应用概率统计　第 36 卷
第 5 期　2020 年 10 月

Chinese Journal of Applied Probability and Statistics
Oct., 2020, Vol. 36, No. 5, pp. 536-550
doi: 10.3969/j.issn.1001-4268.2020.05.008

Asset Allocation and Reinsurance Policy for a Mean-Variance-CVaR Insurer in Continuous-Time *

ZHAO Xia　　SHI Yu⋆

(*School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai, 201620, China*)

Abstract: This paper studies the optimal asset allocation and reinsurance problem under mean-variance-CVaR criteria for an insurer in continuous-time. We obtain the closed-form solution of optimization problem by using martingale method. Numerical results show the trends of optimal wealth, investment and reinsurance strategies with various parameter values.

Keywords: asset allocation; reinsurance policy; mean-variance-CVaR criteria; martingale method

2010 Mathematics Subject Classification: 91B30; 91G10; 60G44

Citation: ZHAO X, SHI Y. Asset allocation and reinsurance policy for a mean-variance-CVaR insurer in continuous-time [J]. Chinese J Appl Probab Statist, 2020, 36(5): 536–550.

§1. Introduction

Asset allocation and reinsurance business are two important issues in insurance industry. Reinsurance is an effective way to diversify risk for large losses while asset allocation is an increasing requirement to achieve its management objective. Therefore, optimization problems with various objectives have attracted much attention in actuarial science and risk management in recent decades. These objectives mainly include the utility of wealth, return and risk of the strategies. For example, in terms of minimizing the ruin probability, Promislow and Young [1] consider the optimal investment-reinsurance problems for an insurer and Meng et al. [2] study an optimal reinsurance problem in which insurance risk is partially transferred to two reinsurers. Yang and Zhang [3] focus on seeking the optimal strategies for an insurer with jump-diffusion risk process in order to maximize the expected exponential utility from terminal wealth. Zhao et al. [4] consider maximizing the expected discounted exponential utility of the consumption and terminal wealth of

*The project was partially supported by the National Natural Science Foundation of China (Grant Nos. 71671104; 11971301) and the Project of Humanities Social Sciences of Research of the Ministry of Education, China (Grant No. 16YJA910003).

⋆Corresponding author, E-mail: shiyudl@foxmail.com.

Received June 24, 2019. Revised August 19, 2019.

an insurer. Measuring the risk by Capital-at-Risk (CaR), Zeng and Li [5] investigate an optimal mean-CaR investment-reinsurance problem. Sun and Guo [6] consider an optimal investment-reinsurance problem for a mean-variance insurer with stochastic volatility. Among all optimization criteria, mean-risk models have obtained more and more attraction since it can control the return and risk directly.

As a typical mean-risk model, mean-variance (MV) model pioneered by Markowitz [7] has long been recognized as the milestone of modern portfolio theory and has stimulated numerous extensions. Bai and Zhang [8] solve an optimal investment-reinsurance problem for a mean-variance insurer in a classical risk model and a diffusion model. Zeng and Li [9] find the optimal time-consistent strategies of an investment-reinsurance problem and an investment-only problem for mean-variance insurers respectively. Considering time-consistent mean-variance portfolio selection with only risky assets, Pun [10] obtains the exact analytical solution in a continuous-time setting. In a non-Markovian regime-switching framework, Wang and Wei [11] obtain the optimal investment policy based on mean-variance criteria. For more literatures on this topic, one may refer to Kolm et al. [12] who review the development, challenges and trends of MV optimization problems.

However, variance, as a risk measure, can only limit the volatility around the expected return, but ignore the loss that occurs in the worst-case scenario. To measure the risk concerned with the left tails of distributions, some risk measures such as expectile, Value-at-Risk (VaR), CaR and Conditional Value-at-Risk (CVaR) are introduced into the study of optimal strategy in finance and insurance management, see [13–18] and the reference therein. Additionally, CVaR has attracted more attention and widely accepted since it has excellent theoretical properties consistent with practice. As we know, the computation of a risk measure is central in all problems related to risk measure and risk management. In the above literature, normal assumption on asset or return is generally supposed, and hence VaR, CaR and CVaR can be written as a linear combination of mean and variance in a closed analytical form, which can be incorporated into optimization problem easily. Under non-normal situation, their computation is still a problem till the year of 2000. Rockafellar and Uryasev [19, 20] propose a minimization formulation in which the value of CVaR can be found through a convex optimization problem. These desirable aspects of CVaR have paved the way of its use in risk management and non-normal/nonlinear portfolio optimization, see [21] and the references therein.

Different risk measures emphasize different aspects of random loss. Variance measures the deviation of the random variable from the expected value while CVaR is an average of losses over a certain threshold (VaR). The optimal strategy generated from mean-CVaR portfolio model could induce a very large variance while the CVaR of the portfolio generating from the traditional MV model could be also unacceptably large. To balance the portfolio policies generated from the MV and mean-CVaR models, Roman et al. [22] propose a static multi-objective optimization model on the basis of three statistics: expected value, variance and CVaR. Li et al. [23] study investment policy of China sovereign wealth fund based on mean-variance-CVaR model, and Younes et al. [24] devote to solve this model via a linear weighted sum method. Assuming the market coefficients are deterministic, Gao et al. [25] expand the above static mean-variance-CVaR model to dynamic portfolio selection and derive the analytical forms of the portfolio policy for mean-variance-CVaR optimization models. They also find that the portfolio policies of mean-variance-CVaR model exhibit a feature of curved V-shape by some illustrative examples.

In this paper, we construct an optimal asset allocation and reinsurance model under mean-variance-CVaR criterion in continuous-time. Specifically, the surplus process of the insurer is described by a diffusion model, which is an approximation of the classical Cramér-Lundberg model. The insurer can invest in a financial market with one risk-free asset and multiple risky assets whose returns follow geometric Brownian motions and purchase proportional reinsurance or acquire new business. We construct a dynamic optimization problem in the sense of minimizing a linear combination of variance and CVaR. We obtain the closed-form expressions of optimal investment and reinsurance strategies and optimal wealth value for the optimization problem by using martingale approach. The main theoretical contribution of this paper is that variance and CVaR are embedded into the dynamic optimal asset allocation and reinsurance problem which is not confined by any distribution assumption.

