应用概率统计　第 36 卷
第 6 期　2020 年 12 月

Chinese Journal of Applied Probability and Statistics
Dec., 2020, Vol. 36, No. 6, pp. 586-604
doi: 10.3969/j.issn.1001-4268.2020.06.003

Parameter Estimation of Multivariate Regression Model with Fuzzy Random Errors *

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Abstract: In many real-world problems, observations are usually described by approximate values due to fuzzy uncertainty, unlike probabilistic uncertainty that has nothing to do with experimentation. The combination of statistical model and fuzzy set theory is helpful to improve the identification and analysis of complex systems. As an extension of statistical techniques, this study is an investigation of the relationship between fuzzy multiple explanatory variables and fuzzy response with numeric coefficients and the fuzzy random error term. In this work we describe a parameter estimation procedure carrying out the least-squares method in a complete metric space of fuzzy numbers to determine the coefficients based on the extension principle. We demonstrate how the fuzzy least squares estimators present large sample statistical properties, including asymptotic normality, strong consistency and confidence region. The estimators are also examined via asymptotic relative efficiency concerning traditional least squares estimators. Different from the construction of error term in Kim et al. [21], it is more reasonable in the proposed model since the problems of inconsistency in referring to fuzzy variable and producing the negative spreads may be avoided. The experimental study verifies that the proposed fuzzy least squares estimators achieve the meaning consistent with the theory identification for large sample data set and better generalization regarding one single variable model.

Keywords: fuzzy least squares estimator; fuzzy number; asymptotic properties; fuzzy random error

2010 Mathematics Subject Classification: 62F12; 62J86

Citation: ZHANG X L, LU Q J. Parameter estimation of multivariate regression model with fuzzy random errors [J]. Chinese J Appl Probab Statist, 2020, 36(6): 586–604.

§1. Introduction

Conventional regression is considered as a very useful data analysis tool that has been widely applied in various areas of applied statistics [1]. However, in many expert and intelligence systems applications, the relationship between a set of explanatory variables (also

*The project was supported by the First-Class Discipline (System Science) Construction Project of Shanghai Province (Grant No. XTKX2012), the Priority Discipline Project of Shanghai Province (Grant No. T0502) and the Foundation of Hujiang (Grant No. B14005).

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Received May 20, 2019. Revised January 19, 2020.

called independent variables, inputs or predictors) $X_1, X_2, \cdots, X_p$ and response variable Y (also called dependent variable or output) can't be precisely measured due to some unexpected situations, where necessary assumptions for statistical regression analysis can not be met because they are fuzzy, not based on random uncertainty. Zadeh [2] described this fuzzy uncertainty as imprecision, ambiguity or vagueness, and introduced the theory of fuzzy sets to allow the incorporation of inaccurate or incomplete information and uncertainty on parameters, properties, geometry, initial conditions, etc. For all fuzzy regression models, explanatory and/or response variables are represented as fuzzy numbers to establish fuzzy regression models using the extension principle, as opposed to the numeric values used in statistical regression models.

Fuzzy regression is an extension of conventional regression analysis to fuzzy environments, which can be employed as an efficient and useful tool for analyzing complex systems in fuzzy situations, such as business systems, socio-economic systems, and environmental systems. Fuzzy regression techniques are usually classified into two distinct categories that are not competitive with each other but complementary. Possibilistic regression model, suggested by Tanaka et al. [3], focused on inclusion relations between actual and estimated outputs to explain the relationship and formulated a linear programming problem to determine the regression coefficients as fuzzy numbers. Diamond [4] developed a fuzzy least-squares method from one single variable fuzzy linear model using the distance between the α-level compact sets of triangular fuzzy numbers. Its regression interval derived from the minimized distance to the given data in a given metric space is narrower than Tanaka et al.'s approaches. Therefore, from the prediction point of view, the fuzzy least-squares method is superior. For more on other approaches concerning regression analysis such as nonparametric methods and some recent works, see [5–9].

Many researchers have revised and modified least-squares approaches. Chang and Stanley [10] introduced a generalized fuzzy weighted least squares regression, giving weights to modes and spreads. D'Urso and Gastaldi [11] took into account a possible linear relationship among modes and spreads and constructed a doubly linear adaptive fuzzy regression model. Hong et al. [12] considered a new class of fuzzy regression model using shape-preserving operations on LR-fuzzy numbers for least-squares fitting. Kao and Chyu [13] proposed a fuzzy regression model that contains two-stage approaches: defuzzifying the fuzzy observations to show the general trend of the data and determining the error term to give the model the best explanatory power for the data. D'Urso et al. [14] constructed a robust fuzzy linear regression model based on the least median squares-weighted least squares estimation procedure. In the work of Hose and Hanss [15], estimating the fuzzy valued parameters of the model was accomplished utilizing a sensible fuzzification procedure of the crisp data points and the application of an exact inverse fuzzy arithmetic.

On the other hand, there is a great interest in studying the connections between fuzzy and probabilistic concepts, and from this convergence arises fuzzy random variable that uses this kind of information. Körner and Näther [16] discussed three approaches of linear regression with random fuzzy variables: extended classical estimates, best linear estimates, and least square estimates. Näther [17] summarized some results on random fuzzy variables of second-order and applied these notions in developing linear statistical inference with fuzzy data. Näther [18] presented different approaches to deal with regression analysis: a purely descriptive approach, statistical regression when the output was modeled fuzzy random variable and regression between two fuzzy random variables. Couso and Sánchez [19] demonstrated a higher-order possibility model that contained the imprecise information provided by fuzzy random variables. González-Rodríguez et al. [20] dealt with a generalized linear regression probabilistic model based on fuzzy-arithmetic, in which input and output were obtained from fuzzy random variables, considering the possibility of fuzzy valued random errors.

Kim et al. [21] have dealt with the asymptotic properties of the least-squares estimators for data sets completely fuzzy, in which the model is restricted into the simple case and the authors imposed imprecision and randomness condition to the error term. The main aim of the current paper is to develop the model and analyze the estimators within the more general context of multivariate regression, specifically, a generalization of the work by Kim et al. [21] is considered because the inferences for model with a single input restriction are more complex and less efficient. Furthermore, differing from the construction of error term introduced by Kim et al. [21], in the proposed model the left and right endpoints of error term are not taken into consideration but by means of the left and right spreads on the fuzzy variable. It is shown that the error term is more reasonable since the problems of inconsistency in referring to fuzzy variables and producing the negative spreads may be avoided.

