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Optimal Investment Strategies for an Insurer with SAHARA Utility *

ZHAO Qian⋆ ZHU Shaohui

(*School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai, 201620, China*)

Abstract: In this paper, we consider the optimal investment strategy which maximizes the utility of the terminal wealth of an insurer with SAHARA utility functions. This class of utility functions has non-monotone absolute risk aversion, which is more flexible than the CARA and CRRA utility functions. In the case that the risk process is modeled as a Brownian motion and the stock process is modeled as a geometric Brownian motion, we get the closed-form solutions for our problem by the martingale method for both the constant threshold and when the threshold evolves dynamically according to a specific process. Finally, we show that the optimal strategy is state-dependent.

Keywords: optimal investment strategy; SAHARA utility function; insurer; martingale approach

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§1. Introduction

In recent years, there is a vast literature on the optimal investment problems for an insurer. Maximizing the utility of the terminal wealth of the insurer is a popular criterion. To obtain closed-form solutions, the CARA and CRRA utility functions are mostly used in solving this kind of problems. Browne [1] considers a model in which the aggregate claims are modeled by a Brownian motion with drift, and the risky asset is modeled by a geometric Brownian motion. Yang and Zhang [2] considers the same problem with the assumption that the risk process of the insurer is a jump diffusion process. By the martingale and dual methods, Wang et al. [3] discusses a more general model in which the risk process of the insurer is a Lévy process. Besides some other criterions, e.g. minimizing the ruin probability and mean-variance criterion, the exponential utility function is adopted in all of these papers.

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⋆Corresponding author, E-mail: zhaoqian@suibe.edu.cn.

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Nowadays, there is an increasing interest in solving the optimal investment problems under different utility functions. Berkelaar et al. [4] considers the portfolio selection problem under a specific two-piece power utility function. Considering specific HARA utility functions, Sotomayor and Cadenillas [5] obtains the explicit solution to the consumption-investment problem in a regime switching model. Chen et al. [6] introduces a new class of utility functions which is called Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA) class. Under this class of utility functions, they obtained the closed-form solution to the optimal investment problem for an ordinary investor (without the risk process).

Similar to the exponential utility functions, the SAHARA utility functions are also defined on the whole real line. Sometimes this is useful and convenient, for example, there are many papers (see, e.g. [7] and [8], among others) assume that the insurer can still run the business for a certain time when the surplus becomes negative.

As we will see, the power and exponential utility functions are limiting cases of SAHARA utility functions. For every SAHARA utility function, there exists a level of wealth, which is called threshold wealth, where the absolute risk aversion attains a finite maximal value. When the level of wealth is above the threshold wealth, the risk aversion decreases as increasing the wealth; when the level of wealth is below the threshold wealth, the risk aversion decreases as decreasing the wealth. Thus, unlike the exponential and power utility functions, the SAHARA utility functions have a distinguishing feature that they allow the flexibility that the absolute risk aversion to be non-monotone and implement the assumption that agents may become less risk averse for very low values of wealth. However, the risk aversion can never be negative, i.e., the investor with SARAHA utility never becomes risk-seeking, which happens in the prospect theory (see, [9] and [10]).

In this paper we consider the optimal investment problem for an insurer with the SAHARA utility function. Similar to Barberis et al. [11], we model the risk process of the insurer by a Brownian motion with drift and the stock price by a geometric Brownian motion. In contrast to Chen et al. [6] which considers the zero threshold wealth, we consider a dynamically updated threshold wealth. This is a more general and more reasonable case, since the investor will update her threshold wealth (or reference point) throughout time, depending on the performance of his wealth process (see, e.g. [4], [11] and [12]). This is completed in two steps. First, we consider the SAHARA utility function with constant threshold wealth, and by the similar techniques of [3], we obtain the closed-form solution for the optimal investment strategy as well as the optimal terminal wealth. Second, given a specific dynamics of the threshold wealth, we show that it is equivalent to solve the problem considered in the first step with some specific constant threshold wealth. Our results show that the optimal strategy is state-dependent.

As we all know that the utility functions are widely used in actuarial science, such as premium principle and indifference pricing of insurance products, etc. Usually, the CARA and CRRA utility functions are considered. This paper shows that we can still get the closed-form solution even we consider a more flexible utility function.

This paper is organized as follows. In Section 2, we introduce the SAHARA utility functions as well as the financial market model. In Section 3, we derive our main results for the optimal investment problem. In Section 4, we illustrate our results by some numerical examples.

§2. Preliminaries

2.1 SAHARA Utility Functions

The class of SAHARA utility functions was first proposed by [6]. We list the definition and some useful properties in this subsection.

Definition 1 A utility function U with domain $\mathbb{R}$ is of the SAHARA class if its absolute risk aversion function $A(x) = -U''(x)/U'(x)$ is well defined on its entire domain and satisfies

$$A(x) = \frac{\alpha}{\sqrt{\gamma^2 + (x-\kappa)^2}} > 0$$

for a given $\gamma > 0$ (the scale parameter), $\alpha > 0$ (the risk aversion parameter) and $\kappa \in \mathbb{R}$ (the threshold wealth).

It is worth noting that the threshold wealth κ plays a role as the reference point in the prospect theory. In other words, when the wealth is above κ, the investor is in the domain of gains; when the wealth is below κ, the investor is in the domain of losses. Motivated by this feature of κ, we will consider a dynamically updated threshold wealth in this paper.

Note that the power and exponential utility functions are limiting cases of SAHARA utility functions. Specifically, when $\kappa = 0$ and $\gamma \downarrow 0$, it becomes the class of HARA utilities with the risk aversion function $A(x) = \alpha/x$, $x > 0$ for all $\alpha \in (0, 1)$; whereas when taken $\kappa = 0$, $\alpha = \theta\gamma$ and $\gamma \to \infty$, it leads to an exponential utility function with constant absolute risk aversion parameter θ.

We now introduce some properties for the class of SAHARA utility functions.