The remaining of this paper is organized as follows. Section 2 introduces the model and the formulation of the problem. Section 3 shows the optimal strategy of the investment-reinsurance problem under the mean-variance-CVaR criterion by using the martingale method. Section 4 provides a numerical analysis and discusses the trend of the optimal wealth, asset allocation and reinsurance policy in terms of different coefficient assumptions and market conditions. Section 5 concludes the paper.

§2. The Model

Let $T > 0$ be a fixed time horizon. All the randomness is modeled by a complete filtered probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathsf{P}\}$, on which a one-dimensional Brownian motion $W_0(t)$ and an n-dimensional Brownian motion $W(t) = (W_1(t), W_2(t), \cdots, W_n(t))'$ are defined. We assume that $W(t)$ and $W_0(t)$ are independent of each other and denote by $\widetilde{W}$ the $n+1$ dimensional standard Brownian motion $\widetilde{W}(t) = (W_0(t), W_1(t), \cdots, W_n(t))'$.

The insurer's surplus process is described by a diffusion approximation model. By Grandell [26], the classical Cramér-Lundberg model can be approximated by the following diffusion model, which works well for large insurance portfolios:

$$\mathrm{d}R(t) = \mu_0 \mathrm{d}t + \sigma_0 \mathrm{d}W_0(t), \qquad t \geqslant 0,$$

where $\mu_0 > 0$ denotes the premium return rate of insurance business, $\sigma_0 > 0$ measures risk level of insurance business.

To manage risk effectively, the insurer is assumed to purchase proportional reinsurance or acquire new business with the retention level $a(t) \geqslant 0$ for $t \in [0, T]$. For convenience, we call the process of risk exposure $\{a(t)\}_{t\in[0,T]}$ as a reinsurance policy. Specifically, a chosen reinsurance policy $a(t) \in [0, 1]$ denotes the portion of the claims retained by the insurer and shows that the cedent should divert part of the premium to the reinsurer at the rate of $[1 - a(t)]\theta$, where $\theta \geqslant \mu_0$ can be regarded as the premium return rate of the reinsurer; $a(t) > 1$ means that the insurer acquires a new reinsurance business. When $a(t)$ is adopted, the corresponding diffusion approximation dynamics for the surplus process $\{R(t)\}_{t\in[0,T]}$ becomes

$$\mathrm{d}R(t) = \{\mu_0 - [1 - a(t)]\theta\}\mathrm{d}t + \sigma_0 a(t)\mathrm{d}W_0(t).$$

We consider a financial market with one risk-free asset and n risky assets, which can be traded continuously within a time horizon $[0, T]$. The price process of the risk-free asset $S_0(t)$ follows

$$\mathrm{d}S_0(t) = r_0 S_0(t)\mathrm{d}t, \qquad S_0(0) = s_0 > 0,$$

where $r_0 > 0$ is the constant risk-free return rate.

The price process of the i-th risky asset $S_i(t)$ $(i = 1, 2, \cdots, n)$ satisfies the following stochastic differential equation

$$\mathrm{d}S_i(t) = S_i(t)\Big[\mu_i(t)\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}(t)\mathrm{d}W_j(t)\Big], \qquad S_i(0) = s_i > 0,$$

where the mean rate of return $\mu_i(t)$ and the dispersion $\sigma_{ij}(t)$ are positive continuous bounded deterministic functions. Let $\sigma(t) = (\sigma_{ij}(t))_{n\times n}$ satisfy the nondegeneracy condition, i.e., $\sigma(t)\sigma'(t) > \epsilon I_{n\times n}$, for all $t \geqslant 0$ and some $\epsilon > 0$, and $I_{n\times n}$ is the identity matrix.

Suppose that the insurer can dynamically purchase proportional reinsurance/acquire new business and invest in the financial market over the time interval $[0, T]$, and that there is no transaction cost in the financial market and insurance market.

Let $\mathscr{L}^2_{\mathscr{F}}(0, T; \mathbb{R}^n)$ be the set of all $\mathbb{R}^n$-valued, $\mathscr{F}_t$-adapted and square integrable stochastic processes, and $\mathscr{L}^2_{\mathscr{F}_T}(\Omega; \mathbb{R}^n)$ be the set of all $\mathbb{R}^n$-valued, $\mathscr{F}_T$-measurable random variables. Define strategies process as $\pi'(t) = \{(a(t), b'(t))\} \in \mathscr{L}^2_{\mathscr{F}}(0, T; \mathbb{R}^{n+1})$, where

$a(t) \geqslant 0$ corresponds to the value of risk exposure at time t, $b(t) := (b_1(t), b_2(t), \cdots, b_n(t))'$, $b_i(t)$ is the dollar amount invested in the i-th risky asset at time t.

Denote by $X(t)$ the wealth at time t under strategy $\pi(t)$.Then the wealth process $X(t)$ is given by

$$\begin{cases} \mathrm{d}X(t) = \Big\{r_0 X(t) + \sum\limits_{i=1}^{n}[\mu_i(t) - r_0]b_i(t) + \theta a(t) + \mu_0 - \theta\Big\}\mathrm{d}t \\ \qquad\qquad + \sigma_0 a(t)\mathrm{d}W_0(t) + b(t)'\sigma(t)\mathrm{d}W(t), \\ X(0) = x_0. \end{cases}$$

Let $h = \mu_0 - \theta$, $r(t) = (\theta, \mu_1(t) - r_0, \mu_2(t) - r_0, \cdots, \mu_n(t) - r_0)'$,

$$\widetilde{\sigma}(t) = \begin{bmatrix} \sigma_0 & 0 & \cdots & 0 \\ 0 & \sigma_{11}(t) & \cdots & \sigma_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \sigma_{n1}(t) & \cdots & \sigma_{nn}(t) \end{bmatrix}$$

which satisfies the nondegeneracy condition. Then the wealth process $X(t)$ can be rewritten as

$$\begin{cases} \mathrm{d}X(t) = [r_0 X(t) + r(t)'\pi(t) + h]\mathrm{d}t + \pi(t)'\widetilde{\sigma}(t)\mathrm{d}\widetilde{W}(t), \\ X(0) = x_0. \end{cases} \tag{1}$$

