应用概率统计　第 36 卷
第 4 期　2020 年 8 月

Chinese Journal of Applied Probability and Statistics
Aug., 2020, Vol. 36, No. 4, pp. 331-341
doi: 10.3969/j.issn.1001-4268.2020.04.001

Limit Theorem and Parameter Estimation for a Critical Branching Process with Mixing Immigration *

LI Doudou　　ZHANG Mei*

(*School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China*)

Abstract: We consider a critical branching process with ψ-mixing immigration and prove a functional limit theorem, improving the results in previous literatures. As applications, we obtain central limit theorems for an estimator of the offspring mean.

Keywords: branching processes; mixing; immigration; limit theorem; parameter estimation

2010 Mathematics Subject Classification: 60J80; 60F10

Citation: LI D D, ZHANG M. Limit theorem and parameter estimation for a critical branching process with mixing immigration [J]. Chinese J Appl Probab Statist, 2020, 36(4): 331–341.

§1. Introduction

Suppose $\{X_{ni}, n, i \geqslant 1\}$ are independent and identically distributed (i.i.d.) non-negative integer-valued random variables. $\{\xi_n, n \geqslant 1\}$ is another sequence of non-negative integer-valued random variables, which is independent of $\{X_{ni}, n, i \geqslant 1\}$. Define $\{Z(n), n \geqslant 0\}$ recursively as

$$Z(n) = \sum_{i=1}^{Z(n-1)} X_{ni} + \xi_n, \quad n \geqslant 1; \qquad Z(0) = 0. \tag{1}$$

Suppose $A := \mathsf{E} X_{ni} = 1$ and $B := \mathsf{Var}\,(X_{ni}) \in (0, \infty)$. Define $\mathsf{E}\,\xi_n = \alpha(n)$, $\mathsf{Var}\,(\xi_n) = \beta(n)$. $\{Z(n), n \geqslant 0\}$ is called a critical Galton-Watson (GW) branching process with immigration.

There have been many research works on the limit theorems of branching processes with immigration. For instance, Wei and Winnicki [1] studied the model (1) where $\{\xi_n\}$ is an i.i.d. sequence with finite variance, and proved a functional limit theorem for $Z([nt])/n$ ($t \geqslant 0$). Ispány et.al. [2] investigated a sequence of nearly critical GW branching processes with immigration, where the offspring variance tends to 0. Rahimov [3] also considered

*The project was supported by the National Natural Science Foundation of China (Grant No. 11871103).

⋆Corresponding author, E-mail: meizhang@bnu.edu.cn.

Received March 21, 2019.

the process defined by (1), where $\{\xi_n\}$ are independent and $\alpha(n) \to \infty$, $\beta(n) \to \infty$. Three different limit theorems were obtained, depending on the relation of $\alpha(n)$ and $\beta(n)$. Furthermore, Guo and Zhang [4] generalized one of the results in [3], by supposing $\{\xi_n\}$ are N-dependent ($N \geqslant 2$) and $\beta(n) = o(n\alpha(n))$.

In this paper, we are interested in the process $\{Z(n),\, n \geqslant 0\}$ defined by (1), where the immigration satisfies the following ψ-mixing condition:

Definition 1 Suppose $\{Y_i\}$ is a sequence of random variables. Let $F_n^m = \sigma(Y_i : n \leqslant i \leqslant m)$, $1 \leqslant n \leqslant m \leqslant \infty$. Define

$$\psi(m) = \sup\Big\{\Big|\frac{\mathsf{P}(AB)}{\mathsf{P}(A)\mathsf{P}(B)} - 1\Big| : A \in F_1^k,\ B \in F_{m+k}^\infty,\ \mathsf{P}(A)\mathsf{P}(B) \neq 0,\ k \geqslant 1\Big\}.$$

The sequence $\{Y_n,\, n \geqslant 1\}$ is said to be ψ-mixing, if $\psi(n) \to 0$ as $n \to \infty$.

