应用概率统计 第 37 卷
第 1 期 2021 年 2 月

Chinese Journal of Applied Probability and Statistics
Feb., 2021, Vol. 37, No. 1, pp. 37-46
doi: 10.3969/j.issn.1001-4268.2021.01.004

Moderate Deviation for the Rightmost Position in a Branching Brownian Motion *

SHI Wanlin

(*School of Mathematics and Physics, North China Electric Power University, Beijing, 102206, China*)

Abstract: We study the moderate deviation probability of the position of the rightmost particle in a branching Brownian motion and obtain its moderate deviation function. Firstly, Chauvin and Rouault studied the large deviation probability for the rightmost position in a branching Brownian motion. Recently, Derrida and Shi considered lower deviation for the same model. By contrast, Our main result is more extensive.

Keywords: branching Brownian motion; moderate deviation probability

2010 Mathematics Subject Classification: 60J80; 60F10

Citation: SHI W L. Moderate deviation for the rightmost position in a branching Brownian motion [J]. Chinese J Appl Probab Statist, 2021, 37(1): 37–46.

§1. Introduction

Branching Brownian motion (BBM, for short) unites two classical continuous-time Markov processes: the random branching and the Brownian motion, or the Winer process. The branching Brownian motion was extensively studied in the past [1–3], and it continues to attach much attention among physicists. Here we consider the branching Brownian motion in one dimension. Suppose that a particle starts at the origin and performs standard Brownian motion with variance σ^2 at time 1, and branches at rate 1 into two independent Brownian motion which themselves branch at rate 1 independently, and so on (see Figure 1). We assume that the exponential random variables and the Brownian motions are independent. One of the issues that matters most is how to determine the distribution of the position $R(t)$ of the rightmost particle of a branching Brownian motion at time t.

*The project was supported by the Fundamental Research Funds for the Central Universities (Grant No. 2020 MS043).

E-mail: shiwanlin@mail.bnu.edu.cn.

Received September 23, 2019. Revised February 4, 2020.

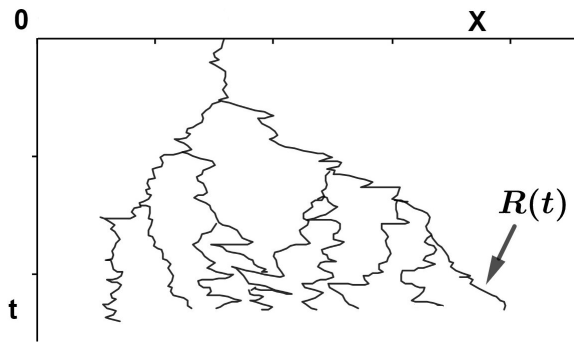

Figure 1 BBM model

The link between BBM and partial differential equations is provided by the following observation due to McKean [3]: if one denotes by

$$u(x,t) = \mathsf{P}(R(t) \leqslant x),$$

the law of the rightmost position, a renewal argument shows that $u(x,t)$ solves the Kolmogorov-Petrovsky-Piscounov or Fisher (F-KPP) equation,

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u \tag{1}$$

with the initial condition $u(x,0) = 1_{\{x \geqslant 0\}}$. The main result of Kolmogorov, Petrovsky and Piscounov has it that this solution settles down to a "traveling wave" with velocity $\sqrt{2}\sigma$ for large t; thus,

$$\lim_{t\to\infty} u(x + m_t\sigma, t) = \omega(x), \qquad \forall\, x \in \mathbb{R}, \tag{2}$$

where

$$\frac{\sigma^2}{2}\omega'' + \sqrt{2\sigma^2}\omega' + \omega^2 - \omega = 0$$

and

$$\sigma m_t = \mathrm{med} R(t) \sim \sqrt{2}\sigma t, \tag{3}$$

where $\mathrm{med}(X) = \sup\{x : \mathsf{P}(X \leqslant x) \leqslant 1/2\}$ is the median of the random variable X. In (3) and everywhere below, the symbol $\sim$ means that $\lim\limits_{t\to\infty} m_t/(\sqrt{2}t) = 1$. By exploiting the connection between the branching Brownian motion and the KPP equation, Bramson [1] showed that the centering term m_t satisfies $m_t = \sqrt{2}t - (3/2\sqrt{2})\ln t + O(1)$, and by using somewhat different techniques based on the Feynman-Kac formula, Bramson [2] improved this by showing that

$$m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\ln t + \text{constant} + o(1).$$