In this paper, both variance and CVaR are chosen to measure the risk generated from the wealth process. Let $\mathscr{V}[X(T)]$ denote the variance of the terminal wealth $X(T)$, and the loss function of the terminal wealth is given by $L(X(t)) = X(0) - X(T)$. We adopt the CVaR definition of the loss given by Rockafellar and Uryasey [19, 20], which is a unified definition applied for all loss functions with a continuous or a discrete distribution. Define the cumulative distribution function of $L(X(t))$ as $\psi(y) = \mathsf{P}(L(X(t)) \leqslant y)$. For a given confidence level β, the correspondent β-tail distribution of the loss function $L(X(t))$ is given by

$$\psi_\beta(y) = \begin{cases} 0, & \text{if } y < \mathrm{VaR}_\beta; \\ [\psi(y) - \beta]/(1 - \beta), & \text{if } y \geqslant \mathrm{VaR}_\beta, \end{cases}$$

where $\mathrm{VaR}_\beta = \inf\{y \,|\, \psi(y) \geqslant \beta\}$.

According to Rockafellar and Uryasey [19, 20], the CVaR of the loss function $L(X(T))$ can be obtained as follows,

$$\mathrm{CVaR}_\beta[L(X(T))] = \min_\alpha \Big\{\alpha + \frac{1}{1-\beta}\mathsf{E}[(X(0) - X(T) - \alpha)^+]\Big\}, \tag{2}$$

where α is an auxiliary variable and $(x)^+ = \max\{x, 0\}$.

Now we formulate the dynamic mean-variance-CVaR (MVC) optimization problem as follows,

$$\mathscr{P}_{\mathrm{mvc}}(\omega): \min_{\pi(\cdot)\in\mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^{n+1})} \mathscr{V}[X(T)] + \omega \mathrm{CVaR}_\beta[L(X(T))], \tag{3}$$

$$\text{s.t.} \quad \mathsf{E}[X(T)] = d, \tag{4}$$

$$\mathrm{d}X(t) = [r_0 X(t) + r(t)'\pi(t) + h]\mathrm{d}t + \pi(t)'\widetilde{\sigma}(t)\mathrm{d}\widetilde{W}(t), \tag{5}$$

$$X(T) \geqslant 0, \tag{6}$$

$$X(0) = x_0, \tag{7}$$

where d is the target value of the expected terminal wealth and $\omega \geqslant 0$ is a weighting parameter, which balances the importance of the two risk measures. Equation (6) indicates that the insurer has to obtain a positive wealth value at terminal period.

§3. Main Results

Letting $\widetilde{X}(t) = X(t) + h/r_0$, we convert equation (1) into

$$\begin{cases} \mathrm{d}\widetilde{X}(t) = [r_0\widetilde{X}(t) + r(t)'\pi(t)]\mathrm{d}t + \pi(t)'\widetilde{\sigma}(t)\mathrm{d}\widetilde{W}(t), \\ \widetilde{X}(0) = x_0 + h/r_0. \end{cases}$$

Subsequently, considering equation (2) and $\mathscr{P}_{\mathrm{mvc}}(\omega)$, we can get the following problem with a fixed value of α,

$$\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha): \min_{\pi(\cdot)\in\mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^{n+1})} \mathscr{V}[\widetilde{X}(T)] + \widehat{\omega}\mathsf{E}[(q-\widetilde{X}(T))^+],$$

$$\text{s.t.} \quad \mathsf{E}[\widetilde{X}(T)] = d + \frac{h}{r_0},$$

$$\mathrm{d}\widetilde{X}(t) = [r_0\widetilde{X}(t) + r(t)'\pi(t)]\mathrm{d}t + \pi(t)'\widetilde{\sigma}(t)\mathrm{d}\widetilde{W}(t),$$

$$\widetilde{X}(T) \geqslant \frac{h}{r_0},$$

$$\widetilde{X}(0) = x_0 + \frac{h}{r_0},$$

where $q = x_0 + h/r_0 - \alpha$ and $\widehat{\omega} = \omega/(1-\beta)$.

Following the idea in [25], we use martingale approach to solve $\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha)$.

At first, we solve a static optimization problem to determine the optimal terminal wealth $\widetilde{X}^*(T)$. Let $\delta(t) = \widetilde{\sigma}^{-1}r(t)$ denote the market price of risk. We define the deflator process $\mathrm{d}z(t) = -z(t)(r_0\mathrm{d}t + \delta'(t)\mathrm{d}\widetilde{W}(t))$ with $z(0) = 1$, or equivalently, the exponential martingale,

$$z(t)=\exp\Big\{-\int_0^t\Big[r_0+\frac{1}{2}\|\delta(\tau)\|^2\Big]\mathrm{d}\tau-\int_0^t\delta'(\tau)\mathrm{d}\widetilde{W}(\tau)\Big\},\tag{8}$$

with $z(0)=1$ (see, e.g., [27]).

We define the lower and upper bounds of $z(T)$ as

$$\underline{\xi}=\inf\{c\in\mathbb{R}:\mathsf{P}(z(T)\leqslant c)>0\},\qquad \overline{\xi}=\sup\{c\in\mathbb{R}:\mathsf{P}(z(T)\geqslant c)>0\}.$$

Under the assumption that the market parameters $r(t)$ and $\sigma(t)$ are deterministic functions of t and r_0 is a constant, we have $\underline{\xi}=0$ and $\overline{\xi}=+\infty$ (see, e.g., [27]). The deflator process $z(t)$ changes the wealth process to a martingale, i.e., $z(t)\widetilde{X}(t)=\mathsf{E}[z(s)\widetilde{X}(s)\,|\,\mathscr{F}_t]$, for any $0\leqslant t<s\leqslant T$. Thus, the optimal terminal wealth $\widetilde{X}^*(T)$ of the problem $\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha)$ can be found by the following static optimization problem:

$$\begin{aligned}
&\widetilde{\mathscr{P}}^{\mathrm{aux}}_{\mathrm{mvc}}:\ \min_{\widetilde{X}(T)\in\mathscr{L}^2_{\mathscr{F}_T}(\Omega;\mathbb{R})}\ \mathsf{E}[\widetilde{X}^2(T)-d^2]+\widehat{\omega}\mathsf{E}[(q-\widetilde{X}(T))^+],\\
&\text{s.t.}\ \ \mathsf{E}[\widetilde{X}(T)]=d+\frac{h}{r_0},\\
&\mathsf{E}[z(T)\widetilde{X}(T)]=x_0+\frac{h}{r_0},\\
&\widetilde{X}(T)\geqslant\frac{h}{r_0}.
\end{aligned}$$

And the above static optimization problem can be solved by Lagrangian multiplier.

