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A Class of Strong Limit Theorems for Markov Chains in Bi-infinite Random Environment *

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Abstract: In this paper, we prove a strong limit theorem for Markov chains in bi-infinite random environment. As corollary, we obtain the strong law of large numbers for nonhomogeneous Markov chains. Finally, we derive the strong limit theorem of harmonic mean of stochastic transition probabilities for Markov chains in bi-infinite random environment.

Keywords: strong limit theorem; harmonic mean; Markov chains; random environment

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§1. Introduction

Markov chains in random environment (MCRE) is an important topic of stochastic process developed in recent years. Many achievements have been made by predecessors on the research of Markov chains in random environment. Smith and Wilkinson [1] has first proposed the concept of random environment in 1969. Nawrotzki [2,3] has established the foundation of a general theory for MCRE. Cogburn [4–6] has introduced the concept of Markov chains in bi-infinite random environment, and has obtained a series of theorems for Markov chains in random environment, in which the ergodic theorem, the central limit theorem, the relationship between direct convergence and transition function, and the existence of invariant probability measure. Hu [7–9] has studied the equivalence theorems of Markov process in random environment, and proved the existence and uniqueness of q-process in random environment. Liu et al. [10] have obtained the strong limit theorem for the average of ternary functions of Markov chains in bi-infinite random environment by constructing an nonnegative martingale. Shi et al. [11,12] has derived the strong law

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of large numbers and the asymptotic equipartition property of Markov chains in single infinite Markovian environment on countable state space.

In this paper, we study the strong limit theorems for Markov chains in bi-infinite random environment and for nonhomogeneous Markov chains. Finally, we derive the limit theorem of harmonic mean of stochastic transition probabilities for Markov chains in bi-infinite Markov chains.

The rest of this paper is organized as follows. In Section 2, some notations and the definition of Markov chains in random environment are introduced. In Section 3, the strong limit theorems for Markov chains in random environment and for nonhomogeneous Markov chains are obtained, and a strong limit theorem of harmonic mean of stochastic transition probabilities for Markov chains in random environment is derived.

§2. Definition and Notations

Let Z be the set of integers and Z_+ be the set of nonnegative integers, and $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space. Let $\mathscr{A}$ be σ-field produced by all subsets of a finite state space $\chi = \{1, 2, \cdots, N\}$, and $(\Theta, \mathscr{B})$ be an arbitrary measurable space. $(\overrightarrow{\chi}, \overrightarrow{\mathscr{A}})$ and $(\overrightarrow{\Theta}, \overrightarrow{\mathscr{B}})$ are two product measurable spaces where $\overrightarrow{\chi} = \chi^{Z_+}$, $\overrightarrow{\mathscr{A}} = \mathscr{A}^{Z_+}$, $\overrightarrow{\Theta} = \Theta^Z$, $\overrightarrow{\mathscr{B}} = \mathscr{B}^Z$. Let $\overrightarrow{\xi} = \{\xi_n,\, n = \cdots, -1, 0, 1, \cdots\}$ and $\overrightarrow{X} = \{X_n,\, n = 0, 1, 2, \cdots\}$ be two collections of random variables on $(\Omega, \mathscr{F}, \mathsf{P})$ taking values in Θ and χ, respectively. $\overrightarrow{\theta} = \{\theta_n,\, \theta_n \in \Theta,\, n \in Z\}$, and $\overrightarrow{\theta}$ ($\overrightarrow{\theta} \in \overrightarrow{\Theta}$) is the realization of $\overrightarrow{\xi}$. Denote $\mathscr{F}_n = \sigma\{X_0, X_1, \cdots, X_n, \overrightarrow{\xi}\}$, $n \geqslant 0$.