Proposition 2 Let U be a SAHARA utility function with scale parameter $\gamma > 0$, risk aversion parameter $\alpha > 0$, and threshold wealth $\kappa \in \mathbb{R}$. Then

(i) There exists constants c_1 and c_2 such that $U(x) = c_1 + c_2\widehat{U}(x)$ with

$$\widehat{U}(x) = \begin{cases} \dfrac{1}{1-\alpha^2}\big[(x-\kappa)+\sqrt{\gamma^2+(x-\kappa)^2}\,\big]^{-\alpha} \\ \qquad \times\big[(x-\kappa)+\alpha\sqrt{\gamma^2+(x-\kappa)^2}\,\big], & \alpha \neq 1; \\ \dfrac{1}{2}\big[(x-\kappa)+\sqrt{\gamma^2+(x-\kappa)^2}\,\big] \\ \qquad +\dfrac{1}{2}\gamma^{-2}(x-\kappa)\big[\sqrt{\gamma^2+(x-\kappa)^2}-(x-\kappa)\big], & \alpha = 1. \end{cases}$$

We take $c_1 = 0$ and $c_2 = 1$ in the following context for simplicity.

(ii) The first-order derivative of U is given by

$$U'(x) = \big[(x-\kappa)+\sqrt{\gamma^2+(x-\kappa)^2}\,\big]^{-\alpha} = \gamma^{-\alpha}\exp\Big\{-\alpha\,\mathrm{arcsinh}\Big(\frac{x-\kappa}{\gamma}\Big)\Big\} > 0.$$

The proof is similar to [6] which considers a zero threshold wealth, i.e., $\kappa = 0$. For more propositions on the SAHARA utility functions, we refer the readers to [6].

Remark 3 The above results are based on the assumption $\gamma < \infty$. By (ii), the second-order derivative satisfies $U''(x) < 0$, which implies that U is a strictly increasing, strictly concave and second-order continuously differential function with $U'(\kappa) := \lim\limits_{x\to\kappa} U'(x) = \gamma^{-\alpha} < \infty$ and $U'(\infty) := \lim\limits_{x\to\infty} U'(x) = 0$.

2.2 The Market Model

We consider a financial market consisting of one bond and one stock. Let $T > 0$ be a fixed time horizon. The prices of the bond and the stock are denoted by $B(t)$ and $S(t)$, respectively. Assume that $B(t)$ satisfies

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t, \qquad B(0) = 1,$$

where r is the risk-free interest rate, and the dynamics of the stock price is given by

$$\mathrm{d}S(t) = S(t)[\mu\mathrm{d}t + \sigma\mathrm{d}W(t)], \qquad 0 \leqslant t \leqslant T, \tag{1}$$

where μ is the appreciation rate and $\sigma > 0$ is the volatility. W is a standard Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Denote by $R(t)$ the risk process of the insurer and assume it follows

$$\mathrm{d}R(t) = \lambda\mathrm{d}t + \beta\mathrm{d}W(t), \qquad 0 \leqslant t \leqslant T, \tag{2}$$

where λ and $\beta > 0$ are constants.

The insurer is allowed to invest in the stock as well as in the bond. Let $\pi(t)$ be the dollar amount invested in the stock at time t. We say $\pi(t)$ is an admissible investment strategy if it is an $\{\mathscr{F}_t\}$-progressively measurable, real-valued process satisfying

$$\mathsf{E}\Big[\int_0^T \pi^2(t)\mathrm{d}t\Big] < \infty. \tag{3}$$

We denote by Π the set of all admissible investment strategies.

The wealth process $X^{x,\pi}$ corresponding to trading strategy π is given by

$$\begin{aligned}\mathrm{d}X^{x,\pi}(t) &= [X^{x,\pi}(t) - \pi(t)]r\mathrm{d}t + \pi(t)[\mu\mathrm{d}t + \sigma\mathrm{d}W(t)] + \mathrm{d}R(t)\\ &= [rX^{x,\pi}(t) + \pi(t)(\mu - r) + \lambda]\mathrm{d}t + [\sigma\pi(t) + \beta]\mathrm{d}W(t),\end{aligned} \tag{4}$$

with $X^{x,\pi}(0) = x$. Then the discounted wealth process satisfies

$$\mathrm{d}(\mathrm{e}^{-rt}X^{x,\pi}(t)) = \mathrm{e}^{-rt}\{[\pi(t)(\mu - r) + \lambda]\mathrm{d}t + [\sigma\pi(t) + \beta]\mathrm{d}W(t)\}.$$

Thus, we have

$$\begin{aligned}X^{x,\pi}(t) &= \mathrm{e}^{rt}\Big\{x + \int_0^t \mathrm{e}^{-rs}[\pi(s)(\mu - r) + \lambda]\mathrm{d}s + \int_0^t \mathrm{e}^{-rs}[\sigma\pi(s) + \beta]\mathrm{d}W(s)\Big\}\\ &= x\mathrm{e}^{rt} - \frac{\lambda}{r}(1 - \mathrm{e}^{rt}) + (\mu - r)\int_0^t \mathrm{e}^{-r(s-t)}\pi(s)\mathrm{d}s\\ &\quad + \int_0^t \mathrm{e}^{-r(s-t)}[\sigma\pi(s) + \beta]\mathrm{d}W(s).\end{aligned} \tag{5}$$

With the SAHARA utilities, the objective of the insurer is to specify an optimal investment strategy π^* such that the expected utility of the terminal wealth, $\mathsf{E}[U(X^{x,\pi^*}(T))]$, is maximized, i.e.,

$$\mathsf{E}[U(X^{x,\pi^*}(T))] = \sup_{\pi\in\Pi} \mathsf{E}[U(X^{x,\pi}(T))].$$

Since a SAHARA utility function is strictly concave and continuously differentiable on $(-\infty, \infty)$, there exists at most a unique optimal terminal wealth for the company. The following proposition is taken from [3].