The remainder of the article is organized as follows: In Section 2, the basics of statistics theory and fuzzy numbers needed to understand the following discussion will be reviewed. Section 3 is devoted to explaining the derivation of the presented result in parameter identification. Next, Section 4 validates the large sample statistical properties about the fuzzy least squares estimator and benchmarks it against statistical regression. A simulation experiment is used to illustrate the efficiency of the fuzzy least squares estimator in Section 5. Section 6 gives our concluding remarks.

§2. Preliminaries

To facilitate further presentation, in this section, we briefly review basic concepts and theories associated with fuzzy numbers and probability properties of random vector sequences.

2.1 Fuzzy Number

In fuzzy sets, each element is mapped to $[0,1]$ by membership function.

$$\mu_A : X \to [0,1],$$

where $[0,1]$ means real numbers between 0 and 1. Thus fuzzy set A is defined by membership function μ_A. Generally, the set can be represented as $A = (x, \mu_A(x))$ for discrete elements and $A = \int \mu_A(x)/x$ for continuous elements [22]. Let $\mathscr{F}(X)$ denote the set of all fuzzy sets in X.

Definition 1 [23] A fuzzy number A is a fuzzy subset of the real line R^1. Its membership function $\mu_A(x)$ satisfies the following criteria:

- α-cut set of $\mu_A(x)$ is a closed interval;
- $\exists\, x$ such that $\mu_A(x) = 1$;
- convexity such that $\mu_A[\lambda x_1 + (1-\lambda)x_2] \geqslant \min(\mu_A(x_1), \mu_A(x_2))$ for $\lambda \in [0,1]$,

where α-cut set contains all x elements that have a membership grade $\mu_A(x) \geqslant \alpha$.

Let $\widetilde{R}$ denote the set of all fuzzy numbers. In several substantive applications, the most utilized class of fuzzy number is LR-fuzzy number.

Definition 2 [24] LR-fuzzy number A is defined by

$$\mu_A(x) = \begin{cases} L((m-x)/l), & \text{if } x \leqslant m; \\ R((x-m)/r), & \text{if } x > m, \end{cases}$$

where $L, R : R^{1+} \to [0,1]$ are fixed left-continuous and non-increasing functions with $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$. L and R are left and right shape functions of A, where m is called the mode of A and $l, r > 0$ are left and right spread of A, respectively. We denote a LR-fuzzy number by $A = (m, l, r)_{\mathrm{LR}}$ (see Figure 1). The spreads l and r represent the fuzziness of fuzzy number and could be symmetric or non-symmetric. If $l = r = 0$, there is no fuzziness of the number, and so it is a crisp number m.

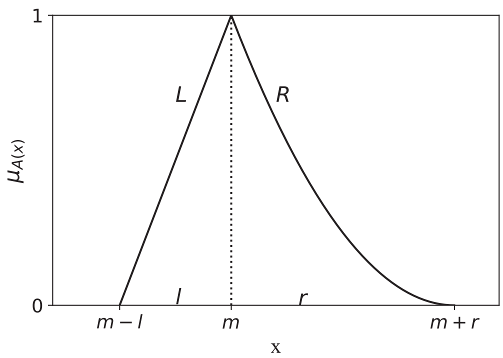


Figure 1 LR-fuzzy number $(m, l, r)_{\mathrm{LR}}$

Let $\widetilde{R}_{\mathrm{LR}}$ denote the set of all LR-fuzzy numbers. We can define different types of fuzzy data, the more utilized LR-fuzzy numbers are the triangular, normal, parabolic and square root fuzzy number. Each case takes into account a different level of uncertainty around the centers of the fuzzy data.

If $L(x) = R(x) = \max\{0, 1-x\}$, then $A = (m, l, r)_{\mathrm{LR}}$ is called a triangular fuzzy number with membership function being

$$\mu_A(x) = \begin{cases} 1-(m-x)/l, & m-l \leqslant x \leqslant m; \\ 1-(x-m)/r, & m \leqslant x \leqslant m+r; \\ 0, & \text{otherwise}. \end{cases}$$

Let $(m, l, r)_T$ denote a triangular fuzzy number and $\widetilde{R}_T$ denote the set of all triangular fuzzy numbers.

Definition 3 [25] Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space. A mapping $X : \Omega \to \widetilde{R}_{\mathrm{LR}}$ is a fuzzy random variable if the s-representation of $X, (X^m, X^l, X^r)_{\mathrm{LR}} : \Omega \to R^1 \times R^{1+} \times R^{1+}$ is a random vector.

It should be noted that A is not an ill-measured real random variable but a random element assuming purely fuzzy values.

Algebraic operations of LR-fuzzy numbers that we use in this paper are derived based on the extension principle of multivariate function as follows.

$$(m_1, l_1, r_1)_{\mathrm{LR}} \oplus (m_2, l_2, r_2)_{\mathrm{LR}} = (m_1+m_2, l_1+l_2, r_1+r_2)_{\mathrm{LR}}$$

and

$$\lambda(m, l, r)_{\mathrm{LR}} = \begin{cases} (\lambda m, \lambda l, \lambda r)_{\mathrm{LR}}, & \text{if } \lambda \geqslant 0; \\ (\lambda m, -\lambda r, -\lambda l)_{\mathrm{RL}}, & \text{if } \lambda < 0. \end{cases}$$

The distance between fuzzy numbers used to measure the goodness of fit for models have an important role in many real applications like data mining, pattern recognition, multivariate data analysis and so on. Diamond [4] defined a distance between two triangular fuzzy numbers as follows.

For all $X, Y \in \widetilde{R}_T$,

$$d(X,Y)^2 = D(\mathrm{supp}X, \mathrm{supp}Y)^2 + [m(X) - m(Y)]^2,$$

where $\mathrm{supp}X$ denotes the compact interval of support of X, and $m(X)$ its mode value. As a result, $(\widetilde{R}_T, d)$ is a complete metric space.

The concept of fuzzy random variable is introduced as an extension of both random variable and fuzzy set.

2.2 Properties of Random Variable Sequence in Probability

This subsection briefly reviews several theorems related to some results in the successive proof we will need.

Theorem 4 [26] Let X_n be a sequence of i.i.d. r.v.'s with mean μ and a finite variance σ^2. Let c_n be a sequence of real vectors $c_n = (c_{n1}, c_{n2}, \cdots, c_{nn})^\mathsf{T}$. If

$$\Big(\max_{1\leqslant i\leqslant n} c_{ni}^2\Big)\Big/\Big(\sum_{i=1}^{n} c_{ni}^2\Big) \to 0, \qquad \text{as } n \to 0,$$

then

$$Z_n = \Big[\sum_{i=1}^{n} c_{ni}(X_i - \mu)\Big]\Big/\Big(\sigma^2 \sum_{i=1}^{n} c_{ni}^2\Big)^{1/2} \xrightarrow{L} \mathrm{N}(0,1),$$

where the notation $\xrightarrow{L}$ stands for convergence in law or in distribution.