Definition 1 Let $\overrightarrow{X} = \{X_n,\, n \geqslant 0\}$ and $\overrightarrow{\xi} = \{\xi_n,\, n = \cdots, -1, 0, 1, \cdots\}$ be two sequences of random variables defined on probability space $(\Omega, \mathscr{F}, \mathsf{P})$ taking values in χ and Θ, respectively, and $P_\theta = \{p(\theta; x, y),\, x, y \in \chi,\, \theta \in \Theta\}$ be transition matrices with parameter θ such that $p(\theta; x, y)$ is measurable on $\mathscr{B}$ for fixed x, y. If $\forall\, x, y \in \chi$, $n \geqslant 0$, we have

$$\mathsf{P}(X_0 = x \mid \overrightarrow{\xi}) = \mathsf{P}(X_0 = x \mid \overrightarrow{\xi}{}_{-\infty}^{0}) \qquad \text{a.e.}, \tag{1}$$

$$\mathsf{P}(X_{n+1} = y \mid \overrightarrow{\xi}, \overrightarrow{X}{}_0^n) = p(\xi_n; X_n, y) \qquad \text{a.e.}, \tag{2}$$

where $\overrightarrow{\xi}{}_{-\infty}^{0} = \{\xi_k,\, k = \cdots, -1, 0\}$ and $\overrightarrow{X}{}_0^n = \{X_k,\, 0 \leqslant k \leqslant n < \infty\}$. Then $\overrightarrow{X}$ will be called Markov chains in bi-infinite random environment $\overrightarrow{\xi}$, or $(\overrightarrow{X}, \overrightarrow{\xi})$ will be called Markov chains in bi-infinite random environment.

Remark 2 If $\overrightarrow{\xi}$ only take a fixed sequence $\overrightarrow{c} = \{c_n,\, n = \cdots, -1, 0, 1, \cdots\}$, then the Markov chains in bi-infinite random environment will be nonhomogeneous Markov chains with transition matrices $P_n = \{p(c_n; x, y),\, n \geqslant 0,\, x, y \in \chi\}$.

§3. Some Strong Limit Theorems and Proofs

In this section, we investigate some strong limit theorems for Markov chains in bi-infinite random environment and for nonhomogeneous Markov chains.

Lemma 3 Let $(\overrightarrow{X}, \overrightarrow{\xi})$ be a Markov chain in bi-infinite random environment defined as Definition 1 with transition matrices P_θ that contains parameter θ, and $g(\overrightarrow{\theta}; x, y)$ be a function taking values in $\overrightarrow{\Theta} \times \chi^2$ such that $g(\cdot; x, y)$ is Borel measurable functions on $\overrightarrow{\mathscr{B}}$ for fixed x, y. Assuming that $\forall\, n \geqslant 1$, $g(\overrightarrow{\xi}; X_{n-1}, X_n)$ is integrable. Then we have

$$\mathsf{E}[g(\overrightarrow{\xi}; X_{n-1}, X_n) \,|\, \mathscr{F}_{n-1}] = \sum_{x_n \in \chi} g(\overrightarrow{\xi}; X_{n-1}, x_n) p(\xi_{n-1}; X_{n-1}, x_n). \tag{3}$$

Proof Let $\overrightarrow{B} = \{B_n,\, B_n \in \mathscr{B},\, n = \cdots, -1, 0, 1, \cdots\}$. By (2), we have

$$\begin{aligned}
&\mathsf{E}\big(I_{\{\overrightarrow{\xi} \in \overrightarrow{B}\}} I_{\{X_{n-1}=x_{n-1}\}} I_{\{X_n = x_n\}} \,|\, \mathscr{F}_{n-1}\big) \\
&= I_{\{\overrightarrow{\xi} \in \overrightarrow{B}\}} I_{\{X_{n-1}=x_{n-1}\}} \mathsf{E}(I_{\{X_n = x_n\}} \,|\, \mathscr{F}_{n-1}) \\
&= I_{\{\overrightarrow{\xi} \in \overrightarrow{B}\}} I_{\{X_{n-1}=x_{n-1}\}} p(\xi_{n-1}; X_{n-1}, x_n).
\end{aligned} \tag{4}$$