Moreover, Bramson [1,2] imply that

$$\lim_{t\to\infty} \frac{R(t)}{t} = \sqrt{2\sigma^2}, \qquad \text{a.s..}$$

For an account of general properties of BBM, see [4].

For the rightmost position of branching Brownian motion, Chauvin and Rouault [5,6] first studied the large deviation probability. Recently, Derrida and Shi [7,8] considered both the large deviation and lower deviation. Here we are interested in its moderate deviation probability, in other words, the convergence rate of

$$\mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t), \tag{4}$$

for any positive function $t \mapsto \ell_t$ on $[0,\infty)$ such that $\lim\limits_{t\to\infty} \ell_t/t = \ell^* \in [0,\infty)$.

Let us introduce some notations. Let $\mathscr{N}(t)$ denote the set of all particles alive at time t and let $N(t) := \#\mathscr{N}(t)$. For any $v \in \mathscr{N}(t)$ let $X_v(t)$ be the position of v at time t; and for any $s < t$, let $X_v(s)$ be the position of the unique ancestor of v that was alive at time s. We define

$$R(t) := \max_{u\in\mathscr{N}(t)} X_u(t),$$

which stands for the rightmost position of branching Brownian motion. For any $s > 0$ and each particle $u \in \mathscr{N}(s)$, the shifted subtree generated by u is

$$\mathscr{N}^u(t) := \{v \in \mathscr{N}(t+s),\, u \preceq v\}, \qquad \forall\, t \geqslant 0,$$

where $u \preceq v$ indicates that v is a descendant of u or is u itself. Further, for any $v \in \mathscr{N}^u(t)$, let

$$X_v^u(t) := X_v(t+s) - X_u(s),$$

be its shifted position. Similarly, we set $R^u(t) := \max_{v\in\mathscr{N}^u(t)}$.

For notational simplification, we write $m_t := \sqrt{2}t - (3/2\sqrt{2})\ln t$ in the rest of this article. Below is our result, which indicates that the moderate deviation probability of the position of the rightmost particle in a branching Brownian motion.

Theorem 1 Let $R(t)$ denote the rightmost position of the BBM at time t. Then for any positive function $t \mapsto \ell_t$ on $[0,\infty)$ such that $\ell_t \uparrow \infty$ and that $\ell^* := \lim\limits_{t\to\infty} \ell_t/t$ exists with $\ell^* \in [0,\infty)$, we have

$$\lim_{t\to\infty} \frac{1}{\ell_t} \ln \mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t) = -\varphi(\ell^*),$$

where

$$\varphi(\ell^*) := \begin{cases} \dfrac{1}{\sigma}(2-\sqrt{2}), & 0 \leqslant \ell^* \leqslant 2\sigma; \\[2mm] \dfrac{2}{\ell^*} + \dfrac{\ell^*}{2\sigma^2} - \dfrac{\sqrt{2}}{\sigma}, & \ell^* \geqslant 2\sigma. \end{cases} \tag{5}$$

Remark 2 Specifically, we take $\ell_t = (1-\alpha)\sqrt{2\sigma^2}t$ in Theorem 1, then we can obtain Theorem 1 in [8].

Remark 3 From (5), we know that $\ell^* \mapsto \varphi(\ell^*)$ is a continuous function on $[0,\infty)$. Set $\sigma = 1$. The following is the functional graph of φ (see Figure 2).