Theorem 5 [27] Let X_n and Y_n be two sequences of random vectors such that $X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{\mathsf{P}} c$, where c is a constant. Let $g(x,y)$ be a continuous function, then $g(X_n, Y_n) \xrightarrow{L} g(X, c)$, where the notation $\xrightarrow{\mathsf{P}}$ means converges in probability.

Definition 6 [28] The sequence of random variables X_n is bounded in probability if, for all $\epsilon > 0$, there exit a constant $B_\epsilon > 0$ and an integer N_ϵ such that

$$n \geqslant N_\epsilon \Rightarrow \mathsf{P}(|X_n| \leqslant B_\epsilon) \geqslant 1 - \epsilon.$$

Theorem 7 [28] Let X_n be a sequence of random variables bounded in probability and let Y_n be a sequence of random variables that converges to 0 in probability, then $X_n Y_n \xrightarrow{\mathsf{P}} 0$.

Theorem 8 [29] Let $S_n = \sum\limits_{i=1}^{n} X_i$, $n \geqslant 1$, be a martingale such that for $k \geqslant 1$, $\mathsf{E}|X_k|^p < \infty$ $(1 \leqslant p \leqslant 2)$. Suppose that $\{b_n\}$ is a sequence of positive constants increasing to ∞ as $n \to \infty$, and $\sum\limits_{i=1}^{\infty} \mathsf{E}(X_i^2)/b_i^2 < \infty$, then $S_n/b_n \stackrel{\text{a.s.}}{\to} 0$, where the notation $\stackrel{\text{a.s.}}{\to}$ means converges in almost surely.

§3. Parameter Estimation of Fuzzy Linear Regression

Consider a classical multivariate linear regression model. In matrix notion, the model is

$$Y_c = X_c\beta + \epsilon, \tag{1}$$

where $Y_c = (y_1, y_2, \cdots, y_n)^\mathsf{T}$ is a vector of observable response variables, X_c is a $(n \times (k+1))$ matrix of known constants x_{ij}, $\beta = (\beta_0, \beta_1, \cdots, \beta_k)^\mathsf{T}$ denotes the vector of unknown parameters, and $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)^\mathsf{T}$ is a n vector of unobservable random error assumed to satisfy $\mathsf{E}(\epsilon) = 0$ and $\mathsf{Var}(\epsilon) = \sigma_\epsilon^2 I_n$.

The usual method of estimation in this case is the least squares method, and the least squares estimator is given by $\breve{\beta}_{\text{LS}} = (X_c^\mathsf{T} X_c)^{-1} X_c Y_c$. The least squares estimator has large sample proportions of asymptotically normality and consistency.

In this section, we generalize the crisp class model described above to a class of fuzzy model. The fuzzy multivariate linear regression is given as follows:

$$Y_i = \beta_0 \oplus \beta_1 X_{i1} \oplus \beta_2 X_{i2} \oplus \cdots \oplus \beta_k X_{ik} \oplus \Phi_i, \qquad i = 1, 2, \cdots, n, \tag{2}$$

where $X_{ij} = (x_{ij}, \xi_{ij}^l, \xi_{ij}^r)_T$ $(1 \leqslant j \leqslant k)$, $Y_i = (y_i, \eta_i^l, \eta_i^r)_T$, x_{ij}, y_i represent the modes, ξ_{ij}^l, ξ_{ij}^r are the left and right spread of X_{ij}, η_i^l, η_i^r are the left and right spread of Y_i, respectively. Moreover, $\Phi_i = (\epsilon_i, \theta_i^l, \theta_i^r)_T$ are the fuzzy random errors, in which the modes, left spreads, right spreads ϵ_i, θ_i^l, θ_i^r are crisp random variables.

Model (2) is an extension of the multivariate regression model, where input and output observations are all given by fuzzy numbers. Let $P = \{j \,|\, \beta_j \geqslant 0,\ j = 1, 2, \cdots, k\}$ and $N = \{j \,|\, \beta_j < 0,\ j = 1, 2, \cdots, k\}$, hence, model (2) can be translated into the following form.

$$\begin{cases} y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \epsilon_i, \\ \eta_i^l = \sum\limits_{j \in P} \beta_j \xi_{ij}^l - \sum\limits_{j \in N} \beta_j \xi_{ij}^r + \theta_i^l, \\ \eta_i^r = \sum\limits_{j \in P} \beta_j \xi_{ij}^r - \sum\limits_{j \in N} \beta_j \xi_{ij}^l + \theta_i^r. \end{cases} \tag{3}$$

The optimal solution of fuzzy multivariate linear regression model is obtained by minimizing the residual sum of squares in the least squares senses.

We will use a modified metric d_H which is defined in the work of Kim et al.[21]:

$$\begin{aligned} Q(\beta_0,\beta_1,\cdots,\beta_k) &= \sum_{i=1}^{n} d_H^2(Y_i,\beta_0\oplus\beta_1X_{i1}\oplus\beta_2X_{i2}\oplus\cdots\oplus\beta_kX_{ik}) \\ &= \frac{1}{9}\sum_{i=1}^{n}\Big[(3y_i+\eta_i^r-\eta_i^l)-\Big(3\beta_0+3\sum_{j=1}^{k}\beta_jx_{ij}-\sum_{j=1}^{k}\beta_j\xi_{ij}^l+\sum_{j=1}^{k}\beta_j\xi_{ij}^r\Big)\Big]^2. \end{aligned}$$

Next, we introduce matrix to solve this problem by letting

$$Y=\begin{pmatrix} 3y_1-\eta_1^l+\eta_1^r \\ 3y_2-\eta_2^l+\eta_2^r \\ \vdots \\ 3y_n-\eta_n^l+\eta_n^r \end{pmatrix},\ X=\begin{pmatrix} 3 & 3x_{11}-\xi_{11}^l+\xi_{11}^r & \cdots & 3x_{1k}-\xi_{1k}^l+\xi_{1k}^r \\ 3 & 3x_{21}-\xi_{21}^l+\xi_{21}^r & \cdots & 3x_{2k}-\xi_{2k}^l+\xi_{2k}^r \\ \vdots & \vdots & \ddots & \vdots \\ 3 & 3x_{n1}-\xi_{n1}^l+\xi_{n1}^r & \cdots & 3x_{nk}-\xi_{nk}^l+\xi_{nk}^r \end{pmatrix},\ \beta=\begin{pmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_k\end{pmatrix}.$$

So, we rewrite model (3) and the matrix form is

$$Y=X\beta+\epsilon^*, \tag{4}$$

where $\epsilon^*=(\epsilon_1^*,\epsilon_2^*,\cdots,\epsilon_n^*)$, $\epsilon_i^*=3\epsilon_i-\theta_i^l+\theta_i^r$, $1\leqslant i\leqslant n$.