We shall base our discussion on the following condition:

(H) The two sequences $\{X_{ni},\, n, i \geqslant 1\}$ and $\{\xi_n,\, n \geqslant 1\}$ are independent with each other. $\{\xi_n,\, n \geqslant 1\}$ is a sequence of ψ-mixing random variables satisfying $\sum\limits_{k=1}^{\infty} \psi(k) < \infty$.

And we always assume

$$\lim_{n\to\infty} \frac{n\alpha(n)}{\beta(n)} = 0. \tag{2}$$

In the paper, under (H) and (2), we shall prove a functional central limit theorem of $Z(n)$, generalizing the results of [3] (independent immigration case) and [4] (N-dependent immigration case). As applications, we obtain some central limit theorems of the estimator $\widehat{A}_n$ of the offspring mean A. We shall see that the limit is normal or the change of a normal variable, according to the relation of $\alpha(n)$ and $\beta(n)$.

To get the proofs, we shall split the normalized process of $Z(n)$ into two parts: reproduction process and immigration process (see (11) and (12)). For the first part, we prove that it converges weakly to 0 by martingale central limit theorem. This proof is inspired by Rahimov [3]. For the second part, in our case, the immigration at different generation are not independent of each other. To deal with it, we use the invariance principle for mixing sequences (see for example, [5–7]). One shall see that under the assumption (2), the immigration variance $\beta(n)$ plays an important role in the proofs. We mention that our technical tool of the proof of main theorem can also be used to deal with the case $\beta(n) = o(n\alpha(n))$, which was discussed in [4] by different decomposition.

The remainder of the paper is organized as follows. The main result is given in Section 2. Section 3 contains some preliminary lemmas and their proofs. Section 4 is devoted to the proof of the main theorem. Finally, applications in parameter estimate are given in Section 5.

In what follows, we write $a_n \sim b_n$ if and only if $\lim\limits_{n\to\infty} a_n/b_n = 1$; $a_n = o(b_n)$ if and only if $\lim\limits_{n\to\infty} a_n/b_n = 0$. '$\xrightarrow{D}$', '$\xrightarrow{d}$' and '$\xrightarrow{\mathsf{P}}$' denote the convergence of random functions in Skorokhod space, the convergence of random variables in distribution, and convergence in probability, respectively.

§2. Functional Limit Theorem

For each $k \geqslant 0$, let $\mathscr{F}(k) = \sigma\{Z(i),\, 1 \leqslant i \leqslant k\}$ and $\mathscr{F}_m^n = \sigma\{Z(i),\, m \leqslant i \leqslant n\}$. For each $n \geqslant 1$, suppose $\alpha(n)$ and $\beta(n)$ are regularly varying functions as $n \to \infty$, i.e.

$$\alpha(n) = n^{\alpha} L_{\alpha}(n), \qquad \beta(n) = n^{\beta} L_{\beta}(n), \tag{3}$$

where $\alpha, \beta \geqslant 0$, and $L_{\alpha}(n)$ and $L_{\beta}(n)$ are slowly varying functions as $n \to \infty$. Denote

$$A(n) = \mathsf{E}[Z(n)], \qquad B^2(n) = \mathsf{Var}\,[Z(n)].$$

By (1), for $n \geqslant 1$,

$$A(n) = \mathsf{E}\Big(\sum_{i=1}^{Z(n-1)} X_{ni}\Big) + \mathsf{E}\,\xi_n = \mathsf{E}[Z(n-1)] + \alpha(n) = \cdots = \sum_{k=1}^{n} \alpha(k),$$

and

$$B^2(n) = \Delta^2(n) + \sigma^2(n) + 2\sum_{i=1}^{n-1}\sum_{k=i+1}^{n} \mathsf{Cov}\,(\xi_i, \xi_k) := \Delta^2(n) + \widetilde{\sigma}^2(n),$$

where

$$\Delta^2(n) = B\sum_{k=1}^{n-1} A(k), \qquad \sigma^2(n) = \sum_{k=1}^{n} \beta(k). \tag{4}$$

For $t \geqslant 0$, define

$$Y_n(t) = \frac{Z([nt]) - A([nt])}{B(n)}, \qquad n \geqslant 1, \tag{5}$$

and for each $\varepsilon > 0$, define

$$\delta_n(\varepsilon) = \frac{1}{\sigma^2(n)} \sum_{k=1}^{n} \mathsf{E}\{|\xi_k - \alpha(k)|^2;\ |\xi_k - \alpha(k)| > \varepsilon\sigma(n)\},$$

where $[nt]$ is the largest integer before nt.