Proposition 1 Under the assumption that $0<(x_0+h/r_0)/d<\mathsf{E}[z(T)]$, the optimal terminal wealth $\widetilde{X}^*(T)$ of problem $\widetilde{\mathscr{P}}^{\mathrm{aux}}_{\mathrm{mvc}}$ is given as follows,

$$\widetilde{X}^*(T)=\begin{cases}(\lambda-\eta z(T))/2, & \text{if } \eta z(T)\leqslant\lambda-2q;\\ q, & \text{if } \lambda-2q<\eta z(T)<\lambda-2q+\widehat{\omega};\\ (\lambda-\eta z(T)+\widehat{\omega})/2, & \text{if } \lambda-2q+\widehat{\omega}\leqslant\eta z(T)\leqslant\lambda+\widehat{\omega}-2h/r_0;\\ h/r_0, & \text{if } \eta z(T)>\lambda+\widehat{\omega}-2h/r_0,\end{cases}\tag{9}$$

where λ and $\eta>0$ solves the following system of equations:

$$\begin{cases}\mathsf{E}[(\lambda-2q-\eta z(T))^+]-\mathsf{E}[(\lambda-2q+\widehat{\omega}-\eta z(T))^+]\\ \quad+\mathsf{E}[(\lambda+\widehat{\omega}-2h/r_0-\eta z(T))^+]=2(d+h/r_0),\\ \mathsf{E}[z(T)(\lambda-2q-\eta z(T))^+]-\mathsf{E}[z(T)(\lambda-2q+\widehat{\omega}-\eta z(T))^+]\\ \quad+\mathsf{E}[z(T)(\lambda+\widehat{\omega}-2h/r_0-\eta z(T))^+]=2(x_0+h/r_0).\end{cases}$$

The proof of the above proposition is similar to Proof of Proposition 3.2 and 3.3 in [25].

The deflator process $z(t)$ defined in equation (8) implies that $z(T)/z(t)$ follows a log-normal distribution, i.e., $\ln(z(T)/z(t)) \sim \mathrm{N}(m(t), v^2(t))$, where

$$m(t) := -\int_t^T \Big(r_0 + \frac{1}{2}\|\delta(\tau)\|^2\Big)\mathrm{d}\tau, \qquad t \in [0,T],$$
$$v^2(t) := \int_t^T \|\delta(\tau)\|^2 \mathrm{d}\tau, \qquad t \in [0,T].$$

Since the market parameters $r(t)$ and $\sigma(t)$ are deterministic functions of t and r_0 is a constant, we have $\mathsf{E}[z(T)] = \mathrm{e}^{-r_0 T}$. Thus the condition $\underline{\xi} < (x_0 + h/r_0)/d < \mathsf{E}[z(T)]$ becomes $d > (x_0 + h/r_0)\mathrm{e}^{r_0 T}$, which implies that the expected target terminal wealth d should be larger than the terminal wealth generated from keeping all wealth in the risk free account. To simplify the notations, we denote the first and second moment of $z(T)/z(t)$ as $A(t) := \mathsf{E}[z(T)/z(t)] = \mathrm{e}^{m(t)+v(t)^2/2} = \mathrm{e}^{-\int_t^T r_0 \mathrm{d}s}$ and $B(t) := \mathsf{E}[(z(T)/z(t))^2] = \mathrm{e}^{2m(t)+2v(t)^2}$, respectively.

Once the optimal terminal wealth $\widetilde{X}^*(T) \in \mathscr{L}^2_{\mathscr{F}_T}(\Omega;\mathbb{R})$ is known, the optimal wealth process $\widetilde{X}^*(t)$ and optimal strategy $\pi^*(t)$ can be obtained by solving the following backward stochastic differential equation (BSDE),

$$\begin{cases} \mathrm{d}\widetilde{X}(t) = [r_0\widetilde{X}(t) + \delta(t)'y(t)]\mathrm{d}t + y(t)'\mathrm{d}\widetilde{W}(t), \\ \widetilde{X}(T) = \widetilde{X}^*(T), \end{cases}$$

where $y(\cdot) = \widetilde{\sigma}(\cdot)\pi(\cdot)$. The optimal wealth process can be expressed as $\widetilde{X}^*(t) = \mathsf{E}[(z(T)/z(t))\widetilde{X}^*(T) \,|\, \mathscr{F}_t]$ because of martingale property. Subsequently, the optimal strategy $\pi^*(t)$ can be given by

$$\pi^*(t) = -(\widetilde{\sigma}(t)\widetilde{\sigma}(t)')^{-1} r(t) \frac{\partial \widetilde{X}^*(t)}{\partial z(t)}. \tag{10}$$

According to the martingale property $\widetilde{X}^*(t) = \mathsf{E}[(z(T)/z(t))\widetilde{X}^*(T) \,|\, \mathscr{F}_t]$ and equation (9), we have

$$\begin{aligned} \widetilde{X}^*(t,\alpha) = {} & \frac{1}{2}\mathsf{E}\Big[\frac{z(T)}{z(t)}(\lambda - \eta z(T))\mathbf{1}_{z(T)\leqslant(\lambda-2q)/\eta} \,\Big|\, \mathscr{F}_t\Big] \\ & + q\mathsf{E}\Big[\frac{z(T)}{z(t)}\mathbf{1}_{(\lambda-2q)/\eta<z(T)<(\lambda-2q+\widehat{\omega})/\eta} \,\Big|\, \mathscr{F}_t\Big] \\ & + \frac{1}{2}\mathsf{E}\Big[\frac{z(T)}{z(t)}(\lambda + \widehat{\omega} - \eta z(T))\mathbf{1}_{(\lambda-2q+\widehat{\omega})/\eta\leqslant z(T)\leqslant[r_0(\lambda+\widehat{\omega})+2h]/(\eta r_0)} \,\Big|\, \mathscr{F}_t\Big] \\ & + \frac{h}{r_0}\mathsf{E}\Big[\frac{z(T)}{z(t)}\mathbf{1}_{z(T)>[r_0(\lambda+\widehat{\omega})+2h]/(\eta r_0)} \,\Big|\, \mathscr{F}_t\Big], \qquad t \in [0,T]. \end{aligned}$$

Applying Proposition 7.1 in [25], we can obtain the explicit expression of $\widetilde{X}^*(t,\alpha)$, and then $\pi^*(t,\alpha)$ can be computed based on equation (10).