We should minimize the following target with respect to β:

$$Q(\beta)=\frac{1}{9}(Y-X\beta)^\mathsf{T}(Y-X\beta).$$

It is easy to show that its minimum is obtained at the solution to the normal equation:

$$-X^\mathsf{T}Y+X^\mathsf{T}X\beta=0.$$

Obviously, the solution is termed the fuzzy least squares estimators of β. Then

$$\widehat{\beta}_{\mathrm{FLS}}=(X^\mathsf{T}X)^{-1}X^\mathsf{T}Y. \tag{5}$$

Special cases:

(i) If $\xi_{ij}^l=\xi_{ij}^r=\eta_i^l=\eta_i^r=\theta_i^l=\theta_i^r=0$, $1\leqslant i\leqslant n$, then all explanatory variables and response variable are crisp. In this situation, we have a particular case of our fuzzy linear regression model: the classical multivariate linear regression model, in fact, $\widehat{\beta}_{\mathrm{FLS}}$ becomes $\breve{\beta}_{\mathrm{LS}}$.

(ii) Notice that as particular case, by applying the equation (5) with $k=1$, we obtain the following fuzzy least squares estimators, thisis confirmed by the work of Kim et al. [21].

$$\widehat{\beta}_1 = \frac{\sum\limits_{i=1}^{n}(3y_i + \eta_i^r - \eta_i^l)[3(x_i - \overline{x}_n) + (\xi_i^r - \overline{\xi}_n^r) - (\xi_i^l - \overline{\xi}_n^l)]}{\sum\limits_{i=1}^{n}[3(x_i - \overline{x}_n) + (\xi_i^r - \overline{\xi}_n^r) - (\xi_i^l - \overline{\xi}_n^l)]^2},$$

$$\widehat{\beta}_0 = (\overline{y}_n - \widehat{\beta}_1\overline{x}_n) + \frac{1}{3}(\overline{\eta}_n^r - \widehat{\beta}_1\overline{\xi}_n^r) - \frac{1}{3}(\overline{\eta}_n^l - \widehat{\beta}_1\overline{\xi}_n^l). \tag{6}$$

§4. Asymptotic Properties

4.1 Asymptotic Normality on the Regression Parameters

Assumption

(A1) ϵ_i are i.i.d. random variables with $\mathsf{E}(\epsilon_i) = 0$, $\mathsf{Var}\,(\epsilon_i) = \sigma_\epsilon^2$ $(< \infty)$.

(A2) θ_i^l, θ_i^r are i.i.d. random variables with $\mathsf{E}(\theta_i^l) = \mathsf{E}(\theta_i^r) = \mu_i$ $(\geqslant 0)$, $\mathsf{Var}\,(\theta_i^l) = \sigma_l^2$ $(< \infty)$, $\mathsf{Var}\,(\theta_i^r) = \sigma_r^2$ $(< \infty)$.

(A3) ϵ_i, θ_i^l and θ_i^r are mutually independent.

(A4) $\lim\limits_{n\to\infty} Q_n = Q$, where $Q_n = n(X^\mathsf{T}X)^{-1}_{(k+1)\times(k+1)}$, and Q is $(k+1)$ order positive definite matrix.

(A5) $\sum\limits_{i=1}^{n}(3x_{ij} + \xi_{ij}^r - \xi_{ij}^l)^2/i^2 < \infty$ as $n \to \infty$, $j = 1, 2, \cdots, k$.

Theorem 9 Suppose that Assumption (A1)−(A4) are fulfilled, then the fuzzy least squares estimator $\widehat{\beta}_{\mathrm{FLS}}$ has an asymptotically normal distribution in the sense that

$$\sqrt{n}\,(\widehat{\beta}_{\mathrm{FLS}} - \beta) \xrightarrow{L} \mathrm{N}(0, \sigma_{\epsilon^*}^2 Q).$$

Proof From Assumption (A1)−(A3), we can conclude that ϵ_i^* are i.i.d. random variables with $\mathsf{E}(\epsilon_i^*) = 0$, $\mathsf{Var}\,(\epsilon_i^*) = 9\sigma_\epsilon^2 + \sigma_l^2 + \sigma_r^2 \triangleq \sigma_{\epsilon^*}^2$.

From $Y = X\beta + \epsilon^*$ and (5), we obtain

$$\widehat{\beta}_{\mathrm{FLS}} - \beta = (X^\mathsf{T}X)^{-1}X^\mathsf{T}\epsilon^*.$$

Let $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_k)^\mathsf{T} \in R^{k+1}$, $\lambda \neq 0$, and we have

$$\lambda^\mathsf{T}\sqrt{n}\,(\widehat{\beta}_{\mathrm{FLS}} - \beta) = \lambda^\mathsf{T}\sqrt{n}\,(X^\mathsf{T}X)^{-1}X^\mathsf{T}\epsilon^*.$$

Then, let

$$e_n^\mathsf{T} = \lambda^\mathsf{T}\sqrt{n}\,(X^\mathsf{T}X)^{-1}X^\mathsf{T} = (e_{n1}, e_{n2}, \cdots, e_{nn}),$$

which implies that

$$\lambda^{\mathsf{T}}\sqrt{n}\,(\widehat{\beta}_{\mathrm{FLS}}-\beta)=e_n^{\mathsf{T}}\epsilon^*=\sum_{i=1}^{n}(e_{ni}\epsilon_i^*).$$

Because $e_n^{\mathsf{T}}=(1/\sqrt{n})\lambda^{\mathsf{T}}Q_nX^{\mathsf{T}}$,

$$e_{ni}^2=\frac{1}{n}\lambda^{\mathsf{T}}Q_n\begin{pmatrix}3\\3x_{i1}-\xi_{i1}^l+\xi_{i1}^r\\3x_{i2}-\xi_{i2}^l+\xi_{i2}^r\\\vdots\\3x_{ik}-\xi_{ik}^l+\xi_{ik}^r\end{pmatrix}$$
$$\times\begin{pmatrix}3 & 3x_{i1}-\xi_{i1}^l+\xi_{i1}^r & 3x_{i2}-\xi_{i2}^l+\xi_{i2}^r & \cdots & 3x_{ik}-\xi_{ik}^l+\xi_{ik}^r\end{pmatrix}Q_n\lambda,$$

which means that $\lim\limits_{n\to\infty}\max\limits_{1\leqslant i\leqslant n}e_{ni}^2=0$. In addition,

$$\sum_{i=1}^{n}e_{ni}^2=e_n^{\mathsf{T}}e_n=\lambda^{\mathsf{T}}Q_n\lambda,$$

so that

$$\lim_{n\to\infty}\Big[\Big(\max_{1\leqslant i\leqslant n}e_{ni}^2\Big)\Big/\Big(\sum_{i=1}^{n}e_{ni}^2\Big)\Big]=0.$$