Moreover, we always assume

$$\lim_{n\to\infty} \frac{\sigma(n)}{\widetilde{\sigma}(n)} = c^* \tag{6}$$

holds for some positive constant c^*. The following is our main theorem of this paper.

Theorem 2 Suppose condition (H), (2) and (6) hold. If $\delta_n(\varepsilon) \to 0$ for any $\varepsilon > 0$, then

$$Y_n(t) \xrightarrow{D} W(t^{1+\beta}), \qquad n \to \infty$$

in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$, where $W(t)$ is the standard Brownian motion.

§3. Preliminary Results

Lemma 3 ([8; Lemma 1.2.11]) Let $\{X_n,\, n \geqslant 1\}$ be a sequence of ψ-mixing random variables. Let $X \in \mathscr{F}_1^k$, $Y \in \mathscr{F}_{k+n}^\infty$, $\mathsf{E}|X| < \infty$, $\mathsf{E}|Y| < \infty$. Then, $\mathsf{E}|XY| < \infty$ and

$$|\mathsf{E}XY - \mathsf{E}X\mathsf{E}Y| \leqslant \psi(n)\mathsf{E}|X|\mathsf{E}|Y|.$$

Remark 4 Consider the immigration sequence $\{\xi_n\}$. Using Lemma 3, we get

$$\begin{aligned}
\widetilde{\sigma}^2(n) &= \mathsf{Var}\,\{[\xi_1 - \alpha(1)] + [\xi_2 - \alpha(2)] + \cdots + [\xi_n - \alpha(n)]\} \\
&= \sum_{k=1}^n \mathsf{Var}\,(\xi_k) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n \mathsf{Cov}\,(\xi_i - \alpha(i), \xi_j - \alpha(j)) \\
&\leqslant \sum_{k=1}^n \mathsf{Var}\,(\xi_k) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n \psi(j-i)\mathsf{E}|\xi_i - \alpha(i)|\mathsf{E}|\xi_j - \alpha(j)| \\
&\leqslant \sum_{k=1}^n \mathsf{Var}\,(\xi_k) + \Big[2\sum_{k=1}^\infty \psi(k)\Big]\sum_{i=1}^n \mathsf{E}[\xi_i - \alpha(i)]^2 \\
&\leqslant \Big[1 + 2\sum_{k=1}^\infty \psi(k)\Big]\sigma^2(n).
\end{aligned}$$

Hence the assumption (6) is reasonable. Particularly, if the sequence $\{\xi_n,\, n \geqslant 1\}$ is positively associated, then (6) holds.

Lemma 5 ([3; Lemma 3]) Assume $\Delta^2(n)$ and $\sigma^2(n)$ are defined by (4). As $n \to \infty$,

(i)
$$\Delta^2(n) \sim \frac{Bn^2\alpha(n)}{(\alpha+1)(\alpha+2)}, \qquad \sigma^2(n) \sim \frac{n\beta(n)}{\beta+1};$$

(ii) For $\gamma \geqslant 0$,
$$\sum_{k=1}^n A^\gamma(k) \sim \frac{nA^\gamma(n)}{\gamma\alpha+\gamma+1}, \qquad A(n) \sim \frac{n\alpha(n)}{1+\alpha}.$$