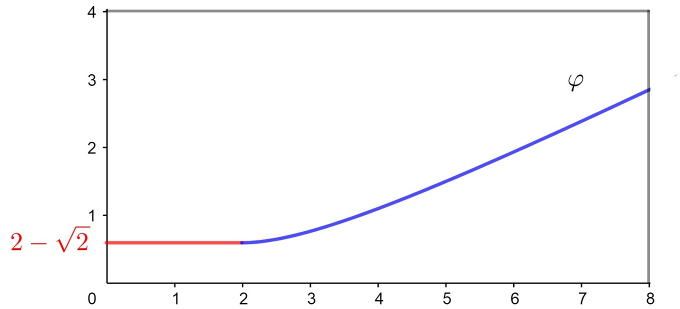

Figure 2 The moderate deviation function of the position of the rightmost particle of a branching Brownian motion

The rest of the article is organized as follows. In Section 2 and Section 3, we prove the lower bound and the upper bound of Theorem 1, respectively. Let c denote positive constant which might change from line to line. As usual, we write $f(t) = o(1)g(t)$ if $f(t)/g(t)$ converges to 0 as t tends to ∞. $f(t) = O(g(t))$ means that $f(t) \leqslant Cg(t)$ for some C.

§2. Lower Bound

For the lower bound, our arguments are largely inspired by [8].

We prove the lower bound in the deviation probability, by considering a special event described as follows: The initial particle does not produce any offspring during time interval $[0,\tau]$ and is positioned at

$$y \in \Big(-\infty, \sigma m_t - \ell_t - \sqrt{2\sigma^2}(t-\tau) + \frac{3}{2\sqrt{2}}\sigma\ln(t-\tau) - 1\Big]$$

at time τ; then, at time t, the maximal position lies in $(-\infty, \sigma m_t - \ell_t)$. Let $\tau := a\ell_t \in (0,t]$, so $a \in (0, 1/\ell^*)$ for sufficiently large t. Therefore, we have

$$\begin{aligned}\mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t) \geqslant \mathrm{e}^{-a\ell_t}\int_{-\infty}^{A_t} &\frac{\mathrm{d}y}{\sqrt{2\pi\sigma^2 a\ell_t}}\mathrm{e}^{-y^2/(2\sigma^2 a\ell_t)} \\ &\times \mathsf{P}(R(t-a\ell_t) \leqslant \sigma m_t - \ell_t - y),\end{aligned} \tag{6}$$

where

$$A_t := \sigma m_t - \ell_t - \sqrt{2\sigma^2}(t - a\ell_t) + \frac{3}{2\sqrt{2}}\sigma \ln(t - a\ell_t) - 1 \leqslant 0.$$

It remains to bound the above probability. Note that for

$$y \in \Big(-\infty, \sigma m_t - \ell_t - \sqrt{2\sigma^2}(t - a\ell_t) + \frac{3}{2\sqrt{2}}\sigma \ln(t - a\ell_t) - 1\Big],$$

we obtain

$$\sigma m_t - \ell_t - y \geqslant \sqrt{2\sigma^2}(t - a\ell_t) - \frac{3}{2\sqrt{2}}\sigma \ln(t - a\ell_t) + 1.$$

Hence,

$$\mathsf{P}(R(t - a\ell_t) \leqslant \sigma m_t - \ell_t - y) \geqslant \mathsf{P}\Big(R(t - a\ell_t) \leqslant \sqrt{2\sigma^2}(t - a\ell_t) - \frac{3}{2\sqrt{2}}\sigma \ln(t - a\ell_t) + 1\Big).$$

Recall that $m_t := \sqrt{2}t - (3/2\sqrt{2}) \ln t$. Using (2), for any $z \in \mathbb{R}$, $\mathsf{P}(R(s) \leqslant m_s\sigma + z)$ converges, as $s \to \infty$, to a positive limit. This yields that there exists a constant $c > 0$ such that

$$\mathsf{P}\Big(R(t - a\ell_t) \leqslant \sqrt{2\sigma^2}(t - a\ell_t) - \frac{3\sigma}{2\sqrt{2}} \ln(t - a\ell_t) + 1\Big) \geqslant c. \tag{7}$$