Proposition 2 Under the assumption that $d > (x_0 + h/r_0)\mathrm{e}^{r_0 T}$, the optimal wealth process $\widetilde{X}^*(t,\alpha)$ and the optimal portfolio policy $\pi^*(t,\alpha)$ of problem $\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha)$ are given respectively as follows,

$$\begin{aligned}\widetilde{X}^*(t,\alpha) &= \frac{A(t)}{2}\Big[(\lambda-2q)\Phi(k_1(t)) - (\lambda-2q+\widehat{\omega})\Phi(k_2(t)) \\ &\quad + \Big(\lambda+\widehat{\omega}-\frac{h}{r_0}\Big)\Phi(k_3(t))\Big] - \frac{z(t)\eta B(t)}{2}[\Phi(k_1(t)-v(t)) \\ &\quad - \Phi(k_2(t)-v(t)) + \Phi(k_3(t)-v(t))], \quad t\in[0,T], \qquad (11)\\ \pi^*(t,\alpha) &= \frac{1}{2}(\widetilde{\sigma}(t)\widetilde{\sigma}(t)')^{-1}r(t)\Big\{\frac{A(t)}{v(t)}\Big[(\lambda-2q)\phi(k_1(t)) \\ &\quad - (\lambda-2q+\widehat{\omega})\phi(k_2(t)) + \Big(\lambda+\widehat{\omega}-\frac{h}{r_0}\Big)\phi(k_3(t))\Big] + z(t)\eta B(t) \\ &\quad \times\{[\Phi(k_1(t)-v(t)) - \Phi(k_2(t)-v(t)) + \Phi(k_3(t)-v(t))] \\ &\quad - \frac{1}{v(t)}[\phi(k_1(t)-v(t)) - \phi(k_2(t)-v(t)) + \phi(k_3(t)-v(t))]\}\Big\}, \quad t\in[0,T], \quad (12)\end{aligned}$$

where $k_1(t) = [\ln(((\lambda-2q)/\eta)^+) - \ln z(t) - m(t)]/v(t) - v(t)$, $k_2(t) = [\ln(((\lambda-2q+\widehat{\omega})/\eta)^+) - \ln z(t) - m(t)]/v(t) - v(t)$, and $k_3(t) = [\ln(((r_0\lambda + r_0\widehat{\omega} - 2h)/(r_0\eta))^+) - \ln z(t) - m(t)]/v(t) - v(t)$. The Lagrange multipliers λ and η are the solutions to the following system of equations,

$$\begin{cases}(\lambda-2q)\Phi(k_1(0)+v(0)) - (\lambda-2q+\widehat{\omega})\Phi(k_2(0)+v(0)) \\ \quad + (\lambda+\widehat{\omega}-h/r_0)\Phi(k_3(0)+v(0)) - \eta A(0) \\ \quad \times[\Phi(k_1(0)) - \Phi(k_2(0)) + \Phi(k_3(0))] = 2(d+h/r_0), \\ A(0)[(\lambda-2q)\Phi(k_1(0)) - (\lambda-2q+\widehat{\omega})\Phi(k_2(0)) \\ \quad + (\lambda+\widehat{\omega}-h/r_0)\Phi(k_3(0))] - \eta B(0)[\Phi(k_1(0)-v(0)) \\ \quad - \Phi(k_2(0)-v(0)) + \Phi(k_3(0)-v(0))] = 2(x_0+h/r_0).\end{cases} \qquad (13)$$

In the following, we minimize the value function of $\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha)$ over α to get the optimal strategies of problem $\mathscr{P}_{\mathrm{mvc}}(\omega)$. That is, to minimize the CVaR term in objective function of $\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha)$. Applying Proposition 7.1 in [25], we can obtained the explicit expression of the CVaR term based on equation (2) and (9).

Proposition 3 The optimal wealth process $\widetilde{X}^*(t)$ and the optimal strategy $\pi^*(t)$ of problem $\mathscr{P}_{\mathrm{mvc}}(\omega)$ are given by

$$\begin{cases}\widetilde{X}^*(t) = \widetilde{X}^*(t,\alpha^*), \\ \pi^*(t) = \pi^*(t,\alpha^*),\end{cases}$$

for some λ^*, η^* and α^*. Here $\alpha^* = \operatorname*{argmin}_{\alpha} \mathscr{G}_{\rm CVaR}(\alpha)$ with

$$\mathscr{G}_{\rm CVaR}(\alpha) = \begin{cases} [(-r_0\lambda - r_0\widehat{\omega} + 2h)/2r_0][\Phi(k_3(0)+v(0)) - \Phi(k_2(0)+v(0))] \\ \quad + (\eta A(0)/2)[\Phi(k_3(0)) - \Phi(k_2(0))] \\ \quad + (x_0 + h/r_0 - \alpha)[1 - \Phi(k_2(0)+v(0))], & \text{if } \alpha < x_0 + h/r_0, \\ \alpha, & \text{if } \alpha \geqslant x_0 + h/r_0, \end{cases} \tag{14}$$

where λ^* and η^* solve equation (13) with $\alpha = \alpha^*$. Furthermore, under the optimal portfolio policy $\pi^*(t)$, the correspondent CVaR of the loss function is given by $\mathrm{CVaR}_\beta[L(\widetilde{X}(T))] = \mathscr{G}_{\rm CVaR}(\alpha^*)$.