It follows from Theorem 4 that

$$\Big(\sum_{i=1}^{n}e_{ni}\epsilon_i^*\Big)\Big/\Big(\sigma_{\epsilon^*}^2\sum_{i=1}^{n}e_{ni}^2\Big)^{1/2}\xrightarrow{L}\mathrm{N}(0,1),$$

using (A4) and Theorem 5, we obtain

$$\Big(\sum_{i=1}^{n}e_{ni}\epsilon_i^*\Big)\Big/\big(\sigma_{\epsilon^*}\sqrt{\lambda^{\mathsf{T}}Q\lambda}\,\big)\xrightarrow{L}\mathrm{N}(0,1),$$

so that

$$\lambda^{\mathsf{T}}\sqrt{n}\,(\widehat{\beta}_{\mathrm{FLS}}-\beta)\xrightarrow{L}\mathrm{N}(0,\lambda^{\mathsf{T}}Q\lambda\sigma_{\epsilon^*}^2).$$

Therefore, the proof is completed. □

It is mentioned that when the proposed model contains one explanatory variable, matrix Q can be reduced to matrix $\sum$ given in [21]. For this, it is easy to check that the proposed theorem is an extension of the corresponding result in [21].

4.2 Consistency on the Regression Parameters

Theorem 10 Suppose that Assumption (A1)−(A4) are fulfilled, then $\widehat{\beta}_{\mathrm{FLS}}$ is weakly consistent for β, that is

$$\widehat{\beta}_{\mathrm{FLS}}\xrightarrow{\mathsf{P}}\beta.$$

Proof From Definition 6, $\sqrt{n}\,(\widehat{\beta}_{\rm FLS}-\beta)$ is random variable sequences and bounded in probability.

From Theorem 7, as $n\to\infty$, $\widehat{\beta}_{\rm FLS}-\beta\xrightarrow{\mathsf{P}}0$. □

Theorem 11 Suppose that Assumption (A1)–(A5) are fulfilled, then $\widehat{\beta}_{\rm FLS}$ is strongly consistent for β, that is

$$\widehat{\beta}_{\rm FLS}\xrightarrow{\rm a.s.}\beta.$$

Proof

$$\widehat{\beta}_{\rm FLS}-\beta=Q_n\frac{X^{\mathsf{T}}\epsilon^*}{n}=Q_n\cdot\begin{pmatrix}3\sum\limits_{i=1}^{n}\dfrac{\epsilon_i^*}{n}\\ \sum\limits_{i=1}^{n}\dfrac{(3x_{i1}-\xi_{i1}^l+\xi_{i1}^r)\epsilon_i^*}{n}\\ \sum\limits_{i=1}^{n}\dfrac{(3x_{i2}-\xi_{i2}^l+\xi_{i2}^r)\epsilon_i^*}{n}\\ \vdots\\ \sum\limits_{i=1}^{n}\dfrac{(3x_{ik}-\xi_{ik}^l+\xi_{ik}^r)\epsilon_i^*}{n}\end{pmatrix}\triangleq Q_n\cdot B.$$

From Assumption (A4), we will approve that all the components of vector B converge to 0 a.s. For the first component, owning to strong law of large numbers, we note that $\overline{\epsilon}_n^*\xrightarrow{\rm a.s.}0$.

Let $T_{n,j}=\sum\limits_{i=1}^{n}(3x_{ij}-\xi_{ij}^l+\xi_{ij}^r)\epsilon_i^*$, then for $n\geqslant1$,

$$T_{(n+1),j}=T_{n,j}+(3x_{n+1,j}-\xi_{n+1,j}^l+\xi_{n+1,j}^r)\epsilon_{n+1}^*,$$

so that

$$\mathsf{E}(T_{n+1,j}\,|\,\epsilon_1^*,\epsilon_2^*,\cdots,\epsilon_n^*)=T_{n,j}.$$

which indicates that $\{T_{n,j}\}$ is a martingale.

In addition, for $k\geqslant1$,

$$\begin{aligned}&\mathsf{E}|(3x_{kj}+\xi_{kj}^r-\xi_{kj}^l)\epsilon_k^*|=|(3x_{kj}+\xi_{kj}^r-\xi_{kj}^l)|\cdot|\mathsf{E}(\epsilon_k^*)|=0<\infty,\\ &\mathsf{E}|(3x_{kj}+\xi_{kj}^r-\xi_{kj}^l)\epsilon_k^*|^2=(3x_{kj}+\xi_{kj}^r-\xi_{kj}^l)\cdot\sigma_{\epsilon^*}^2+[\mathsf{E}|(3x_{kj}+\xi_{kj}^r-\xi_{kj}^l)\epsilon_k^*|]^2<\infty,\end{aligned}$$

so for $1\leqslant p\leqslant2$,

$$\mathsf{E}|(3x_{kj}-\xi_{kj}^l+\xi_{kj}^r)\epsilon_k^*|^p<\infty.$$

Let $c_i=i$, as $n\to\infty$ using Assumption (A5),

$$\sum_{i=1}^{n}\frac{\mathsf{E}|(3x_{ij}-\xi_{ij}^l+\xi_{ij}^r)\epsilon_i^*|^2}{c_i^2}=\sigma_{\epsilon^*}^2\sum_{i=1}^{n}\frac{(3x_{ij}+\xi_{ij}^r-\xi_{ij}^l)^2}{c_i^2}<\infty.$$

Thus, $T_{n,j}/c_n$ converge to 0 a.s. by Theorem 8, which accomplishes the proof of the theorem. □

4.3 Confidence Region and Asymptotic Relative Efficiency

Theorem 12 Under the conditions of Theorem 9,

$$G_n(\widehat{\beta}_{\rm FLS}) = \frac{n}{\sigma^2_{\epsilon^*}}(\widehat{\beta}_{\rm FLS}-\beta)^{\mathsf T}Q_n^{-1}(\widehat{\beta}_{\rm FLS}-\beta)$$

has a asymptotically a chi-squared distribution with $(k+1)$ degrees of freedom.