Applying Lemma 5, (2) and (6), as $n \to \infty$,

$$\widetilde{\sigma}(n) \sim B(n), \qquad \sigma(n) \sim c^* B(n). \tag{7}$$

Lemma 6 If $\alpha(n) \to \infty$ as $n \to \infty$, then

(i) $$B^{-4}(n)\mathsf{Var}\Big[\sum_{k=1}^{[nt]} Z(k)\Big] \to 0;$$

(ii) $$B^{-4}(n)\sum_{k=1}^{[nt]} \mathsf{E}Z^2(k) \to 0.$$

Proof For part (i), by Hölder's inequality, the upper bound

$$\begin{aligned} B^{-4}(n)\mathsf{Var}\Big[\sum_{k=1}^{[nt]} Z(k)\Big] &= B^{-4}(n)\sum_{k=1}^{[nt]} B^2(k) + 2B^{-4}(n)\sum_{i=1}^{[nt]-1}\sum_{j=i+1}^{[nt]} \mathsf{Cov}\,(Z(i), Z(j)) \\ &\leqslant B^{-4}(n)\sum_{k=1}^{[nt]} B^2(k) + 2B^{-4}(n)\sum_{i=1}^{[nt]} B(i)\sum_{j=1}^{[nt]} B(j) \\ &\leqslant \frac{C_t n^2 B^2(n)}{B^4(n)} \to 0 \end{aligned}$$

holds as $n \to \infty$, where C_t is a positive constant depending only on t.

For part (ii), in fact,

$$\mathsf{E}Z^2(k) = B^2(k) + A^2(k).$$

Using Lemma 5, we complete the proof. □

For $n \geqslant 1$, let

$$M(n) = Z(n) - Z(n-1) - \alpha(n), \tag{8}$$

and for $k \geqslant 1$,

$$T(k) = \sum_{i=1}^{Z(k-1)} (X_{ki} - 1), \tag{9}$$

Here we denote $\sum\limits_{i=1}^{0} = 0$. Then

$$M(n) = T(n) + \xi_n - \alpha(n). \tag{10}$$

By (1), (8) and (10), $Y_n(t)$ given by (5) can be written as

$$Y_n(t) = B^{-1}(n)\sum_{k=1}^{[nt]} M(k) = Y_n^{(1)}(t) + Y_n^{(2)}(t), \tag{11}$$

where

$$Y_n^{(1)}(t) = B^{-1}(n)\sum_{k=1}^{[nt]} T(k), \qquad Y_n^{(2)}(t) = B^{-1}(n)\sum_{k=1}^{[nt]} [\xi_k - \alpha(k)]. \tag{12}$$

Next, we consider the property of $T(k)$ defined by (9).

Lemma 7 For $i \geqslant 1$,

$$\mathsf{E}[T(i)T(j)] = \begin{cases} BA(i-1), & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof Considering the independence of reproduction process and immigration, we know that

$$\mathsf{E}[T(k) \,|\, \mathscr{F}(k-1)] = Z(k-1)\mathsf{E}(X_{11}-1) = 0, \tag{13}$$

and

$$\mathsf{E}[T^2(k) \,|\, \mathscr{F}(k-1)] = Z(k-1)\mathsf{E}(X_{11}-1)^2 = BZ(k-1).$$

Using the property of conditional expectation, we get

$$\mathsf{E}[T(i)T(j)] = \mathsf{E}\{T(i)\,\mathsf{E}[T(j) \,|\, \mathscr{F}(j-1)]\} = 0$$

for $j > i$, and

$$\mathsf{E}[T^2(i)] = \mathsf{E}\{\mathsf{E}[T^2(i) \,|\, \mathscr{F}(i-1)]\} = BA(i-1). \qquad \square$$

For $\varepsilon > 0$ and $t \geqslant 0$, let

$$I(n) = B^{-2}(n) \sum_{k=1}^{[nt]} \mathsf{E}[T^2(k)\mathbb{1}_{\{|T(k)|>\varepsilon B(n)\}} \,|\, \mathscr{F}(k-1)].$$

Lemma 8 Assume (2) and (6) hold. Then

$$I(n) \xrightarrow{\mathsf{P}} 0, \qquad n \to \infty. \tag{14}$$

Proof Obviously,

$$I(n) \leqslant \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} \mathsf{E}[T^2(k) \,|\, \mathscr{F}(k-1)].$$