Combining (6) with (7), we get that for all $\tau \in (0, t]$,

$$\mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t) \geqslant c\mathrm{e}^{-a\ell_t} \int_{-\infty}^{A_t} \frac{\mathrm{d}y}{\sqrt{2\pi\sigma^2 a\ell_t}} \mathrm{e}^{-y^2/(2\sigma^2 a\ell_t)}. \tag{8}$$

Let $\varepsilon > 0$ be small constant such that $A_t \geqslant (\sqrt{2\sigma^2}a - 1 - \varepsilon)\ell_t$. As a consequence,

$$\begin{aligned}
\mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t) &\geqslant c\mathrm{e}^{-a\ell_t} \int_{-\infty}^{(\sqrt{2\sigma^2}a-1-\varepsilon)\ell_t} \frac{\mathrm{d}y}{\sqrt{2\pi\sigma^2 a\ell_t}} \mathrm{e}^{-y^2/(2\sigma^2 a\ell_t)} \\
&\geqslant c\Big[\mathrm{e}^{-a\ell_t} \frac{-(\sqrt{2\sigma^2}a - 1 - \varepsilon)\ell_t}{1 + (\sqrt{2\sigma^2}a - 1 - \varepsilon)^2\ell_t^2} \frac{1}{\sqrt{2\pi\sigma^2 a\ell_t}} \mathrm{e}^{-(\sqrt{2\sigma^2}a-1-\varepsilon)^2\ell_t/(2\sigma^2 a)}\Big] \\
&= c\Big[\frac{-(\sqrt{2\sigma^2}a - 1 - \varepsilon)\ell_t}{1 + (\sqrt{2\sigma^2}a - 1 - \varepsilon)^2\ell_t^2} \frac{1}{\sqrt{2\pi\sigma^2 a\ell_t}} \mathrm{e}^{-(\sqrt{2\sigma^2}a-1-\varepsilon)^2\ell_t/(2\sigma^2 a)-a\ell_t}\Big],
\end{aligned} \tag{9}$$

where the second inequality follows from the standard Gaussian tail estimate

$$\int_u^\infty \mathrm{e}^{-x^2/2} \geqslant u(1 + u^2)^{-1}\mathrm{e}^{-u^2/2}.$$

Taking the logarithm and the lower limit of both sides in (9), one has

$$\liminf_{t\to\infty} \frac{1}{\ell_t} \ln \mathsf{P}(R(t) \leqslant \sigma m(t) - \ell_t) \geqslant -a - \frac{(\sqrt{2\sigma^2}a - 1 - \varepsilon)^2}{2\sigma^2 a}.$$

Together with the fact that r.h.s. is independent of a, we obtain

$$\liminf_{t\to\infty}\frac{1}{\ell_t}\ln \mathsf{P}(R(t)\leqslant \sigma m(t)-\ell_t)\geqslant \sup_{a\in(0,1/(\ell^*\vee\sqrt{2}\sigma)]}\Big\{-a-\frac{(\sqrt{2\sigma^2}a-1-\varepsilon)^2}{2\sigma^2 a}\Big\}. \tag{10}$$

To calculate the above supremum, we just need to consider the minimum of the function $g(a;\varepsilon):=a+(\sqrt{2\sigma^2}a-1-\varepsilon)^2/(2\sigma^2 a)$, with $a\in(0,1/(\ell^*\vee\sqrt{2}\sigma)]$. By elementary calculation, we obtain the minimum of the function g as follows:

$$g_{\min}(\ell^*;\varepsilon):=\begin{cases}\dfrac{(2-\sqrt{2})(1-\varepsilon)}{\sigma}, & 0\leqslant \ell^*\leqslant \dfrac{2\sigma}{\varepsilon+1};\\ \dfrac{2}{\ell^*}+\dfrac{(1+\varepsilon)\ell^*}{2\sigma^2}-\dfrac{\sqrt{2}(1-\varepsilon)}{\sigma}, & \ell^*\geqslant \dfrac{2\sigma}{\varepsilon+1}.\end{cases} \tag{11}$$