Proposition 4 If there exists a strategy $\pi^* \in \Pi$ such that

$$\mathsf{E}[U'(X^{x,\pi^*}(T))X^{x,\pi}(T)] \text{ is constant over } \pi \in \Pi, \tag{6}$$

then π^* is the optimal trading strategy.

§3. Main Results

3.1 Constant Threshold Wealth

In the case of SAHARA utility with constant threshold wealth κ, we have

$$U'(x)=\gamma^{-\alpha}\exp\Big[-\alpha\operatorname{arcsinh}\Big(\frac{x-\kappa}{\gamma}\Big)\Big],$$

and the condition (6) can be expressed as

$$\begin{aligned}&\mathsf{E}\Big\{\exp\Big[-\alpha\operatorname{arcsinh}\Big(\frac{X^{x,\pi^*}(T)-\kappa}{\gamma}\Big)\Big]\\&\times\Big[\int_0^T(\mu-r)\mathrm{e}^{-rs}\pi(s)\mathrm{d}s+\int_0^T\sigma\mathrm{e}^{-rs}\pi(s)\mathrm{d}W(s)\Big]\Big\}\text{ is constant over }\pi\in\Pi.\end{aligned}\tag{7}$$

Next we will conjecture the form of the optimal strategies π^* from condition (7), and then verify it.

Put

$$Z^*(t):=\mathsf{E}\Big\{\exp\Big[-\alpha\operatorname{arcsinh}\Big(\frac{X^{x,\pi^*}(T)-\kappa}{\gamma}\Big)\Big]\,\Big|\,\mathscr{F}_t\Big\},\qquad t\in[0,T],$$

then

$$Z^*(T)=\exp\Big[-\alpha\operatorname{arcsinh}\Big(\frac{X^{x,\pi^*}(T)-\kappa}{\gamma}\Big)\Big],$$

and $Z^*(\tau)=\mathsf{E}[Z^*(T)\,|\,\mathscr{F}_\tau]$, a.s. for any stopping time $\tau\leqslant T$.

Lemma 5 Let $\pi^*\in\Pi$, then π^* satisfies condition (7) if and only if the triple (X^{x,π^*},π^*,Z^*) solves the following forward-backward stochastic differential equation (FBSDE)

$$\begin{cases}\mathrm{d}X(t)=[rX(t)+\pi(t)(\mu-r)+\lambda]\mathrm{d}t+[\sigma\pi(t)+\beta]\mathrm{d}W(t),\\X(0)=x,\\\mathrm{d}Z(t)=-\dfrac{\mu-r}{\sigma}Z(t)\mathrm{d}W(t),\\Z(T)=\exp\Big[-\alpha\operatorname{arcsinh}\Big(\dfrac{X(T)-\kappa}{\gamma}\Big)\Big],\end{cases}\tag{8}$$

for $(X,\pi,Z)\in L^2_{\mathscr{F}}\times\Pi\times L^2_{\mathscr{F}}$. Here $L^2_{\mathscr{F}}$ denotes the set of all $\mathscr{F}_t$-adapted process $X(t)$ with cadlag paths such that $\mathsf{E}\big[\sup\limits_{0\leqslant t\leqslant T}|X(t)|^2\big]<\infty$.

Proof Assume that π^* satisfies condition (7). It is clear that $X^{x,\pi^*}\in L^2_{\mathscr{F}}$ is a solution to the forward SDE in (8) for X and that $Z^*(t)$ is a square-integrable martingale.

Let $\pi^\tau(t) = \mathbf{1}_{\{t\leqslant\tau\}}$ for any stopping time $\tau \leqslant T$, then $\pi^\tau \in \Pi$. Substituting π^τ into (7), we obtain

$$\begin{aligned}
&\mathsf{E}\Big\{Z^*(T)\Big[\int_0^\tau(\mu-r)\mathrm{e}^{-rs}\mathrm{d}s+\int_0^\tau\sigma\mathrm{e}^{-rs}\mathrm{d}W(s)\Big]\Big\}\\
&=\mathsf{E}\Big\{\mathsf{E}[Z^*(T)\,|\,\mathscr{F}_\tau]\Big[\int_0^\tau(\mu-r)\mathrm{e}^{-rs}\mathrm{d}s+\int_0^\tau\sigma\mathrm{e}^{-rs}\mathrm{d}W(s)\Big]\Big\}\\
&=\mathsf{E}\Big\{Z^*(\tau)\Big[\int_0^\tau(\mu-r)\mathrm{e}^{-rs}\mathrm{d}s+\int_0^\tau\sigma\mathrm{e}^{-rs}\mathrm{d}W(s)\Big]\Big\}
\end{aligned}$$

is constant over all stopping time $\tau \leqslant T$ a.s., which implies that

$$Z^*(t)\Big[\int_0^t(\mu-r)\mathrm{e}^{-rs}\mathrm{d}s+\int_0^t\sigma\mathrm{e}^{-rs}\mathrm{d}W(s)\Big]\ \text{ is a martingale.} \tag{9}$$

Since $Z^*(t)$ is a square-integrable martingale, then by the martingale representation theorem (see e.g. [3; Lemma 2.1]), there exists an $\mathscr{F}_t$-predictable, $\mathbb{R}$-valued process $\theta(t)$ satisfying

$$\mathsf{E}\Big[\int_0^T|\theta(t)|^2\mathrm{d}t\Big]<\infty,$$

such that

$$\mathrm{d}Z^*(t)=\theta(t)\mathrm{d}W(t),\qquad \forall\, t\in[0,T].$$

Therefore by Itô formula, we obtain

$$\begin{aligned}
&\mathrm{d}\Big(Z^*(t)\Big[\int_0^t(\mu-r)\mathrm{e}^{-rs}\mathrm{d}s+\int_0^t\sigma\mathrm{e}^{-rs}\mathrm{d}W(s)\Big]\Big)\\
&=[\sigma\theta(t)+(\mu-r)Z^*(t)]\mathrm{e}^{-rt}\mathrm{d}t+\sigma Z^*(t)\mathrm{e}^{-rt}\mathrm{d}W(t),
\end{aligned}$$

which together with (9) implies $\sigma\theta(t)+(\mu-r)Z^*(t)=0$, i.e.,

$$\theta(t)=-\frac{\mu-r}{\sigma}Z^*(t).$$

Therefore we know that Z^* solves the backward SDE in (8) for Z. Hence it follows that (X^{x,π^*},π^*,Z^*) solves the FBSDE (8).

