(u,v)&=\max(0,u+v-1,1/3-(1/2-u)^+-(2/3-v)^+);\\
C_7^6(u,v)&=\min(u,v,(u-1/2)^++(v-1/3)^+);\\
C_8^6(u,v)&=\max(0,u+v-1,1/6-(1/3-u)^+-(1/2-v)^+);\\
C_9^6(u,v)&=\min(u,v,1/3+(u-2/3)^++(v-1/2)^+);\\
C_{10}^6(u,v)&=\max(0,u+v-1,1/2-(1/2-u)^+-(2/3-v)^+);\\
C_{11}^6(u,v)&=\min(u,v,1/2+(u-5/6)^++(v-2/3)^+);\\
C_{12}^6(u,v)&=\max(0,u+v-1,2/3-(2/3-u)^+-(5/6-v)^+).
\end{aligned}$$

Theorem 3 Let $\mathbf{C}_6=\{C|C$ is a copula and $\delta(C)=1/2\}$. Then

(a) $C\in\boldsymbol{C}_6$ if and only if C meets any of the following conditions:

(i) $C(1/6,5/6)=1/6$, $C(5/6,1/6)=0$; $C(1/6,4/6)=1/6$, $C(4/6,1/6)=0$; $C(1/6,1/2)=1/6$, $C(1/2,1/6)=0$; $C(1/6,1/3)=1/6$, $C(1/3,1/6)=0$;

(ii) $C(1/3,1/2)=1/3$, $C(1/2,1/3)=1/6$;

(iii) $C(1/3,1/2)=1/6$, $C(1/2,1/3)=0$;

(iv) $C(1/3,5/6)=1/3$, $C(5/6,1/3)=1/6$ or $C(1/2,2/3)=1/3$, $C(2/3,1/2)=1/6$;

(v) $C(1/2,2/3)=1/2$, $C(2/3,1/2)=1/3$ or $C(1/2,5/6)=1/2$, $C(5/6,1/2)=1/3$;

(vi) $C(2/3,5/6)=2/3$, $C(5/6,2/3)=1/2$.

(b) For every $C \in \boldsymbol{C}_6$, C is satisfies the conditions (i) of (a) if and only if $C_2^6 \prec C \prec C_1^6$ or $C_2^{6\top} \prec C \prec C_1^{6\top}$; C satisfies (ii) if and only if $C_4^6 \prec C \prec C_3^6$, or $C_4^{6\top} \prec C \prec C_3^{6\top}$; C satisfies (iii) if and only if $C_6^6 \prec C \prec C_5^6$, or $C_6^{6\top} \prec C \prec C_5^{6\top}$; C satisfies (iv) if and only if $C_8^6 \prec C \prec C_7^6$, or $C_8^{6\top} \prec C \prec C_7^{6\top}$; C satisfies (v) if and only if $C_{10}^6 \prec C \prec C_9^6$, or $C_{10}^{6\top} \prec C \prec C_9^{6\top}$; C satisfies (vi) if and only if $C_{12}^5 \prec C \prec C_{11}^5$ or $C_{12}^{5\top} \prec C \prec C_{11}^{5\top}$.

So there are twelve best possible upper and lower bounds for the set of copulas with given degree of non-exchangeability $t = 3/5$, denoted by respectively

$$\boldsymbol{C}_U^6 = \{C_1^6, C_3^6, C_5^6, C_7^6, C_9^6, C_{11}^6; C_1^{6\top}, C_3^{6\top}, C_5^{6\top}, C_7^{6\top}, C_9^{6\top}, C_{11}^{6\top}\},$$
$$\boldsymbol{C}_L^6 = \{C_2^6, C_4^6, C_6^6, C_8^6, C_{10}^6, C_{12}^6; C_2^{6\top}, C_4^{6\top}, C_6^{6\top}, C_8^{6\top}, C_{10}^{6\top}, C_{12}^{6\top}\}.$$

§4. A Comparison of the Bounds

To measure the effectiveness in narrowing the Fréchet-Hoeffding bounds of the degree of non-exchangeability, we use the function

$$m(t) = 1 - 6\int\int_{\boldsymbol{I}^2} [\overline{A}_t(u,v) - \underline{A}_t(u,v)]\mathrm{d}u\mathrm{d}v, \tag{8}$$

where $\overline{A}_t(u,v)$ and $\underline{A}_t(u,v)$ are the bounds element of the set $\boldsymbol{C}_U(\boldsymbol{C}_U^5, \boldsymbol{C}_U^6)$ and $\boldsymbol{C}_L(\boldsymbol{C}_L^5, \boldsymbol{C}_L^6)$. $m(t)$ represents the volume between copulas $z = \underline{A}_t(u,v)$ and $z = \overline{A}_t(u,v)$ in $\boldsymbol{I}^3$. When there is no improvement in the bounds, $m(t) = 0$, i.e., $\underline{A}_t = W$ and $\overline{A}_t = M$, and when the bounds coincide, $m(t) = 1$,