Substituting this into (10) yields that

$$\liminf_{t\to\infty}\frac{1}{\ell_t}\ln \mathsf{P}(R(t)\leqslant \sigma m(t)-\ell_t)\geqslant -g_{\min}(\ell^*;\varepsilon). \tag{12}$$

Letting $\varepsilon\downarrow 0$ gives that

$$\liminf_{t\to\infty}\frac{1}{\ell_t}\ln \mathsf{P}(R(t)\leqslant \sigma m(t)-\ell_t)\geqslant -\varphi(\ell^*), \tag{13}$$

where

$$\varphi(\ell^*):=\begin{cases}\dfrac{1}{\sigma}(2-\sqrt{2}), & 0\leqslant \ell^*\leqslant 2\sigma;\\ \dfrac{2}{\ell^*}+\dfrac{\ell^*}{2\sigma^2}-\dfrac{\sqrt{2}}{\sigma}, & \ell^*\geqslant 2\sigma.\end{cases} \tag{14}$$

§3. Upper Bound

For the upper bound, the proof strategy is motivated by [9].

Let

$$T_t=\inf\{a\geqslant 0: N(a\ell_t)\geqslant \ell_t^2\}$$

and for $\delta>0$ and $\varepsilon>0$ small enough set

$$F(\delta)=\Big\{\delta,2\delta,\cdots,\Big\lceil\frac{1}{\delta(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)}\Big\rceil\delta\Big\}.$$

According to the definition of T_t, we split the probability $\mathsf{P}(R(t)\leqslant \sigma m_t-\ell_t)$ into two parts as

$$\begin{aligned}\mathsf{P}(R(t)\leqslant \sigma m_t-\ell_t)\leqslant{}& \mathsf{P}\Big(R(t)\leqslant \sigma m_t-\ell_t; N\Big(\frac{\ell_t}{(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)}\Big)\leqslant \ell_t^2\Big)\\ &+\mathsf{P}\Big(R(t)\leqslant \sigma m_t-\ell_t; N\Big(\frac{\ell_t}{(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)}\Big)\geqslant \ell_t^2\Big)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathsf{P}\Big(N\Big(\frac{\ell_t}{(\sqrt{2}\sigma \vee \ell^*)(1+2\varepsilon)}\Big) \leqslant \ell_t^2\Big) \\
&\quad + \mathsf{P}\Big(R(t) \leqslant \sigma m_t - \ell_t; N\Big(\frac{\ell_t}{(\sqrt{2}\sigma \vee \ell^*)(1+2\varepsilon)}\Big) \geqslant \ell_t^2\Big) \\
&\leqslant \mathsf{P}\Big(N\Big(\frac{\ell_t}{(\sqrt{2}\sigma \vee \ell^*)(1+2\varepsilon)}\Big) \leqslant \ell_t^2\Big) \\
&\quad + \sum_{a\in F(\delta)} \mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t; T_t \in (a-\delta, a]). \qquad (15)
\end{aligned}
$$