Conversely, suppose that there exists $Z^*\in L^2_{\mathscr{F}}$ such that (X^{x,π^*},π^*,Z^*) solves FBSDE (8). It is easy to check by Itô formula that for any $\pi\in\Pi$, $Z^*(t)M^\pi(t)$ is a local martingale, where

$$M^\pi(t):=\int_0^t(\mu-r)\mathrm{e}^{-rs}\pi(s)\mathrm{d}s+\int_0^t\sigma\mathrm{e}^{-rs}\pi(s)\mathrm{d}W(s). \tag{10}$$

Furthermore, for any $\pi\in\Pi$, it is clear that $M^\pi\in L^2_{\mathscr{F}}$. It follows that

$$\mathsf{E}\Big[\sup_{0\leqslant t\leqslant T}|Z^*(t)M^\pi(t)|\Big]\leqslant\sqrt{\mathsf{E}\Big[\sup_{0\leqslant t\leqslant T}|Z^*(t)|^2\Big]\mathsf{E}\Big[\sup_{0\leqslant t\leqslant T}|M^\pi(t)|^2\Big]}<\infty.$$

Therefore, the family $\{Z^*(\tau)M^\pi(\tau) : \tau$ is a stopping time and $\tau \leqslant T\}$ is uniformly integrable and thus Z^*M^π is a martingale.

Then we have $\mathsf{E}[Z^*(T)M^\pi(T)] = 0$ for any $\pi \in \Pi$, which implies that π^* satisfies condition (7). $\square$

In what follows, we will solve the FBSDE (8) by two steps: first, we conjecture the form of the solution and then we verify it.

Step 1: Let us define

$$A_1(t) := \exp\Big[\int_0^t a_1(s)\mathrm{d}s\Big], \quad A_2(t) := \exp\Big[\int_0^t a_2(s)\mathrm{d}s\Big], \qquad t \in [0,T],$$

where $a_1(t)$ and $a_2(t)$ are non-random Lebesgue-integrable functions to be determined.

By Itô formula, we have that

$$\begin{aligned} Z^{p_i}(T) = {}& \frac{Z^{p_i}(0)}{A_i(T)} - \frac{(\mu - r)}{\sigma}\int_0^T \frac{p_i A_i(s)Z^{p_i}(s)}{A_i(T)}\mathrm{d}W(s) \\ &+ \int_0^T \Big[\frac{1}{2}p_i(p_i-1)\Big(\frac{\mu-r}{\sigma}\Big)^2 + a_i(s)\Big]\frac{A_i(s)Z^{p_i}(s)}{A_i(T)}\mathrm{d}s, \qquad i = 1,2, \end{aligned}$$

where $p_1 = -1/\alpha$ and $p_2 = 1/\alpha$.

Therefore,

$$\begin{aligned} &\frac{\gamma}{2}\big[Z^{-1/\alpha}(T) - Z^{1/\alpha}(T)\big] + \kappa \\ ={}& \frac{\gamma}{2}\Big[\frac{Z^{-1/\alpha}(0)}{A_1(T)} - \frac{Z^{1/\alpha}(0)}{A_2(T)}\Big] + \kappa \\ &+ \frac{\gamma(\mu-r)}{2\sigma\alpha}\int_0^T \Big[\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) + \frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\Big]\mathrm{d}W(s) \\ &+ \frac{\gamma}{2}\int_0^T \Big[\Big(\frac{\alpha+1}{2\alpha^2}\Big)\Big(\frac{\mu-r}{\sigma}\Big)^2 + a_1(s)\Big]\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) \\ &+ \Big[\Big(\frac{\alpha-1}{2\alpha^2}\Big)\Big(\frac{\mu-r}{\sigma}\Big)^2 - a_2(s)\Big]\frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\mathrm{d}s. \end{aligned} \tag{11}$$

Let (X,π,Z) be a solution of the FBSDE (8), then there must be that

$$X^{x,\pi}(T) = \frac{\gamma}{2}\big[Z^{-1/\alpha}(T) - Z^{1/\alpha}(T)\big] + \kappa.$$

Comparing the $\mathrm{d}W(t)$-terms in (11) and (5), it is reasonable to conjecture that

$$\frac{\gamma(\mu-r)}{2\sigma\alpha}\Big[\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) + \frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\Big] = \mathrm{e}^{-r(s-T)}(\sigma\pi(s) + \beta),$$

i.e.,

$$\pi(s) = \frac{\gamma(\mu-r)}{2\sigma^2\alpha}\mathrm{e}^{r(s-T)}\Big[\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) + \frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\Big] - \frac{\beta}{\sigma}. \tag{12}$$

After substituting (12) into (5), we have

$$\begin{aligned}X^{x,\pi}(T) = x^* &+ \frac{\gamma(\mu-r)^2}{2\sigma^2\alpha}\int_0^T \Big[\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) + \frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\Big]\mathrm{d}s \\ &+ \frac{\gamma(\mu-r)}{2\sigma\alpha}\int_0^T \Big[\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) + \frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\Big]\mathrm{d}W(s),\end{aligned}$$

where

$$x^* = x + \Big[x + \frac{\lambda}{r} - \frac{(\mu-r)\beta}{r\sigma}\Big](\mathrm{e}^{rT}-1). \tag{13}$$