Proof Theorem 12 follows immediately from Theorem 9. □

With reference to the limiting distribution of $G_n(\widehat{\beta}_{\rm FLS})$, we define that

$$C^*_{1-\alpha}(\beta) = \{\beta \,|\, (\widehat{\beta}_{\rm FLS}-\beta)^{\mathsf T}Q_n^{-1}(\widehat{\beta}_{\rm FLS}-\beta) \leqslant \delta^*\}, \tag{7}$$

where $\delta^* = (\sigma^2_{\epsilon^*}/n)\chi^2_{1-\alpha}(k+1)$ and $\sigma^2_{\epsilon^*} = 9\sigma^2_\epsilon + \sigma^2_l + \sigma^2_r$ as before, $\chi^2_{1-\alpha}(k+1)$ is the $(1-\alpha)$th the quantile of chi-squared distribution. Then, for n large $C^*_{0.95}(\beta)$ provides a 95% confidence region for $\widehat{\beta}_{\rm FLS}$ when $\alpha = 0.05$.

Note that under certain regularity conditions, the sequence of crisp least square estimators $\breve{\beta}$ has asymptotically a normal distribution in the sense that

$$\sqrt{n}\,(\breve{\beta}_{\rm LS}-\beta) \xrightarrow{L} {\rm N}(0, \sigma^2_\epsilon V),$$

where σ^2_ϵ is the variance of errors ϵ_i in the model and V is given by

$$V_n = n(X_c^{\mathsf T}X_c)^{-1} \to V$$

as $n \to \infty$, where X_c is $n \times (k+1)$ matrix comprised of the mode of input data. Thus, a $100(1-\alpha)$ percent approximate confidence region based on the crisp least square estimator is denoted by

$$C_{1-\alpha}(\beta) = \{\beta \,|\, (\breve{\beta}_{\rm LS}-\beta)^{\mathsf T}V_n^{-1}(\breve{\beta}_{\rm LS}-\beta) \leqslant \delta\},$$

where $\delta = (\sigma^2_\epsilon/n)\chi^2_{1-\alpha}(k+1)$ and $\sigma^2_\epsilon = \mathsf{Var}\,(\epsilon_i)$ as before, $\chi^2_{1-\alpha}(k+1)$ is the $(1-\alpha)$th the quantile of chi-squared distribution. Then, for n large $C^*_{0.95}(\beta)$ provides a 95% confidence region for $\breve{\beta}_{\rm FLS}$ when $\alpha = 0.05$.

Remark 13 It is worth noting that there may be desirable to make the hypotheses testing on fuzzy regression parameters. In this regard, the conventional procedures concerning the definition and analysis associated with linear restriction on regression parameters have been received in fuzzy environment. It is also easy to develop the test statistics and its asymptotic distribution used for rejecting or accepting the null hypothesis at a specific level of significance in investigating the problem of linearity among the model parameters. More discussion including the problem of extending the restriction to an imprecision setting such as fuzzy non-linearity could be considered in the next studies.

In this case, if we choose $\{\xi^l_{ij}\}$ and $\{\xi^r_{i,j}\}$ in the designed input data where $\xi^l_{i,j} = \xi^r_{i,j}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, k$, which means $\{X_{i,j}\}$ are the symmetric triangular fuzzy input data, then we have

$$Q_n = \frac{1}{9} X_c^{\mathsf{T}} X_c = \frac{1}{9} V_n,$$

so that $Q = V/9$. The asymptotic relative efficiency of the fuzzy least square estimators $\widehat{\beta}_{\rm FLS}$ with respect to the classical crisp least square estimators $\breve{\beta}_{\rm LS}$

$$e(F,C) = \Big(\frac{|\sigma^2_\epsilon V|}{|\sigma^2_{\epsilon^*} Q|}\Big)^{1/(k+1)} = \Big(\frac{9^{k+1}\sigma_\epsilon^{2(k+1)}|V|}{\sigma_{\epsilon^*}^{2(k+1)}|Q|}\Big)^{1/(k+1)} = \frac{9\sigma^2_\epsilon}{\sigma^2_{\epsilon^*}} = \frac{9\sigma^2_\epsilon}{9\sigma^2_\epsilon + \sigma^2_l + \sigma^2_r}. \tag{8}$$

Remark 14 This asymptotic relative efficiency in (8), not more than 1, means that the fuzzy least squares estimators are asymptotically less efficient than the crisp least squares estimators, different from the conclusion in [21] which makes it at least less reasonable because fuzzy error term is set inconsistent with other fuzzy variables and the endpoints of fuzzy error term interval are assumed to be normally distributed. It is conforming to human cognition that one fuzzy variable's possibilities are from 0 to 1 and fuzzy least squares estimators possess fuzziness in nature in our work. As it is observed, the amount of asymptotic relative efficiency for the fuzzy least squares estimator is close to 1 provided that the error term has little fuzzy uncertainty.

§5. The Simulation Study

In this section, we investigate the effectiveness of fuzzy multivariate linear regression model through simulations.

5.1 Performance Evaluation for Fuzzy Least Squares Estimators

The performance of fuzzy least squares estimators is evaluated with fuzzy observations through Monte-Carlo simulation under a finite sample. The design for data generation is as follows. To obtain a random sample with n observations, n were selected as the values of 30 for a small sample, 100 for moderate sample, and 500 for a larger sample. We set $k = 2$ in the following simulations, the modes and spreads of input data x_{i1} were chosen as random samples from normal and uniform distributions, $\mathrm{N}(7, 2^2)$, $\mathrm{U}[0, 0.5]$, respectively, and $\mathrm{N}(3,1^2)$, $\mathrm{U}[0,0.4]$ of the input data x_{i2}. Moreover, the modes of fuzzy error terms were normally distributed with zero mean and variance 0.1^2, and the spreads of fuzzy error terms were extracted from a realization of a uniform random variable on the interval, $\mathrm{U}[0,0.05]$. Then the values of the fuzzy dependent variable were computed under the model setting with parameters $\beta = (1.5,2,4)^{\mathrm{T}}$. For a particular sample size n, 1 000 replications of experiments were conducted, thus obtaining 1 000 estimations. The following simulation executed with a program written in Python 3.6. The appropriateness of the estimators of the proposed model is evaluated through the average estimates and average mean squared error (AMSE) over 1 000 simulations, located in Table 1. As predicted, the results in Table 1 show that the fuzzy least squares estimators perform well because of their smaller AMSE (the difference between the true values and the estimates) for a larger sample. That is, the influences of changes in the number of a sample are trivial. This implies that regression parameters in the fuzzy multivariate regression model can be effectively estimated as in one single predictor case, provided that the sample obtained is large enough.