Then, the optimal wealth $X^*(t)$ is given by

$$X^*(t) = \widetilde{X}^*(t) - \frac{h}{r_0}, \qquad t \in [0, T]. \tag{15}$$

§4. Numerical Analysis

In this section, we present a numerical example to investigate the properties of reinsurance policies and portfolio policies derived from $\widetilde{\mathscr{P}}_{\rm mvc}(\omega, \alpha)$ based on results in Section 3. The procedure of numerical analysis is given as follows:

Step 1: Under different parameter settings, compute the values of λ, η and α numerically according to equation (13) and (14).

Step 2: Generate 1 000 discrete values of $z(t)$ randomly based on equation (8).

Step 3: Find the optimal wealth value $X^*(t)$, reinsurance level $a^*(t)$, portfolio level $b^*(t)$ under different assumptions according to equation (11), (12) and (15).

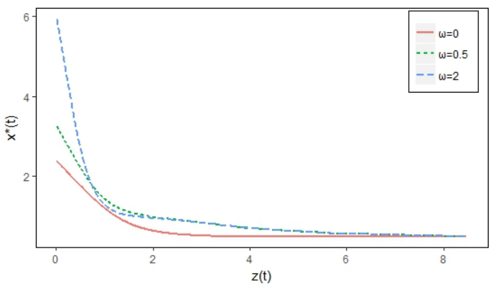
4.1 The Effect of Weighting Coefficient ω

In this subsection, we analyze how the weighting coefficient ω influences the optimal wealth value, reinsurance level and portfolio level. We consider a financial market consisting of one risk-free asset and one risky asset. The annually based market parameters are given as $r_0 = 0.0408$, $r_1 = 0.1068$, and $\sigma_1 = 0.22$, which can be seen in [25]. For insurer's surplus process, we set the parameters to be $\mu_0 = 0.5$, $\theta = 0.52$, and $\sigma_0 = 1$, which can be seen in [5]. And the insurer's initial wealth is assumed to be $X(0) = 1$ (million) and the target feedback is about 10 percent (i.e., $d = 1.1$) of the portfolio return in $T = 1$ year. The insurer guides his strategy through models $\widetilde{\mathscr{P}}_{\rm mvc}(\omega, \alpha)$ (with confidence level being $\beta = 95\%$), where the weighting coefficient ω is set at $\omega = 0$, $\omega = 0.5$ and $\omega = 2$. Following Step 1, we can get the values of parameters λ^*, η^* and α^* in Table 1.

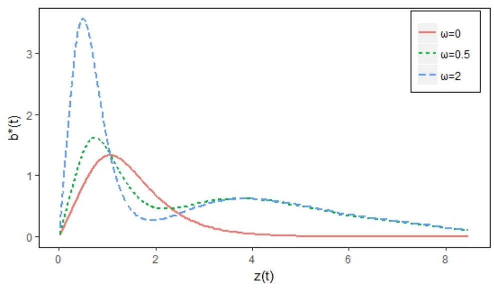
Table 1 λ^*, η^* and α^* for $\widetilde{\mathscr{P}}_{\mathbf{mvc}}(\omega,\alpha)$

ω	α^*	λ^*	η^*
0	0.5092	2.8906	2.1506
0.5	0.5083	4.7068	3.7189
2	0.5068	10.3665	12.2697

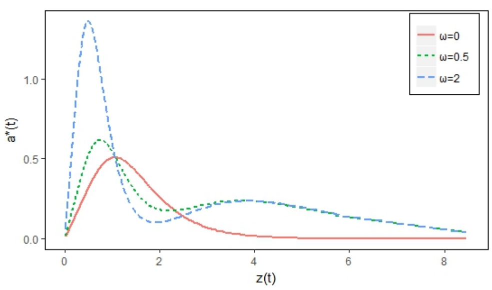
Following Step 2 and Step 3, we can obtain the optimal wealth value $X^*(t)$, reinsurance level $a^*(t)$, portfolio level $b^*(t)$ at $t = 0.5$. Similarly, the optimal terminal wealth $X^*(T)$ can be derived under $T = 1$. Figure 1 shows the optimal wealth $X^*(t)$, reinsurance level $a^*(t)$, portfolio level $b^*(t)$ at $t = 0.5$ and the optimal terminal wealth $X^*(T)$ for $\widetilde{\mathcal{P}}_{\mathrm{mvc}}(\omega,\alpha)$. The X-axis represents the value of $z(t)$. Under $t = 0.5$, $z(t)_{\min} = 0.015$, $\mathsf{E}[z(t)] = 0.81$, $z(t)_{\max} = 8.84$ and 99% of $z(t)$ falls into the interval $(0, 4.79]$, which means that 0.81 is the dividing point between a good market and a bad market. Bigger $z(t)$ means worse market condition and the market is extremely bad if $z(t) > 4.79$.



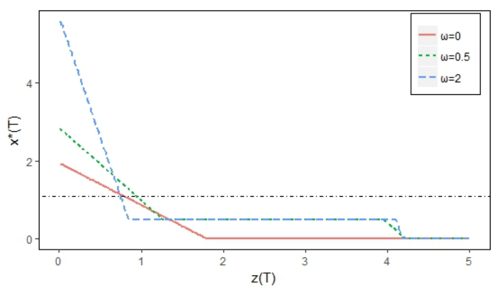
(a) The wealth $X^*(t)$ at $t = 0.5$



(b) The investment policy $b^*(t)$ at $t = 0.5$



(c) The reinsurance policy $a^*(t)$ at $t = 0.5$



(d) The terminal wealth $X^*(T)$

Figure 1 **The results of** $\widetilde{\mathscr{P}}_{\mathbf{mvc}}(\omega,\alpha)$

Problem $\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha)$ degenerates to the dynamic MV portfolio selection model when $\omega = 0$. It can be observed from Figure 1(a) that $X^*(t)$ generated from $\widetilde{\mathscr{P}}_{\mathrm{mvc}}(\omega,\alpha)$ $(\omega > 0)$ is higher than the one from MV portfolio selection model $(\omega = 0)$ when the market condition is good (e.g. $z(t) = 0.5$) and more weight allocated to the risk measure

CVaR results in a higher $X^*(t)$. Therefore, the insurer can increase the value of optimal intermediate wealth by adjusting the weight ω. When the market condition is bad ($z(t)$ is large), $X^*(t)$ generated based on MV portfolio selection model is still lower, but the change of ω has less significant effect on the value of $X^*(t)$. Undoubtedly, from the point of view of optimal wealth $X^*(t)$, mean-variance-CVaR model is better than MV model in most cases.