Then

$$\begin{aligned}&\frac{\gamma}{2}\Big[Z^{-1/\alpha}(T) - Z^{1/\alpha}(T)\Big] + \kappa \\ &= X^{x,\pi}(T) + \frac{\gamma}{2}\Big[\frac{Z^{-1/\alpha}(0)}{A_1(T)} - \frac{Z^{1/\alpha}(0)}{A_2(T)}\Big] + \kappa - x^* \\ &\quad + \int_0^T \frac{\gamma}{2}\Big[\Big(\frac{\alpha+1}{2\alpha^2}\Big)\Big(\frac{\mu-r}{\sigma}\Big)^2 + a_1(s)\Big]\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) \\ &\quad + \frac{\gamma}{2}\Big[\Big(\frac{\alpha-1}{2\alpha^2}\Big)\Big(\frac{\mu-r}{\sigma}\Big)^2 - a_2(s)\Big]\frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\mathrm{d}s \\ &\quad - \frac{\gamma(\mu-r)^2}{2\sigma^2\alpha}\int_0^T \Big[\frac{A_1(s)}{A_1(T)}Z^{-1/\alpha}(s) + \frac{A_2(s)}{A_2(T)}Z^{1/\alpha}(s)\Big]\mathrm{d}s.\end{aligned} \tag{14}$$

If we take

$$\begin{cases} a_1(t) = \dfrac{\alpha-1}{2\alpha^2}\Big(\dfrac{\mu-r}{\sigma}\Big)^2, \\ a_2(t) = -\dfrac{\alpha+1}{2\alpha^2}\Big(\dfrac{\mu-r}{\sigma}\Big)^2 \end{cases}$$

and

$$Z(0) = \mathrm{e}^{-[(\mu-r)/\sigma]^2T/2}\exp\Big[-\alpha\,\mathrm{arcsinh}\Big(\frac{x^*-\kappa}{c\gamma}\Big)\Big],$$

where

$$c = \exp\Big[\frac{(\mu-r)^2}{2\sigma^2\alpha^2}T\Big], \tag{15}$$

we have that

$$\frac{\gamma}{2}\big[Z^{-1/\alpha}(T) - Z^{1/\alpha}(T)\big] + \kappa = X^{x,\pi}(T),$$

i.e.,

$$Z(T) = \exp\Big[-\alpha\,\mathrm{arcsinh}\Big(\frac{X^{x,\pi}(T)-\kappa}{\gamma}\Big)\Big].$$

Step 2: Now we verify the conjectures in Step 1.

Let

$$Z^*(0) = \mathrm{e}^{-[(\mu-r)/\sigma]^2T/2}\exp\Big[-\alpha\,\mathrm{arcsinh}\Big(\frac{x^*-\kappa}{c\gamma}\Big)\Big], \tag{16}$$

then the SDE

$$\mathrm{d}Z(t) = -\frac{\mu - r}{\sigma} Z(t)\mathrm{d}W(t)$$

has a solution

$$Z^*(t) = Z^*(0)\exp\Big[-\frac{\mu - r}{\sigma}W(t) - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 t\Big]. \tag{17}$$

Furthermore, let

$$\begin{aligned} \pi^*(t) &= \frac{\gamma(\mu - r)}{2\sigma^2\alpha}\mathrm{e}^{r(t-T)}\big\{\mathrm{e}^{[(\alpha-1)/(2\alpha^2)][(\mu-r)/\sigma]^2(t-T)}[Z^*(t)]^{-1/\alpha} \\ &\quad + \mathrm{e}^{-[(\alpha+1)/(2\alpha^2)][(\mu-r)/\sigma]^2(t-T)}[Z^*(t)]^{1/\alpha}\big\} - \frac{\beta}{\sigma} \\ &= \frac{\gamma(\mu - r)}{\sigma^2\alpha}\mathrm{e}^{\{r-[1/(2\alpha^2)][(\mu-r)/\sigma]^2\}(t-T)} \\ &\quad \times \cosh\Big(-\frac{1}{\alpha}\ln Z^*(t) + \frac{1}{2\alpha}\Big(\frac{\mu - r}{\sigma}\Big)^2(t-T)\Big) - \frac{\beta}{\sigma}. \end{aligned} \tag{18}$$

Then, by a procedure similar to the one used in (14), it is easy to show that

$$X^{x,\pi^*}(T) = \frac{\gamma}{2}\big\{[Z^*(T)]^{-1/\alpha} - [Z^*(T)]^{1/\alpha}\big\} + \kappa, \tag{19}$$

i.e.,

$$Z^*(T) = \exp\Big[-\alpha\,\mathrm{arcsinh}\Big(\frac{X^{x,\pi^*(T)} - \kappa}{\gamma}\Big)\Big].$$

Therefore, $(X^{x,\pi^*}, \pi^*, Z^*)$ is a solution of the FBSDE (8). We finally get the following theorem.

Theorem 6 The admissible strategy π^* defined in (18) is the optimal strategy for the SAHARA utility with constant threshold wealth κ.

Example 7 When $\kappa = \lambda = \beta = 0$, our problem degenerates to the one considered in [6]. Substituting (17) into (19), it is easy to check that the optimal terminal wealth given by (19) is equal to the one obtained in [6]. Similarly, replacing $Z^*(t)$ in (18) by the right-hand-side of Equation (17) yields that

$$\pi^*(t) = \frac{\gamma(\mu - r)}{\sigma^2\alpha}\mathrm{e}^{\{r-[1/(2\alpha^2)][(\mu-r)/\sigma]^2\}(t-T)}\cosh(Y(t)),$$

where

$$Y(t) = \frac{(\mu - r)^2}{\sigma^2\alpha}t + \frac{\mu - r}{\alpha\sigma}W(t) + \mathrm{arcsinh}\Big(\frac{x^*}{c\gamma}\Big).$$

On the other hand, using that $\mathrm{arcsinh}(x) = \ln(x + \sqrt{1 + x^2})$, we can rewrite the optimal strategy obtained in Theorem 3.2 of [6] as

$$\frac{\gamma(\mu - r)}{\sigma^2\alpha}\mathrm{e}^{\{r-[1/(2\alpha^2)][(\mu-r)/\sigma]^2\}(t-T)}\big[\mathrm{e}^{Y(t)} - \sinh(Y(t))\big].$$

Noting that $\cosh(x) = \mathrm{e}^x - \sinh(x)$, we get the same optimal investment strategy.