Table 1 Estimation results and AMSE under different samples

	$n = 30$	$n = 100$	$n = 500$
$\widehat{\beta}_0$	1.5039	1.5001	1.5000
$\widehat{\beta}_1$	1.9999	1.9999	2.0000
$\widehat{\beta}_2$	3.9999	3.9999	3.9999
AMSE	0.236×10^{-4}	0.612×10^{-5}	0.116×10^{-5}

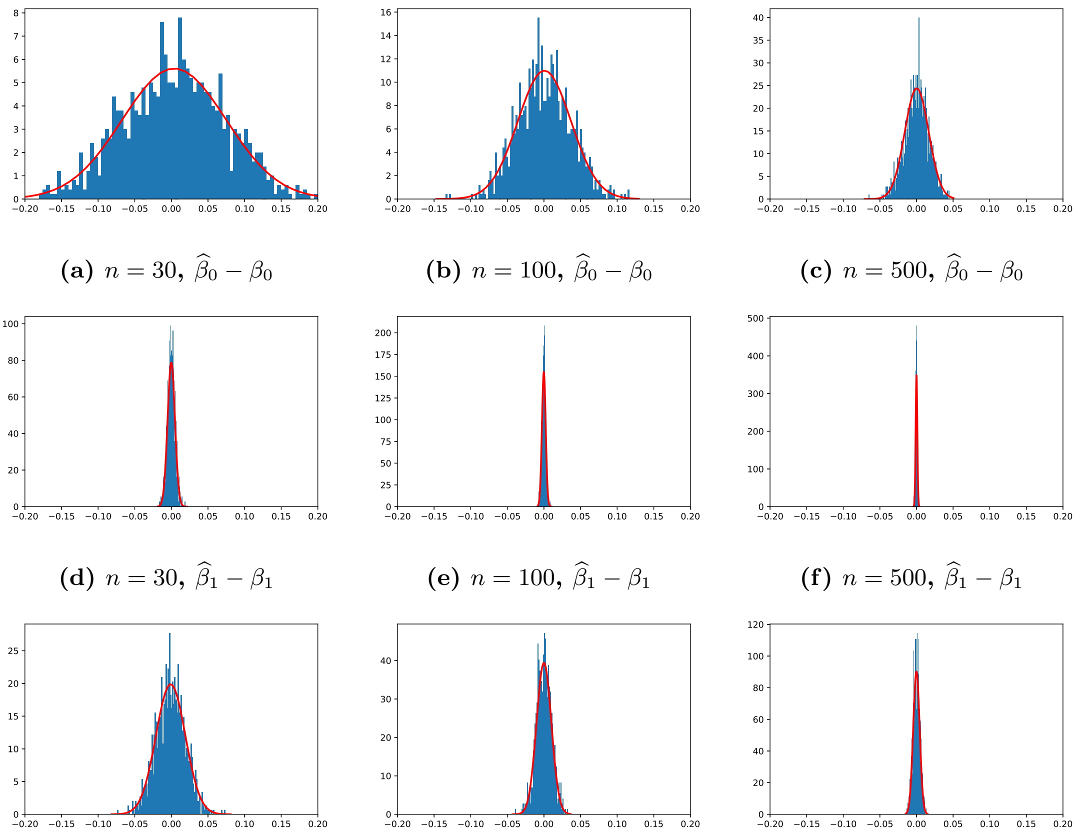
Furthermore, the behavior of the estimators are also examined in Table 2, based on 1 000 replicates through the error distribution of the estimators: the mean, the standard deviation, the minimum and maximum from the 1 000 estimation errors and the various percentiles of the 1 000 estimation errors, first quartile, median, third quartile, 95% point. It is apparent that the larger the sample size, the smaller the error of the estimates. Table 2 shows that the effects of sample size on some factors, such as maximum, mean, percentiles, are fairly significant. A means to improve the precision of estimators in models is to raise the sample size. With the sample size increased to 100 or even more, the variability of the fuzzy least squares estimators diminish remarkably.

The differences between the estimates and true values of regression parameters under three different sample sizes considered for comparison are shown in Figure 2. If the sample size keeps substantial, such as $n = 100$ or even more, the estimates performs better by providing more accurate but less variance and better fitted models. Moreover, it seems that the differences between the estimates and true values of regression parameters further

Table 2 Descriptive statistics corresponding to estimate errors

n	Mean	Std	Minimum	First quartile	Median	Third quartile	95% point	Maximum
$\lvert\widehat{\beta}_0-\beta_0\rvert$								
30	0.0568	0.0430	0.0000	0.0216	0.0487	0.0816	0.1351	0.2749
100	0.0291	0.0217	0.0000	0.0118	0.0246	0.0425	0.0692	0.1344
500	0.0129	0.0100	0.0000	0.0048	0.0107	0.0190	0.0331	0.0652
$\lvert\widehat{\beta}_1-\beta_1\rvert$								
30	0.0040	0.0031	0.0000	0.0016	0.0033	0.0056	0.0103	0.0197
100	0.0020	0.0016	0.0000	0.0007	0.0016	0.0030	0.0052	0.0094
500	0.0009	0.0007	0.0000	0.0003	0.0007	0.0013	0.0023	0.0038
$\lvert\widehat{\beta}_2-\beta_2\rvert$								
30	0.0159	0.0122	0.0000	0.0063	0.0135	0.0229	0.0393	0.0743
100	0.0081	0.0061	0.0000	0.0034	0.0071	0.0113	0.0205	0.0392
500	0.0035	0.0027	0.0000	0.0014	0.0029	0.0050	0.0087	0.0137

demonstrate very close to normal distribution for large data set. These findings in Figure 2 results in a confirmation shown in the previous theoretical analysis.