Figure 1(b) shows the allocation in the risky asset derived from $\widetilde{\mathscr{P}}_{\rm mvc}(\omega, \alpha)$ is more sensitive to the market condition when the market is good, and more risky asset is required in optimal asset allocation. As the market becomes worse, the proportion of risky asset in optimal strategy drops sharply, and even below the result in MV model. This finding tells that the CVaR term is more sensitive to the market condition compared to the variance term. With the market getting extremely bad, ω cannot significantly affect the allocation policy.

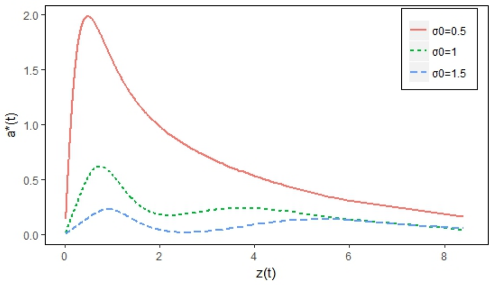
Figure 1(c) shows the reinsurance policies sharing the same trend as the investment part. In good market condition, bigger weight of CVaR results in higher proportion of reinsurance, that is, more new business will be acquired by insurers to make more profit. As the market become worse, the reinsurance policy becomes less and less sensitive to the weight of CVaR in optimization model.

In Figure 1(d), the dot-dash line represents the target terminal return. Bigger weight of CVaR term results in higher terminal wealth when the market condition is good. While $X^*(T)$ from MVC model with a small weight of CVaR term is the last one to reduce to below the target line when the market becomes worse. If the market continues becoming worse, it can be seen that MVC models still dominate to prevent more serious loss. And the three models come to a same result when the market is extremely bad, which can be seen in forgoing figures as well.

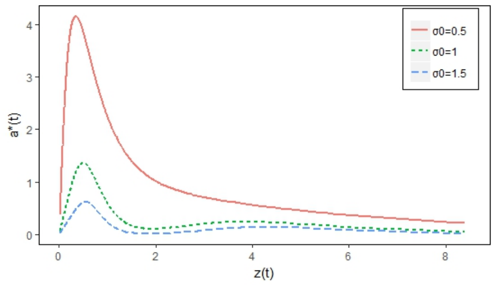
4.2 Sensitivity Analysis

In this subsection, we analyze how the parameters in insurance market impact on the optimal reinsurance policy derived from $\widetilde{\mathscr{P}}_{\rm mvc}(\omega, \alpha)$. Risk parameter in insurance surplus σ_0 takes values in 0.5, 1 and 1.5. The premium return rate of reinsurer θ changes among 0.51, 0.52 and 0.53. The values of other parameters in models are the same as ones in Section 4.1. Following Step 1 – Step 3, the results of $\widetilde{\mathscr{P}}_{\rm mvc}(\omega, \alpha)$ at $t = 0.5$ with $\omega = 0.5$ and $\omega = 2$ can be obtained and shown in Figure 2.

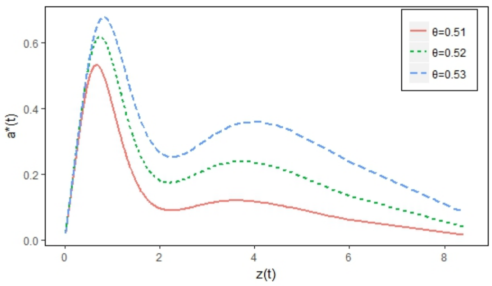
Figure 2(a) displays that the insurer will keep less insurance business in any market conditions when the volatility of the insurer's surplus becomes larger. And the reinsurance policy is a little less sensitive to σ_0 when the market condition is quiet bad. In addition,



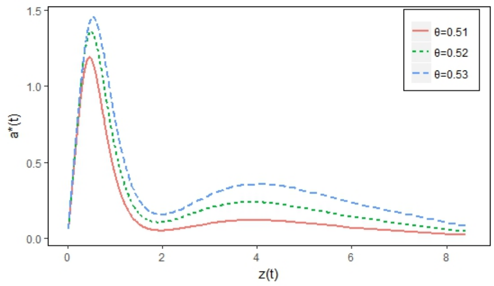
(a) The reinsurance policy $a^*(t)$ at $\omega = 0.5$



(b) The reinsurance policy $a^*(t)$ at $\omega = 2$



(c) The reinsurance policy $a^*(t)$ at $\omega = 0.5$



(d) The reinsurance policy $a^*(t)$ at $\omega = 2$

Figure 2 The results of $\widetilde{\mathscr{P}}_{\mathbf{mvc}}(\omega, \alpha)$ with different σ_0 and θ at $t = 0.5$

compared with Figure 2(b), the trend of $a^*(t)$ is similar, while higher ω corresponds to a more drastic change trend since CVaR term is sensitive to the market condition.

Figure 2(c) and Figure 2(d) reveal that the optimal reinsurance policy $a^*(t)$ changes regarding to the premium return rate θ of the reinsurer. When the market is relatively good, optimal reinsurance policy is less sensitive to the change of premium return rate of the reinsurer. However, it changes greatly, when the market gets worse. If the market becomes extremely bad, the policy turns to be less sensitive again. Additionally, bigger premium return rate of the reinsurance results in more retention level of insurance business.

§5. Conclusion

In this paper, we have introduced two risk measures (variance and CVaR) into asset allocation and reinsurance optimization model for an insurer in continuous-time. The insurer is allowed to invest in a financial market and purchase proportional reinsurance/acquire new business. The surplus process of the insurer is assumed to follow a diffusion approximation model and the financial market consists of one risk-free asset and multiple risky assets whose price processes are governed by geometric Brownian motions. By using Rockafellar and Uryasey's approximation method and following the idea in [25], we have derived the closed-form solutions of this dynamic mean-variance-CVaR (MVC) optimization problem. Moreover, the optimal strategies obtained in our model have been analyzed numerically based on different parameter settings and market conditions. When the market condition is good, the insurer can get a significantly increasing profit level compared with MV model through adjusting a relatively high weight of CVaR term (ω). Furthermore, when the market condition turns to bad, the insurer may choose an appropriate ω to avoid too much loss. That is, through the optimal reinsurance policy and investment policy derived from our MVC model, insurers may manage risk and profit more flexibly and comprehensively.