Example 8 When $\kappa = 0$ and $\alpha = \vartheta\gamma$ in (18), letting $\gamma \to \infty$ and noting that $\cosh(0) = 1$, we get that

$$\pi^*(t) = \frac{\mu - r}{\sigma^2 \vartheta} \mathrm{e}^{r(t-T)} - \frac{\beta}{\sigma},$$

which is the optimal investment strategy for the exponential utility functions with constant absolute risk aversion ϑ. This is consistent with the definition of $f_t^*(x)$ given on Page 954 of [1] with $\rho = 1$. Furthermore, if the interest rate $r = 0$, the above equation becomes the Equation (9) in [1] with $\rho = 1$.

Example 9 When $\kappa = 0$ and $\gamma \to 0$, the SAHARA utility function becomes the power utility function with absolute risk aversion α/x, where $\alpha \in (0, 1)$ and $x \geqslant 0$. In this case, from (18), we know that the optimal investment strategy is $-\beta/\sigma$. It is worth noting that this is consistent with the result given in Section 4 of [1] (with $\rho = 1$), where he considers the strategy in which the ruin probability is minimized. This is reasonable since the power utility function is defined on $[0, \infty)$.

3.2 Dynamically Updated Threshold Wealth

Let $\kappa(t)$ denote the threshold wealth of the insurer at time t. Similar to [4] (see also, [11] and [12]), we assume that the insurer continuously adjusts his initial threshold wealth κ with the constant riskless rate r, weighted by $(1 - \rho)$, and with the change of his wealth $\mathrm{d}X^{x,\pi}(t)$, weighted by ρ. Hence, starting from time $t = 0$, the insurer's threshold wealth evolves dynamically according to the following process:

$$\mathrm{d}\kappa(t) = (1 - \rho)\kappa r \mathrm{d}t + \rho \mathrm{d}X^{x,\pi}(t), \qquad 0 \leqslant t \leqslant T, \tag{20}$$

with $0 \leqslant \rho < 1$.

Note that the threshold wealth is a nondecreasing function of the investors current wealth. This is reasonable by recalling that κ measures both gains and losses. For example, an investor with 100 dollars may take 110 as her threshold wealth, but when her wealth increases to 200 dollars, she would like to take another higher threshold wealth.

After integrating (20), we get the threshold wealth $\kappa(T)$ at the maturity time T as follows:

$$\kappa(T) = \kappa + (1 - \rho)\kappa r T + \rho[X^{x,\pi}(T) - x], \tag{21}$$

with $0 \leqslant \rho < 1$.

Therefore, the SAHARA utility of the terminal wealth $X^{x,\pi}(T)$ with a stochastic threshold wealth is given by

$$U(X^{x,\pi}(T))=\begin{cases}\dfrac{1}{1-\alpha^2}\big\{[X^{x,\pi}(T)-\kappa(T)]+\sqrt{\gamma^2+[X^{x,\pi}(T)-\kappa(T)]^2}\big\}^{-\alpha}\\ \quad\times\big\{[X^{x,\pi}(T)-\kappa(T)]+\alpha\sqrt{\gamma^2+[X^{x,\pi}(T)-\kappa(T)]^2}\big\}, \qquad \alpha\neq 1;\\ \dfrac{1}{2}\gamma^{-2}[X^{x,\pi}(T)-\kappa(T)]\big\{\sqrt{\gamma^2+[X^{x,\pi}(T)-\kappa(T)]^2}-[X^{x,\pi}(T)-\kappa(T)]\big\}\\ \quad+\dfrac{1}{2}\ln\big\{[X^{x,\pi}(T)-\kappa(T)]+\sqrt{\gamma^2+[X^{x,\pi}(T)-\kappa(T)]^2}\big\}, \qquad \alpha=1.\end{cases}\tag{22}$$

To solve the utility maximization problem with a stochastic threshold wealth $\kappa(t)$, we first show that it is equivalent to a utility maximization problem with a constant threshold wealth. Substituting (21) into (22), we can rewrite the utility function as follows:

$$U(X^{x,\pi}(T))=\begin{cases}\dfrac{(1-\rho)^{1-\alpha}}{1-\alpha^2}\big\{[X^{x,\pi}(T)-\kappa_T^*(\rho)]+\sqrt{\gamma_\rho^2+[X^{x,\pi}(T)-\kappa_T^*(\rho)]^2}\big\}^{-\alpha}\\ \quad\times\big\{[X^{x,\pi}(T)-\kappa_T^*(\rho)]+\alpha\sqrt{\gamma_\rho^2+[X^{x,\pi}(T)-\kappa_T^*(\rho)]^2}\big\}, \qquad \alpha\neq 1;\\ \dfrac{1}{2}\gamma_\rho^{-2}[X^{x,\pi}(T)-\kappa_T^*(\rho)]\big\{\sqrt{\gamma_\rho^2+[X^{x,\pi}(T)-\kappa_T^*(\rho)]^2}-[X^{x,\pi}(T)-\kappa_T^*(\rho)]\big\}\\ \quad+\dfrac{1}{2}\ln\big\{[X^{x,\pi}(T)-\kappa_T^*(\rho)]+\sqrt{\gamma_\rho^2+[X^{x,\pi}(T)-\kappa_T^*(\rho)]^2}\big\}\\ \quad+\dfrac{1}{2}\ln(1-\rho), \qquad \alpha=1,\end{cases}\tag{23}$$

where $\gamma_\rho=\gamma/(1-\rho)$, and

$$\kappa_T^*(\rho)=\frac{1}{1-\rho}[\kappa+(1-\rho)\kappa rT-\rho x]\tag{24}$$

is a deterministic constant depending on ρ, κ, r, x and T.