By Lemma 7, Lemma 5 and noticing that $n\alpha(n) = o(\beta(n))$, as $n \to \infty$,

$$\begin{aligned} \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} \mathsf{E}[(T(k))^2] &= \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} BA(k-1) \\ &\sim \frac{B}{\alpha+2} \cdot \frac{[nt]A([nt])}{\Delta^2(n) + \widetilde{\sigma}^2(n)} \\ &\to 0, \end{aligned}$$

hence

$$\frac{1}{B^2(n)} \sum_{k=1}^{[nt]} \mathsf{E}[(T(k))^2 \,|\, \mathscr{F}(k-1)] \xrightarrow{\mathsf{P}} 0. \tag{15}$$

Then, we obtain the result. $\square$

Lemma 9 ([9] and [3]) Let $\{U_k^n,\, k \geqslant 1\}$ for each $n \geqslant 1$ be a sequence of martingale differences with respect to some filtration $\{\widetilde{\mathscr{F}}_k^n,\, k \geqslant 1\}$, such that the conditional Lindeberg condition

$$\sum_{k=1}^{[nt]} \mathsf{E}[(U_k^n)^2 \mathbb{1}_{\{|U_k^n| > \varepsilon\}} \,|\, \widetilde{\mathscr{F}}_{k-1}^n] \xrightarrow{\mathsf{P}} 0$$

holds as $n \to \infty$ for all $\varepsilon > 0$ and $t \in \mathbb{R}_+$. Then

$$\sum_{k=1}^{[nt]} U_k^n \xrightarrow{D} U(t)$$

in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$, where $U(t)$ is a continuous Gaussian martingale with mean 0 and covariance function $C(t)$, $t \in \mathbb{R}_+$, if and only if

$$\sum_{k=1}^{[nt]} \mathsf{E}[(U_k^n)^2 \,|\, \widetilde{\mathscr{F}}_{k-1}^n] \xrightarrow{\mathsf{P}} C(t)$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$.

Lemma 10 ([6; Theorem 1]) Let $\{X_k,\, k \geqslant 1\}$ be a centered ψ-mixing sequence of random variables having finite second moments. Suppose $\{k_n,\, n \geqslant 0\}$ satisfies

$$0 = k_0 < k_1 < \cdots, \qquad \lim_{n\to\infty} \max_{1 \leqslant i \leqslant n} (k_i - k_{i-1})/k_n = 0, \tag{16}$$

and

$$s_n^2 = k_n h(k_n),$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a slowly varying function, and

$$S_n = \sum_{k=1}^{n} X_k, \qquad s_n^2 = \mathsf{E} S_n^2.$$

Define

$$m_n(t) = \max\{i \geqslant 0 : k_i \leqslant t k_n\}, \qquad W_n(t) = S_{m_n(t)}/s_n, \qquad t \in [0,1].$$

If $\{X_k,\, k \geqslant 1\}$ satisfies

$$\lim_{n\to\infty} s_n^{-2} \sum_{k=1}^{n} \mathsf{E}[X_k^2 \mathbb{1}_{\{|X_k| > \varepsilon s_n\}}] = 0 \quad \text{for any } \varepsilon > 0 \tag{17}$$

and

$$\lim_{n\to\infty} s_n^{-1} \Big(\max_{1 \leqslant k \leqslant n} \mathsf{E}|X_k| \Big) \sum_{k=1}^{n} \psi(k) = 0, \tag{18}$$

then

$$W_n(t) \xrightarrow{D} W(t) \quad \text{in } D[0,1], \qquad n \to \infty.$$

Remark 11 If $k_n \uparrow \infty$ as $n \to \infty$, then condition (16) is equivalent to $\lim\limits_{n\to\infty} k_{n+1}/k_n = 1$.

§4. Proofs of Main Theorem

To prove Theorem 2, we need the following propositions.

Proposition 12 Suppose condition (H), (2) and (6) hold. Then

$$Y_n^{(1)}(t) \xrightarrow{D} 0, \qquad n \to \infty$$

in $D(\mathbb{R}_+, \mathbb{R})$.