In view of [3], we know that $N(t)$ follows the geometric distribution with parameter e^{-t}. That is to say, $\mathsf{P}(N(t)=k) = \mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{k-1}$, $k=1,2,\cdots$. Consequently,

$$
\begin{aligned}
\mathsf{P}\Big(N\Big(\frac{\ell_t}{(\sqrt{2}\sigma \vee \ell^*)(1+2\varepsilon)}\Big) \leqslant \ell_t^2\Big) &= \sum_{k=1}^{\lfloor \ell_t^2 \rfloor} \mathsf{P}\Big(N\Big(\frac{\ell_t}{(\sqrt{2}\sigma \vee \ell^*)(1+2\varepsilon)}\Big) = k\Big) \\
&= \sum_{k=1}^{\lfloor \ell_t^2 \rfloor} \mathrm{e}^{-\ell_t/[(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)]}\big(1-\mathrm{e}^{-\ell_t/[(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)]}\big)^{k-1} \\
&\sim 1 - \big(1-\mathrm{e}^{-\ell_t/[(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)]}\big)^{\ell_t^2}. \qquad (16)
\end{aligned}
$$

By means of (16), we have

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{\ell_t} \ln \mathsf{P}\Big(N\Big(\frac{\ell_t}{(\sqrt{2}\sigma \vee \ell^*)(1+2\varepsilon)}\Big) \leqslant \ell_t^2\Big) &= \lim_{t\to\infty} \frac{1}{\ell_t} \ln\big(1-\big(1-\mathrm{e}^{-\ell_t/[(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)]}\big)^{\ell_t^2}\big) \\
&= -\frac{1}{(\sqrt{2}\sigma\vee\ell^*)(1+2\varepsilon)} \qquad (17)
\end{aligned}
$$

and

$$
\limsup_{t\to\infty} \frac{1}{\ell_t} \ln \mathsf{P}(T_t \in (a-\delta, a]) \leqslant \limsup_{t\to\infty} \frac{1}{\ell_t} \ln \mathsf{P}(N((a-\delta)\ell_t) \leqslant \ell_t^2) = -(a-\delta). \qquad (18)
$$

Meanwhile,

$$
\begin{aligned}
&\mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t \,|\, T_t \in (a-\delta, a]) \\
&= \mathsf{P}\Big(\max_{u\in\mathscr{N}(a\ell_t)} X_u(a\ell_t) + R^u(t-a\ell_t) \leqslant \sigma m_t - \ell_t \,\Big|\, T_t \in (a-\delta, a]\Big) \\
&\leqslant \mathsf{P}\Big(\max_{u\in\mathscr{N}(a\ell_t)} B_{a\ell_t} + R^u(t-a\ell_t) \leqslant \sigma m_t - \ell_t \,\Big|\, T_t \in (a-\delta, a]\Big) \\
&\leqslant \mathsf{P}(B_{a\ell_t} \leqslant \sigma m_t - (1-\varepsilon)\ell_t - m_{t-a\ell_t}) \\
&\quad + \mathsf{P}\Big(\max_{u\in\mathscr{N}(a\ell_t)} R^u(t-a\ell_t) \leqslant m_{t-a\ell_t} - \varepsilon\ell_t \,\Big|\, T_t \in (a-\delta, a]\Big) \\
&=: I_1 + I_2, \qquad (19)
\end{aligned}
$$

where in the first inequality we use the fact that $X_u(\cdot)$ and $R^u(\cdot)$ are independent and B_t is one-dimensional standard Brownian motion. Firstly, We estimate I_1. For any $a \in F(\delta)$, one can check that $\sqrt{2}\sigma a-1+\varepsilon<0$ and

$$\sigma m_t-(1-\varepsilon)\ell_t-m_{t-a\ell_t}=\frac{3}{2\sqrt{2}}\sigma\ln\frac{t-a\ell_t}{t}+(\sqrt{2}\sigma a-1+\varepsilon)\ell_t\leqslant(\sqrt{2}\sigma a-1+\varepsilon)\ell_t. \tag{20}$$

Thus

$$I_1\leqslant \mathsf{P}(B_{a\ell_t}\leqslant(\sqrt{2}\sigma a-1+\varepsilon)\ell_t). \tag{21}$$