Equation (23) shows that it is sufficient to consider the SAHARA utility function $U_\rho(x)$ with scale parameter γ_ρ, risk aversion parameter α and threshold wealth $\kappa_T^*(\rho)$. Furthermore, we have

$$U'_\rho(x)=\gamma_\rho^{-\alpha}(1-\rho)^{1-\alpha}\exp\Big[-\alpha\,\mathrm{arcsinh}\Big(\frac{x-\kappa_T^*(\rho)}{\gamma_\rho}\Big)\Big].$$

From the results of last subsection, we have the following results for the SAHARA utility with stochastic threshold wealth $\kappa(t)$ defined by (20).

Theorem 10 For the SAHARA utility function with stochastic threshold wealth $\kappa(t)$ defined by (20), the optimal strategy π_ρ^* is given by

$$\pi_\rho^*(t)=\frac{\gamma_\rho(\mu-r)}{\sigma^2\alpha}\mathrm{e}^{\{r-[1/(2\alpha^2)][(\mu-r)/\sigma]^2\}(t-T)}\cosh\Big[-\frac{1}{\alpha}\ln Z_\rho^*(t)+\frac{1}{2\alpha}\Big(\frac{\mu-r}{\sigma}\Big)^2(t-T)\Big]-\frac{\beta}{\sigma},$$

where

$$Z_\rho^*(t) = Z_\rho^*(0) \exp\Big[-\frac{\mu - r}{\sigma} W(t) - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 t\Big], \tag{25}$$

and

$$Z_\rho^*(0) = \mathrm{e}^{-[(\mu-r)/\sigma]^2 T/2} \exp\Big[-\alpha \operatorname{arcsinh}\Big(\frac{x^* - \kappa_T^*(\rho)}{c\gamma_\rho}\Big)\Big].$$

The optimal terminal wealth is

$$X_\rho^{x,\pi^*}(T) = \frac{\gamma_\rho}{2}\big\{[Z_\rho^*(T)]^{-1/\alpha} - [Z_\rho^*(T)]^{1/\alpha}\big\} + \kappa_T^*(\rho).$$

Here, x^*, c and $\kappa_T^*(\rho)$ are given by (13), (15) and (24), respectively.

§4. Numerical Illustrations

From the results of the last section, we know that the optimal strategy and the optimal terminal wealth are dependent on $Z^*(t)$. However, the insurer can not observe $Z^*(t)$ directly from the market. From (1), (17) and (25), we have

$$Z^*(t) = Z^*(0) \exp\Big\{-\frac{\mu - r}{\sigma^2}\Big[\ln\frac{S(t)}{S(0)} - \Big(\mu - \frac{1}{2}\sigma^2\Big)t\Big] - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 t\Big\},$$

and

$$Z_\rho^*(t) = Z_\rho^*(0) \exp\Big\{-\frac{\mu - r}{\sigma^2}\Big[\ln\frac{S(t)}{S(0)} - \Big(\mu - \frac{1}{2}\sigma^2\Big)t\Big] - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 t\Big\},$$

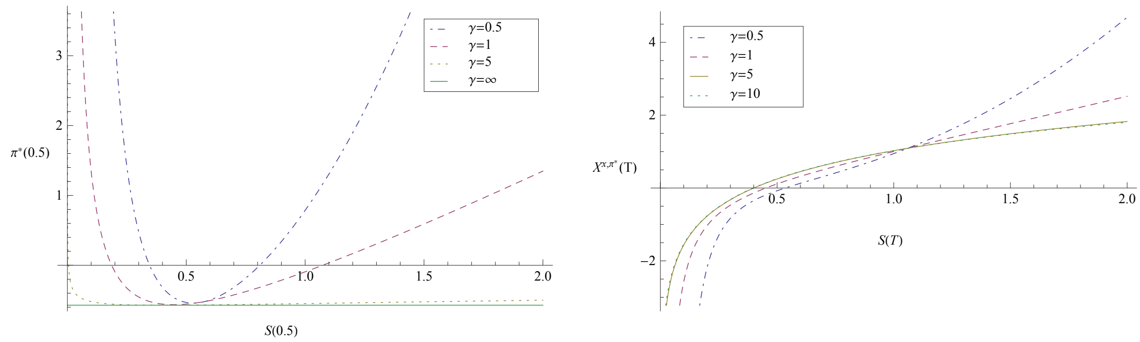
which imply that the insurer can make her decision based on the observations of the stock prices.

For the numerical example, we assume the fixed time horizon $T = 1$. The parameters of the financial market are listed in Table 1.

Table 1 The parameters of financial market

r	μ	σ	λ	β	$X(0)$	$S(0)$
0.03	0.08	0.15	0.1	0.25	1	1

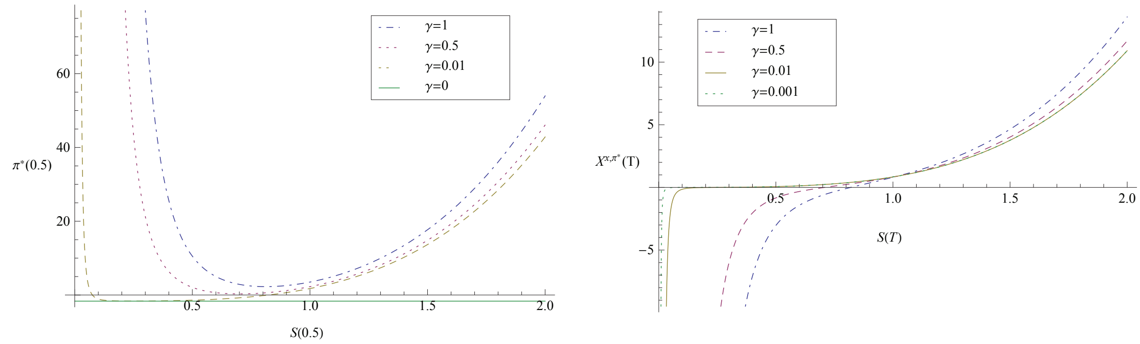
In Figure 1, we set $\kappa = 0$ and $\alpha = 2\gamma$. In sub-figure (a), it illustrates the optimal strategies at time $t = 0.5$, i.e. $\pi^*(0.5)$, versus $S(0.5)$ with γ approaching to infinity. The curve for $\gamma = \infty$ is the optimal strategy under the exponential utility functions with absolute risk aversion $\vartheta = 2$. It is obvious that there exists a threshold price such that an insurer with SAHARA utility invests less money into the risk asset both when the asset price approaches to it from above and below. Furthermore, when $\gamma \to \infty$, this threshold price approaches to the threshold wealth $\kappa = 0$. We also note that the optimal strategy