Proof First, by (13), we know that $\{T(k), k \geqslant 1\}$ is a sequence of martingale differences with respect to $\{\mathscr{F}(k), k \geqslant 1\}$. Let $U_k^n = T(k)/B(n)$ and $\widetilde{\mathscr{F}}_k^n = \mathscr{F}(k)$ in Lemma 9. Combing with (14) and (15), we complete the proof. □

Proposition 13 Suppose condition (H), (2) and (6) hold. If $\delta_n(\varepsilon) \to 0$ for any $\varepsilon > 0$, then

$$Y_n^{(2)}(t) \xrightarrow{D} W(t^{1+\beta}), \qquad n \to \infty$$

in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$, where $\{W(t), t \in \mathbb{R}_+\}$ is the standard Brownian motion.

Proof Let $X_k = \xi_k - \alpha(k)$ in Lemma 10. Then

$$s_n = \widetilde{\sigma}(n) \sim B(n), \qquad n \to \infty. \tag{19}$$

From Lindeberg condition, we know (17) is satisfied. Moreover, by Hölder's inequality, $\mathsf{E}|X_k| \leqslant \sqrt{\mathsf{E}|X_k|^2} = \sqrt{\beta(k)}$, we then have

$$s_n^{-1}\Big(\max_{1\leqslant k\leqslant n} \mathsf{E}|X_k|\Big) \sum_{k=1}^{n} \psi(k) \leqslant \sum_{k=1}^{\infty} \psi(k) \frac{\sqrt{\beta(n)}}{\widetilde{\sigma}(n)}.$$

Recalling (6) and Lemma 5 (i), we get (18). So, conditions of Lemma 10 are satisfied. Let $k_n = n^{1+\beta}$, then $m_n(t) = [nt^{1/(1+\beta)}]$. Thus, we obtain

$$W_n(t) = \frac{S_{[nt^{1/(1+\beta)}]}}{s_n} \xrightarrow{D} W(t) \tag{20}$$

as $n \to \infty$. Setting $u = t^{1/(1+\beta)}$ in (20), we get

$$S_{[nu]}/s_n \xrightarrow{D} W(u^{1+\beta}).$$

Together with (19), we end the proof. □

Proof of Theorem 2 Combining Proposition 12 with Proposition 13, we complete the proof of Theorem 2. □

§5. Applications

Rahimov[10] obtained limit distributions of the conditional least-squares estimator (CLSE) $\widehat{A}_n$ of the offspring mean A ($A = 1$) with independent immigration. In the present paper, $\{\xi_n, n \geqslant 1\}$ are not mutually independent, as a result, the estimator

$$\widehat{A}_n = \frac{\sum\limits_{k=1}^{n} [Z(k) - \alpha(k)]Z(k-1)}{\sum\limits_{k=1}^{n} Z^2(k-1)} \tag{21}$$

is not the CLSE of A any more. However, under condition (H) and some moment conditions, the central limit theorems of $\widehat{A}_n$ still hold. The results are classified into three cases as follows:

$$n\alpha(n) = o(\beta(n)) \quad \text{and} \quad \beta(n) = o(n\alpha^2(n)), \qquad n \to \infty; \tag{22}$$

$$n\alpha^2(n) = o(\beta(n)), \qquad n \to \infty; \tag{23}$$

$$n\alpha^2(n) \sim d_0\beta(n), \qquad d_0 \in (0, \infty), \ \ n \to \infty. \tag{24}$$

Theorem 14 Assume (22). If $\delta_n(\varepsilon) \to 0$ for any $\varepsilon > 0$, then

$$\frac{n^{3/2}\alpha(n)}{\sqrt{\beta(n)}}(\widehat{A}_n - 1) \xrightarrow{d} N(0, a^2), \qquad n \to \infty,$$

where $N(0, a^2)$ is a normal random variable with mean 0 and variance

$$a^2 = \frac{(1+\alpha)^2(2\alpha+3)^2}{(2\alpha+\beta+3)(c^*)^2}.$$

Theorem 15 Suppose (23) and there exists $0 \leqslant \delta < 1$ such that $\mathsf{E}(\xi_k^4) \leqslant k^{\delta+2\beta}$ ($k \geqslant 1$). If $\delta_n(\varepsilon) \to 0$ for any $\varepsilon > 0$, then

$$n(\widehat{A}_n - 1) \xrightarrow{d} \frac{W^2(1) - (c^*)^2}{2\displaystyle\int_0^1 W^2(t^{1+\beta})\mathrm{d}t}, \qquad n \to \infty.$$