Applying the standard Gaussian tail estimate

$$\int_u^\infty \mathrm{e}^{-x^2/2}\leqslant u^{-1}\mathrm{e}^{-u^2/2},$$

the inequality (21) is bounded from above by

$$\frac{1}{-\sqrt{2}\sigma a+1-\varepsilon}\frac{1}{\sqrt{2\pi a\sigma^2\ell_t}}\exp\Big\{-\frac{(\sqrt{2}\sigma a-1+\varepsilon)^2\ell_t}{2a\sigma^2}\Big\}. \tag{22}$$

By (21) and (22), we have

$$\limsup_{t\to\infty}\frac{1}{\ell_t}\mathsf{P}(B_{a\ell_t}\leqslant\sigma m_t-(1-\varepsilon)\ell_t-\sigma m_{t-a\ell_t})\leqslant-\frac{(\sqrt{2}\sigma a-1+\varepsilon)^2}{2\sigma^2 a}.$$

Next, we need to control I_2.

$$\begin{aligned}I_2&=\mathsf{E}\big(\mathsf{P}(R(t-a\ell_t)\leqslant\sigma m_{t-a\ell_t}-\varepsilon\ell_t)^{N(a\ell_t)}\,\big|\,T_t\in(a-\delta,a]\big)\\&\leqslant\mathsf{P}(R(t-a\ell_t)\leqslant\sigma m_{t-a\ell_t}-\varepsilon\ell_t)^{\ell_t^{3/2}}+\mathsf{P}\big(N(a\ell_t)\leqslant\ell_t^{3/2}\,\big|\,T_t\in(a-\delta,a]\big).\end{aligned}$$

By the definition of T_t and the property of branching Brownian motion, we obtain that

$$\mathsf{P}\big(N(a\ell_t)\leqslant\ell_t^{3/2}\,\big|\,T_t\in(a-\delta,a]\big)=0.$$

According to Bramson [1,2], there exists $c\in(0,\infty]$ such that

$$\lim_{t\to\infty}\mathsf{P}(R(t-a\ell_t)\leqslant\sigma m_{t-a\ell_t}-\varepsilon\ell_t)=\mathrm{e}^{-c}<1.$$

Therefore, $I_2\leqslant\mathrm{e}^{-c_1\ell_t^{3/2}}$.

Putting I_1 into I_2, we have

$$\limsup_{t\to\infty}\frac{1}{\ell_t}\ln\mathsf{P}(R(t)\leqslant\sigma m_t-\ell_t\,|\,T_t\in(a-\delta,a])\leqslant-\frac{(\sqrt{2}\sigma a-1+\varepsilon)^2}{2\sigma^2 a}. \tag{23}$$

Going back to (15), together with (17) and (18), one has

$$\limsup_{t\to\infty}\frac{1}{\ell_t}\ln\mathsf{P}(R(t)\leqslant\sigma m_t-\ell_t)$$

$$\leqslant -\frac{1}{(\sqrt{2}\sigma \vee \ell^*)(1+2\varepsilon)} \vee \sup_{a\in F(\delta)} \Big\{ -(a-\delta) - \frac{(\sqrt{2}\sigma a - 1 + \varepsilon)^2}{2\sigma^2 a} \Big\}.$$

Similar to the argument of the lower bound in (10), we have

$$\limsup_{t\to\infty} \frac{1}{\ell_t} \ln \mathsf{P}(R(t) \leqslant \sigma m_t - \ell_t) \leqslant \sup_{a\in(0,1/(\ell^*\vee\sqrt{2}\sigma))} \Big\{ -a - \frac{(\sqrt{2}\sigma a - 1)^2}{2\sigma^2 a} \Big\} := -\psi(\ell^*),$$

$$\psi(\ell^*) := \begin{cases} \dfrac{1}{\sigma}(2-\sqrt{2}), & 0 \leqslant \ell^* \leqslant 2\sigma; \\ \dfrac{2}{\ell^*} + \dfrac{\ell^*}{2\sigma^2} - \dfrac{\sqrt{2}}{\sigma}, & \ell^* \geqslant 2\sigma, \end{cases} \tag{24}$$

which yields the upper bound for the probability in the Theorem 1 because $\psi(\ell^*)$ coincides with $\varphi(\ell^*)$ given in (5). This completes the proof of the upper bound in Theorem 1.