(a) $n=30$, $\widehat{\beta}_0-\beta_0$ **(b)** $n=100$, $\widehat{\beta}_0-\beta_0$ **(c)** $n=500$, $\widehat{\beta}_0-\beta_0$

(d) $n=30$, $\widehat{\beta}_1-\beta_1$ **(e)** $n=100$, $\widehat{\beta}_1-\beta_1$ **(f)** $n=500$, $\widehat{\beta}_1-\beta_1$

(g) $n=30$, $\widehat{\beta}_2-\beta_2$ **(h)** $n=100$, $\widehat{\beta}_2-\beta_2$ **(i)** $n=500$, $\widehat{\beta}_2-\beta_2$

Figure 2 Distribution diagram of $\widehat{\beta}_i-\beta_i$ under three sample sizes

5.2 Confidence Region of Regression Parameters

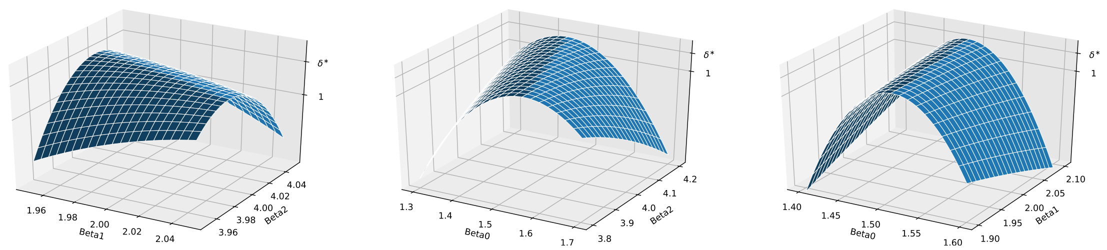
In order to display the confidence region of the fuzzy least estimators in a more visualized way, we decide to use formula (7) to draw picture of it. But, it is difficult demonstrating a 3-parameters confidence area. So, we express two estimated parameters' confidence region at once, turning

$$G_n(\widehat{\beta}_{\mathrm{FLS}}) = \frac{n}{\sigma_{\epsilon^*}^2}(\widehat{\beta}_{\mathrm{FLS}} - \beta)^{\mathsf{T}} Q_n^{-1} (\widehat{\beta}_{\mathrm{FLS}} - \beta)$$

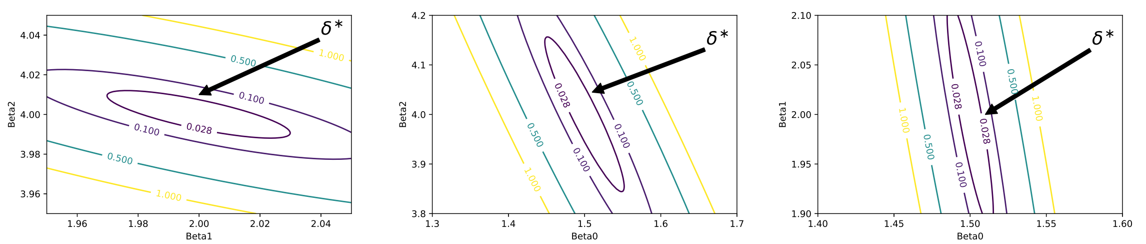
into a two-variable function — keeping the one parameter constant, and draw the function in a picture. For example, when $\beta_0 = 1.5$, the confidence region of β_0 and β_1 expressed in Figure 3 (a). The confidence region means

$$C_{1-\alpha}^*((\beta_1, \beta_2)^{\mathsf{T}}) = \{(\beta_1, \beta_2)^{\mathsf{T}} \,|\, G_n(\widehat{\beta}_{\mathrm{FLS}}) \leqslant \delta^*\},$$

where $\widehat{\beta}_{\mathrm{FLS}} = (1.5, \widehat{\beta}_1, \widehat{\beta}_2)^{\mathsf{T}}$. The surface is the function of G_n, so 95% confidence region is composed by the data below the surface and above δ^*. And its profile line of different value is in the small picture (d). The confidence region is represented by the small area bounded by δ^*.



(a) confidence region for β_1 and β_2 ($\beta_0 = 1.5$) **(b) confidence region for β_0 and β_2 ($\beta_1 = 2$)** **(c) confidence region for β_0 and β_1 ($\beta_2 = 4$)**



(d) profile line of confidence region for β_1 and β_2 **(e) profile line of confidence region for β_0 and β_2** **(f) profile line of confidence region for β_0 and β_1**

Figure 3 Confidence region of regression parameters and its profile line ($\alpha = 0.05$)

In Figure 3, we show the graphical representation of the confidence region of regression parameters along with its profile line ($\alpha = 0.05$). We first investigate the confidence region for regression parameters in which two parameters changes and only the other remain unchanged (see Figure 3 (a), 3 (b), and 3 (c)). Since the position of confidence region is between the curve and horizontal plane at the height δ^*, we convert the direction of the vertical axis for showing a clear illustration. We next mark the profile line of confidence region (see Figure 3 (d), 3 (e), and 3 (f)) to observe the cross-section in the confidence region of regression parameters.

§6. Conclusions

In this paper, we proposed a general framework to obtain a regression model with fuzzy data (both explanatory variables and response) that considers a parametric estimator applying least-squares method in terms of the concept of distance between a few α-cuts, generalizing the discussion of [21] to cope with cases of multivariate explanatory variables. When all fuzzy observations degenerate to one point in their domains, the proposed fuzzy least squares estimators become the conventional least-squares estimators in the crisp environment. Moreover, the proposed fuzzy least-squares estimation procedure is well defined because if there exists only one explanatory variable then our estimates reduce to the estimates in the work of [21].

A simulation example is used to illustrate our estimators and visually demonstrate the confidence region of the regression parameters. Regarding the asymptotic confidence region for the parameters, it is also observed that the proposed estimator is asymptotically less efficient to the crisp least squares estimators, different from the conclusion in [21], which makes it at least more reasonable because the endpoints of fuzzy adjustment term interval are set consistent with other fuzzy variables and fuzzy least squares estimators possess fuzziness in nature in our work. Estimators can be compared only taking into account their properties and importance from the theoretical and practical points of view.

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具有模糊随机误差的回归模型的参数估计

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摘　要: 由于现实世界问题内在的复杂性和可变性, 不同于概率不确定性, 大量模糊不确定信息难以从试验得到其准确的信息, 联合采用统计模型与模糊集理论有助于提高对复杂系统的辨识和分析. 本文针对具有模糊输入、模糊输出和模糊随机误差项的多元线性回归模型, 运用基于模糊数扩张理论的最小二乘法, 研究模型参数的解析表达式. 本文是文献 [21] 模型的推广, 基于模糊数间完备距离得到多元情形下回归模型清晰参数的最小二乘估计. 本文探讨了该估计量的大样本性质、渐近相对效率和回归参数的置信域, 证明了多元情形下估计量的渐近正态性和相合性. 另外, 文中将模糊变量进行统一设定规避了文献 [21] 对模糊随机误差项作端点设定引致的计算不便和负展形问题. 最后, 本文通过随机数值试验模拟来探究两个自变量情形下回归参数模糊最小二乘估计的样本性质和置信域.

关键词: 最小二乘估计; 模糊数; 渐近性质; 模糊随机误差

中图分类号: O212.1