In the future research, we will extend this work to a jump-diffusion surplus process with a stochastic interest rate. It may be also of interest to relax the assumption that the risk source of reinsurance market is independent of the risk source of the financial market.

References

[1] PROMISLOW S D, YOUNG V R. Minimizing the probability of ruin when claims follow brownian motion with drift [J]. *N Am Actuar J*, 2015, **9(3)**: 110–128.

[2] MENG H, ZHOU M, SIU T K. Optimal reinsurance policies with two reinsurers in continuous time [J]. *Econ Model*, 2016, **59**: 182–195.

[3] YANG H L, ZHANG L H. Optimal investment for insurer with jump-diffusion risk process [J]. *Insurance Math Econom*, 2005, **37(3)**: 615–634.

[4] ZHAO Q, WANG R M, WEI J Q. Exponential utility maximization for an insurer with time-inconsistent preferences [J]. *Insurance Math Econom*, 2016, **70**: 89–104.

[5] ZENG Y, LI Z F. Optimal reinsurance-investment strategies for insurers under mean-CaR criteria [J]. *J Ind Manag Optim*, 2012, **8(3)**: 673–690.

[6] SUN Z Y, GUO J Y. Optimal mean-variance investment and reinsurance problem for an insurer with stochastic volatility [J]. *Math Methods Oper Res*, 2018, **88(1)**: 59–79.

[7] MARKOWITZ H. Portfolio selection [J]. *J Finance*, 1952, **7(1)**: 77–91.

[8] BAI L H, ZHANG H Y. Dynamic mean-variance problem with constrained risk control for the insurers [J]. *Math Methods Oper Res*, 2008, **68(1)**: 181–205.

[9] ZENG Y, LI Z F. Optimal time-consistent investment and reinsurance policies for mean-variance insurers [J]. *Insurance Math Econom*, 2011, **49(1)**: 145–154.

[10] PUN C S. Time-consistent mean-variance portfolio selection with only risky assets [J]. *Econ Model*, 2018, **75**: 281–292.

[11] WANG T X, WEI J Q. Mean-variance portfolio selection under a non-Markovian regime-switching model [J]. *J Comput Appl Math*, 2019, **350**: 442–455.

[12] KOLM P N, TÜTÜNCÜ R, FABOZZI F J. 60 years of portfolio optimization: practical challenges and current trends [J]. *European J Oper Res*, 2014, **234(2)**: 356–371.

[13] CAI J, WENG C G. Optimal reinsurance with expectile [J]. *Scand Actuar J*, 2016, **2016(7)**: 624–645.

[14] EMMER S, KLÜPPELBERG C, KORN, R. Optimal portfolios with bounded capital at risk [J]. *Math Finance*, 2001, **11(4)**: 365–384.

[15] LI Z F, NG K W, DENG X T. Continuous-time optimal portfolio selection using mean-CaR models [J]. *Nonlinear Dyn Syst Theory*, 2007, **7(1)**: 35–49.

[16] GUO W J, LI X D. Optimal portfolio selection bounded by VaR for insurer [J]. *J Syst Manag*, 2009, **18(5)**: 583–587. (in Chinese)

[17] LIU B. A new risk measure and its application in portfolio optimization: the SPP-CVaR approach [J]. *Econ Model*, 2015, **51**: 383–390.

[18] WANG L Z, LI J. Research on insurance portfolio under based on risk-adjusted return under the constraint of policy [J]. *Chinese J Management Sci*, 2012, **20(1)**: 16–22. (in Chinese)

[19] ROCKAFELLAR R T, URYASEV S. Optimization of conditional value-at-risk [J]. *J Risk*, 2000, **2(3)**: 21–41.

[20] ROCKAFELLAR R T, URYASEV S. Conditional value-at-risk for general loss distributions [J]. *J Bank Financ*, 2002, **26(7)**: 1443–1471.

[21] ZHU S S, FUKUSHIMA M. Worst-case conditional value-at-risk with application to robust portfolio management [J]. *Oper Res*, 2009, **57(5)**: 1155–1168.

[22] ROMAN D, DARBY-DOWMAN K, MITRA G. Mean-risk models using two risk measures: a multi-objective approach [J]. *Quant Finance*, 2007, **7(4)**: 443–458.

[23] LI J, HUANG H X, XIAO X. The sovereign property of foreign reserve investment in China: a CVaR approach [J]. *Econ Model*, 2012, **29(5)**: 1524–1536.

[24] ELAHI Y, AZIZ M I A. Mean-variance-CvaR model of multiportfolio optimization via linear weighted sum method [J]. *Math Probl Eng*, 2014, **2014**: Article ID 104064, 7 pages.

[25] GAO J J, XIONG Y, LI D. Dynamic mean-risk portfolio selection with multiple risk measures in continuous-time [J]. *European J Oper Res*, 2016, **249(2)**: 647–656.

[26] GRANDELL J. *Aspects of Risk Theory* [M]. New York: Springer-Verlag, 1991.

[27] KARATZAS I, SHREVE S E. *Methods of Mathematical Finance* [M]. New York: Springer-Verlag, 1998.

基于 Mean-Variance-CVaR 准则的保险公司最优资产配置与再保险策略

赵 霞 时 雨

(上海对外经贸大学统计与信息学院, 上海, 201620)

摘 要: 本文研究了连续时间下保险公司基于均值–方差–CVaR 准则选择最优资产配置和再保险策略的问题. 我们运用鞅方法求解优化问题并得到了相应的显示解. 基于数值模拟, 我们分析了在不同参数值下最优财富、资产配置和再保险策略随市场条件变化而变化的趋势.

关键词: 资产配置; 再保险策略; Mean-Variance-CVaR 准则; 鞅方法

中图分类号: O211.6