§4. Remark

All the discussion of Section 2 and Section 3 can be easily generalized to a branching Brownian motion where, at each branching event, the particle branches into m particles instead of 2. For convenience, we take $\sigma^2 = 2$ in the following theorem.

Theorem 4 Let $R(t)$ denote the rightmost position of the BBM at time t. Then for any positive function $t \mapsto \ell_t$ on $[0,\infty)$ such that $\ell_t \uparrow \infty$ and that $\ell^* := \lim_{t\to\infty} \ell_t/t$ exists with $\ell^* \in [0,\infty)$, we have

$$\lim_{t\to\infty} \frac{1}{\ell_t} \ln \mathsf{P}(R(t) \leqslant \sqrt{2} m_t - \ell_t) = -\varphi(\ell^*),$$

where

$$\varphi(\ell^*) := \begin{cases} \sqrt{m} - \sqrt{m-1}, & 0 \leqslant \ell^* \leqslant 2m; \\ \dfrac{m}{\ell^*} + \dfrac{\ell^*}{4} - \sqrt{m-1}, & \ell^* \geqslant 2m. \end{cases} \tag{25}$$

Acknowledgement We would like to thank the referee for carefully reading our paper. Moreover, I am most grateful to my supervisor Professor Hui He for helping me.

References

[1] BRAMSON M D. Maximal displacement of branching Brownian motion [J]. *Comm Pure Appl Math*, 1978, **31(5)**: 531–581.

[2] BRAMSON M D. Convergence of solutions of the Kolmogorov equation to travelling waves [J]. *Mem Amer Math Soc*, 1983, **44(285)**: 1–190.

[3] MCKEAN H P. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov [J]. *Comm Pure Appl Math*, 1975, **28(3)**: 323–331.

[4] BOVIER A. *Gaussian Processes on Trees: From Spin Glasses to Branching Brownian Motion* [M]. Cambridge: Cambridge University Press, 2017.

[5] CHAUVIN B, ROUAULT A. KPP equation and supercritical branching brownian motion in the subcritical speed area. Application to spatial trees [J]. *Probab Theory Related Fields*, 1988, **80(2)**: 299–314.

[6] ROUAULT A. Large deviations and branching processes. Proceedings of the 9th International Summer School on Probability Theory and Mathematical Statistics (Sozopol, 1997) [J]. *Pliska Stud Math Bulgar*, 2000, **13**: 15–38.

[7] DERRIDA B, SHI Z. Slower deviations of the branching Brownian motion and of branching random walks [J]. *J Phys A*, 2017, **50(34)**: 344001, 13 pages.

[8] DERRIDA B, SHI Z. Large deviations for the rightmost position in a branching Brownian motion [M] // PANOV V. (ed.) *Modern Problems of Stochastic Analysis and Statistics: Selected Contributions In Honor of Valentin Konakov. Springer Proceedings in Mathematics & Statistics, Vol. 208.* Cham: Springer, 2017: 303–312.

[9] CHEN X X, HE H. Lower deviation and moderate deviation probabilities for maximum of a branching random walk [OL]. 2018 [2018-07-22]. https://arxiv.org/abs/1807.08263.

分枝布朗运动最右位置的中偏差

石 万 林

(华北电力大学数理学院, 北京, 102206)

摘 要: 我们研究了分枝布朗运动最右粒子位置的中偏差概率, 并且得到了中偏差函数. 首先, Chauvin 和 Rouault 考虑了分枝布朗运动最右位置的大偏差概率. 最近, Derrida 和 Shi 对同样的模型研究了其下偏差. 相比之下, 我们的结果更加广泛.

关键词: 分枝布朗运动; 中偏差概率

中图分类号: O211.4