By (4), (3) follows. $\square$

Theorem 4 Let $\overrightarrow{X}$ be a Markov chain in bi-infinite random environment $\overrightarrow{\xi}$ with transition matrix $P_\theta = \{p(\theta; x, y),\, \theta \in \Theta,\, x, y \in \chi\}$. Assume that $f_k(\overrightarrow{\theta}; x, y)$ is a family of functions defined on $\overrightarrow{\Theta} \times \chi^2$ such that $f_k(\overrightarrow{\theta}; x, y)$ is measurable on $\overrightarrow{\mathscr{B}}$ for fixed x, y, and for any $k \geqslant 1$, $f_k(\overrightarrow{\xi}; X_{k-1}, X_k)$ is integrable. Write $f_k = f_k(\overrightarrow{\xi}; X_{k-1}, X_k)$. If there exists a constant $b > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}(\mathrm{e}^{b|f_k|} \,|\, \mathscr{F}_{k-1}) \leqslant C_1(\omega) < \infty \qquad \text{a.e.}, \tag{5}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \{f_k(\overrightarrow{\xi}; X_{k-1}, X_k) - \mathsf{E}[f_k(\overrightarrow{\xi}; X_{k-1}, X_k) \,|\, \mathscr{F}_{k-1}]\} = 0 \qquad \text{a.e.} \tag{6}$$

Proof Let r ($|r| \leqslant b$) be a constant. Let $M_0(r) = 1$ and

$$M_n(r) = \frac{\mathrm{e}^{r \sum\limits_{k=1}^{n} f_k(\overrightarrow{\xi}; X_{k-1}, X_k)}}{\prod\limits_{k=1}^{n} \mathsf{E}(\mathrm{e}^{r f_k(\overrightarrow{\xi}; X_{k-1}, X_k)} \,|\, \mathscr{F}_{k-1})}. \tag{7}$$

We will prove that $\{M_n(r)\}$ is a nonnegative martingale. For any $n \geqslant 1$

$$\mathsf{E}[M_n(r) \,|\, \mathscr{F}_{n-1}] = \mathsf{E}\left[\frac{\mathrm{e}^{r \sum\limits_{k=1}^{n} f_k(\overrightarrow{\xi}; X_{k-1}, X_k)}}{\prod\limits_{k=1}^{n} \mathsf{E}(\mathrm{e}^{r f_k(\overrightarrow{\xi}; X_{k-1}, X_k)} \,|\, \mathscr{F}_{k-1})} \,\middle|\, \mathscr{F}_{n-1}\right]$$

$$
\begin{aligned}
&= \frac{\mathrm{e}^{r\sum\limits_{k=1}^{n-1} f_k(\overrightarrow{\xi};X_{k-1},X_k)}}{\prod\limits_{k=1}^{n-1}\mathsf{E}(\mathrm{e}^{rf_k}\mid\mathscr{F}_{k-1})}\frac{\mathsf{E}(\mathrm{e}^{rf_n}\mid\mathscr{F}_{n-1})}{\mathsf{E}(\mathrm{e}^{rf_n}\mid\mathscr{F}_{n-1})}\\
&= M_{n-1}(r) \qquad \text{a.e.} \qquad (8)
\end{aligned}
$$

Noticing that $\mathsf{E}[M_n(r)] = \mathsf{E}[M_{n-1}(r)] = \cdots = \mathsf{E}[M_0(r)] = 1$, thus $\{M_n(r),\, n \geqslant 0\}$ is a nonnegative martingale. And $M_n(r)$ almost convergence to a finite nonnegative random variable $M_\infty(r)$ everywhere by Doob martingale convergence theorem, that is

$$
\lim_{n\to\infty} M_n(r) = M_\infty(r) < \infty \qquad \text{a.e.}, \tag{9}
$$

which implies

$$
\limsup_{n\to\infty}\frac{\ln M_n(r)}{n} \leqslant 0 \qquad \text{a.e.} \tag{10}
$$