By Theorem 1, $C_1(u,v)$ and $C_2(u,v)$ defined by Eq. (4) and (5) are the upper and lower bounds of copulas with non-exchangeability $t = 3/4$ satisfying $C(1/2, 1/4) = 0$ and $C(1/4, 1/2) = 1/4$. The support of them see Figure 1. Using Eq. (8), we can calculate the narrowing effectiveness between them as following,

$$m_{1,2}\Big(\frac{3}{4}\Big) = 1 - 6\int\int_{\boldsymbol{I}^2}[C_1(u,v) - C_2(u,v)]\mathrm{d}u\mathrm{d}v = 1 - 6 \times \frac{7}{96} = 0.5625,$$

where

$$\begin{aligned}
&\int\int_{\boldsymbol{I}^2}[C_1(u,v) - C_2(u,v)]\mathrm{d}u\mathrm{d}v \\
= &\int_0^{\frac{1}{8}}\Big[\int_{\frac{1}{4}}^{u+1/4}\Big(v - \frac{1}{4}\Big)\mathrm{d}v + \int_{u+1/4}^{1/2-1} u\mathrm{d}v + \int_{1/2-u}^{1/2}\Big(\frac{1}{2} - v\Big)\mathrm{d}v\Big]\mathrm{d}u \\
&+ \int_{1/8}^{1/4}\Big[\int_{1/4}^{1/2-u}\Big(v - \frac{1}{4}\Big)\mathrm{d}v + \int_{1/2-u}^{1/4+u}\Big(\frac{1}{4} - u\Big)\mathrm{d}v + \int_{1/4+u}^{1/2}\Big(\frac{1}{2} - v\Big)\mathrm{d}v\Big]\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&+\int_{1/4}^{1/2}\Big[\int_{1/2}^{1/4+u}\Big(v-\frac{1}{2}\Big)\mathrm{d}v+\int_{1/4+u}^{5/4-u}\Big(u-\frac{1}{4}\Big)\mathrm{d}v+\int_{5/4-u}^{1}(1-v)\mathrm{d}v\Big]\mathrm{d}u\\
&+\int_{1/2}^{3/4}\Big[\int_{0}^{u-1/2}v\mathrm{d}v+\int_{u-1/2}^{1/2}\Big(u-\frac{1}{2}\Big)\mathrm{d}v+\int_{1/2}^{5/4-u}\Big(u+v-\frac{1}{2}\Big)\mathrm{d}v+\int_{5/4-u}^{3/4}\frac{1}{4}\mathrm{d}v\\
&+\int_{3/4}^{1}(1-v)\mathrm{d}v\Big]\mathrm{d}u+\int_{3/4}^{1}\Big[\int_{0}^{1-u}v\mathrm{d}v+\int_{1-u}^{u}(1-u)\mathrm{d}v+\int_{u}^{1}(1-v)\mathrm{d}v\Big]\mathrm{d}u.
\end{aligned}
$$

Similarly as $m_{12}(3/4)$, we have

$$
m_{3,4}\Big(\frac{3}{4}\Big)=1-6\int\int_{\boldsymbol{I}^2}[C_3(u,v)-C_4(u,v)]\mathrm{d}u\mathrm{d}v=1-6\times\frac{7}{96}=0.5625=m_{12}\Big(\frac{3}{4}\Big),
$$

where C_3 and C_4 are given by (6) and (7). As

$$
\int\int_{\boldsymbol{I}^2}[C_k^{\mathsf{T}}(u,v)-C_{k+1}^{\mathsf{T}}(u,v)]\mathrm{d}u\mathrm{d}v=\int\int_{\boldsymbol{I}^2}[C_k(u,v)-C_{k+1}(u,v)]\mathrm{d}u\mathrm{d}v,\qquad k=1,3.
$$

The $m(t)$ of the upper and lower bounds of transpose copula are omitted here.

Comparing with the conclusion about bounds of the paper [6], where $m(3/4)=3/32=0.09375$, so the bounds in our paper is more accurate.

For the narrowing effectiveness about best-possible bounds of copulas with non-exchangeability $t=3/5$ and $1/2$, we have the following similar conclusions. Let

$$
\begin{aligned}
m_{i,i+1}\Big(\frac{3}{5}\Big)&=1-6\int\int_{\boldsymbol{I}^2}[C_i^5(u,v)-C_{i+1}^5(u,v)]\mathrm{d}u\mathrm{d}v,\qquad i=1,3,5,7;\\
m_{j,j+1}\Big(\frac{3}{5}\Big)&=1-6\int\int_{\boldsymbol{I}^2}[C_j^6(u,v)-C_{j+1}^6(u,v)]\mathrm{d}u\mathrm{d}v,\qquad j=1,3,5,7,9,11.
\end{aligned}
$$

After tedious integral calculation, we have

$$
\begin{gathered}
m_{1,2}\Big(\frac{3}{5}\Big)=0.432,\qquad m_{3,4}\Big(\frac{3}{5}\Big)=0.768,\\
m_{5,6}\Big(\frac{3}{5}\Big)=m_{7,8}\Big(\frac{3}{5}\Big)=m_{1,2}\Big(\frac{3}{5}\Big)=0.432;\\
m_{1,2}\Big(\frac{1}{2}\Big)=m_{11,12}\Big(\frac{1}{2}\Big)=0.361,\qquad m_{3,4}\Big(\frac{1}{2}\Big)=0.444,\\
m_{5,6}\Big(\frac{1}{2}\Big)=m_{7,8}\Big(\frac{1}{2}\Big)=0.5,\qquad m_{9,10}\Big(\frac{1}{2}\Big)=0.389.
\end{gathered}
$$

Comparing with the conclusion about bounds of the paper [8], where $m(3/5)=1/125=0.008$ and $m(1/2)=0$, which means the bounds in our paper are specific practical and effective in narrowing the Fréchet-Hoeffding bounds.

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给定不可交换性度量的三类 Copula 界的比较

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摘　要： 给出当 Copula 的不可交换性度量为 $t = 3/4$, $t = 3/5$ 和 $t = 3/6 = 1/2$ 时的最优界, 研究了这三类 Copula 的结构, 计算了最优下上界间的距离, 说明它们有效地改善了 Fréchet-Hoeffding 上下界.

关键词： 相关结构; 最优界; 不可交换性度量; 变窄效率

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