Theorem 16 Suppose (24) and there exists $0 \leqslant \delta < 1$ such that $\mathsf{E}(\xi_k^4) \leqslant k^{\delta+2\beta}$ ($k \geqslant 1$). If $\delta_n(\varepsilon) \to 0$ for any $\varepsilon > 0$, then

$$n(\widehat{A}_n - 1) \xrightarrow{d} \frac{2^{-1}[W^2(1) - (c^*)^2] + c_0\eta}{c_0^2/(2\alpha+3) + \zeta}, \qquad n \to \infty,$$

where

$$c_0 = \frac{c^*\sqrt{d_0(1+\beta)}}{1+\alpha}, \qquad \eta = (1+\alpha)\int_0^1 [W(1) - W(t^{1+\beta})]t^{\alpha}\mathrm{d}t,$$

$$\zeta = 2c_0\int_0^1 W(t^{1+\beta})t^{1+\alpha}\mathrm{d}t + \int_0^1 W^2(t^{1+\beta})\mathrm{d}t.$$

Remark 17 The proofs of Theorems 14 – 16 are parallel to [10; Theorems 2 – 4]. We only explain the differences here. Our key step is to prove that as $n\to\infty$,

$$\frac{1}{B^2(n)}\sum_{k=1}^{n}M^2(k)\xrightarrow{\mathsf{P}}(c^*)^2. \tag{25}$$

Using Lemma 5 and (2), we derive

$$\frac{1}{B^2(n)}\sum_{k=1}^{n}T^2(k)\xrightarrow{\mathsf{P}}0,\qquad \frac{1}{B^2(n)}\sum_{k=1}^{n}T(k)(\xi_k-\alpha(k))\xrightarrow{\mathsf{P}}0.$$

Then, (25) is equivalent to

$$\frac{1}{B^2(n)}\sum_{k=1}^{n}(\xi_k-\alpha(k))^2\xrightarrow{\mathsf{P}}(c^*)^2. \tag{26}$$

By (4) and (6),

$$\mathsf{E}\Big\{\frac{1}{B^2(n)}\sum_{k=1}^{n}[\xi_k-\alpha(k)]^2\Big\}\to(c^*)^2. \tag{27}$$

Using Lemma 3, as $n\to\infty$,

$$\begin{aligned}&\mathsf{Var}\Big\{\frac{1}{B^2(n)}\sum_{k=1}^{n}[\xi_k-\alpha(k)]^2\Big\}\\ &\leqslant B^{-4}(n)\Big\{\sum_{k=1}^{n}\mathsf{Var}\{[\xi_k-\alpha(k)]^2\}+\Big[2\sum_{k=1}^{\infty}\psi(k)\Big]\sum_{i=1}^{n}\mathsf{E}[\xi_i-\alpha(i)]^4\Big\}\\ &=B^{-4}(n)\Big\{\Big[1+2\sum_{k=1}^{\infty}\psi(k)\Big]\sum_{i=1}^{n}\mathsf{E}[\xi_i-\alpha(i)]^4\Big\}-B^{-4}(n)\sum_{k=1}^{n}\beta^2(k)\\ &\leqslant\frac{C}{B^4(n)}\sum_{k=1}^{n}\mathsf{E}(\xi_k^4)\\ &\to 0.\end{aligned} \tag{28}$$

Therefore, (26) follows from (27) and (28).

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带相依移民的临界分枝过程的极限定理和参数估计

李豆豆　　张　梅

(北京师范大学数学科学学院, 北京, 100875)

摘　要: 我们考虑了一个临界带相依移民的分枝过程, 并证明了它的泛函极限定理, 推广了以往文献的结论. 作为应用, 我们得到了后代均值估计的中心极限定理.

关键词: 分枝过程; 相依; 移民; 极限定理; 参数估计

中图分类号: O211.62