(a) (b)

Figure 1 The optimal strategies and the optimal terminal wealth when $\gamma \to \infty$

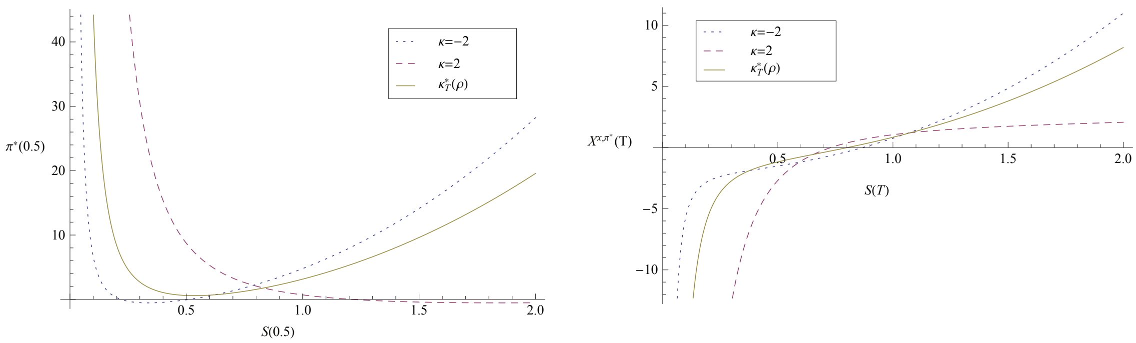
π^* decreases (which means that the SAHARA agent is more risk averse) when γ increases. Sub-figure (b) plots the optimal terminal wealth $X^{x,\pi^*}(T)$ versus the terminal realization of the risky asset $S(T)$ for different values of γ. It shows that the curves for $\gamma = 5$ and $\gamma = 10$ are almost overlapped. Thus, both of these two curves show the optimal terminal wealth in the limiting case, i.e., $\gamma = \infty$. Apparently, there is a positive relation between the optimal terminal wealth $X^{x,\pi^*}(T)$ and $S(T)$ for all the cases. A smaller value of γ (less risk aversion) leads to a larger positive terminal wealth in a flourishing market, also a larger negative final wealth in an unfavorable market.



(a) (b)

Figure 2 The optimal strategies and the optimal terminal wealth when $\gamma \to 0$

In Figure 2, we set $\kappa = 0$ and $\alpha = 0.6$. In sub-figure (a), we show the optimal strategies at time $t = 0.5$ versus $S(0.5)$ with γ approaching to 0. The curve for $\gamma = 0$ is the optimal strategy under the power utility functions with risk aversion function $0.6/x$, $x > 0$. Obviously, the optimal strategy π^* from the power utility always under-performs the SAHARA utility. It also shows that there exists a threshold price such that an insurer with SAHARA utility invests less money into the risk asset both when the asset price approaches to it from above and below. In contrast to Figure 1 (a), the optimal strategy π^* increases (which means that the SAHARA agent is less risk averse) when γ increases. Sub-figure (b) plots the optimal terminal wealth versus $S(T)$ for different values of γ. Similarly, the curves for $\gamma = 0.01$ and $\gamma = 0.001$ are almost overlapped above the horizon axis, and they show the optimal terminal wealth in the limiting case, i.e., $\gamma = 0$. There is also a positive relation between the optimal terminal wealth $X^{x,\pi^*}(T)$ and $S(T)$ for all the cases.



(a) (b)

Figure 3 The optimal strategies and the optimal terminal wealth for different κ

In Figure 3, we set $\alpha = 1$, $\gamma = 0.5$, and $\kappa_T^*(\rho)$ is calculated with $\rho = 0.5$ and initial value 0. Sub-figure (a) plots the optimal strategies versus $S(0.5)$, while sub-figure (b) plots the optimal terminal wealth versus $S(T)$ for both constant and dynamically updated threshold wealth. It shows that for most values of $S(0.5)$ the optimal strategy with dynamically updated threshold wealth (starts form zero) is located between the optimal strategies for a positive threshold $\kappa = 2$ and a negative threshold $\kappa = -2$. This also happens to the optimal terminal wealth. This may imply that comparing with a constant threshold wealth, a dynamically updated threshold wealth is an "average" threshold wealth.

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SAHARA 效用函数下的保险人的最优投资策略

赵 倩 朱绍辉

(上海对外经贸大学统计与信息学院, 上海, 201620)

摘 要: 在本文中, 我们考虑带有 SAHARA 效用函数的保险人的最优投资策略, 目标是最大化其终端财富效用. 这类效用函数拥有非单调的绝对风险厌恶, 比 CARA 和 CRRA 效用函数更加灵活. 在风险过程和股票过程分别由布朗运动和几何布朗运动刻画的情形下, 我们采用鞅方法分别得到了临界值为常数和临界值动态服从一个明确过程的情况下的显示解. 最后, 我们证明最优投资策略是状态独立的.

关键词: 最优投资策略; SAHARA 效用函数; 保险人; 鞅方法

中图分类号: O29; O232; O211.9