By (7) and (10), we have

$$
\limsup_{n\to\infty}\frac{r}{n}\Big[\sum_{k=1}^n f_k - \frac{1}{r}\sum_{k=1}^n \ln\mathsf{E}(\mathrm{e}^{rf_k}\mid\mathscr{F}_{k-1})\Big] \leqslant 0 \qquad \text{a.e.} \tag{11}
$$

By virtue of inequalities $\ln x \leqslant x-1$ $(x>0)$ and $\mathrm{e}^x - 1 - x \leqslant (x^2/2)\mathrm{e}^{|x|}$, noticing that

$$
\sup\{x^2\mathrm{e}^{-hx}; x\geqslant 0\} = \frac{4\mathrm{e}^{-2}}{h^2} \qquad (h>0),
$$

for $0<|r|<b$, we will get

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n[rf_k - \mathsf{E}(rf_k\mid\mathscr{F}_{k-1})]\\
\leqslant\ &\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n[\ln\mathsf{E}[\mathrm{e}^{rf_k}\mid\mathscr{F}_{k-1}) - \mathsf{E}(rf_k\mid\mathscr{F}_{k-1})]\\
\leqslant\ &\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n[\mathsf{E}(\mathrm{e}^{rf_k}\mid\mathscr{F}_{k-1}) - 1 - \mathsf{E}(rf_k\mid\mathscr{F}_{k-1})]\\
\leqslant\ &\frac{r^2}{2}\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\mathsf{E}(f_k^2\mathrm{e}^{|rf_k|}\mid\mathscr{F}_{k-1})\\
=\ &\frac{r^2}{2}\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\mathsf{E}(f_k^2\mathrm{e}^{b|f_k|}\mathrm{e}^{(|r|-b)|f_k|}\mid\mathscr{F}_{k-1})\\
\leqslant\ &\frac{r^2}{2}\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\mathsf{E}\Big[\mathrm{e}^{b|f_k|}\frac{4\mathrm{e}^{-2}}{(b-|r|)^2}\,\Big|\,\mathscr{F}_{k-1}\Big]\\
=\ &\frac{2r^2\mathrm{e}^{-2}}{(b-|r|)^2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\mathsf{E}(\mathrm{e}^{b|f_k|}\mid\mathscr{F}_{k-1})\\
\leqslant\ &\frac{2r^2\mathrm{e}^{-2}}{(b-|r|)^2}C_1(\omega) \qquad \text{a.e.} \qquad (12)
\end{aligned}
$$

Letting $0 < r < b$, dividing two side of (12) by r, we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [f_k - \mathsf{E}(f_k \mid \mathscr{F}_{k-1})] \leqslant \frac{2r\mathrm{e}^{-2}}{(b-|r|)^2} C_1(\omega) < \infty \qquad \text{a.e.}, \tag{13}$$

for $r \to 0^+$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [f_k - \mathsf{E}(f_k \mid \mathscr{F}_{k-1})] \leqslant 0 \qquad \text{a.e.} \tag{14}$$

Letting $-b < r < 0$, we similarly have

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [f_k - \mathsf{E}(f_k \mid \mathscr{F}_{k-1})] \geqslant 0 \qquad \text{a.e.} \tag{15}$$

Equation (6) holds immediately from (14) and (15). □

Corollary 5 [10] Let $\overrightarrow{X}$ be a Markov chain in bi-infinite random environment $\overrightarrow{\xi}$ with transition matrix $P_\theta = \{p(\theta; x, y),\, \theta \in \Theta,\, x, y \in \chi\}$. Let $f_m(\overrightarrow{\xi}; x, y)$ defined as Theorem 4. If there exists a constant $b > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}(f_k^2 \mathrm{e}^{b|f_k|} \mid \mathscr{F}_{k-1}) \leqslant C_2(\omega) < \infty \qquad \text{a.e.}, \tag{16}$$

then (6) holds.

Proof Since

$$\mathrm{e}^{b|f_k|} \leqslant f_m^2 \mathrm{e}^{b|f_k|} + \mathrm{e}^b, \tag{17}$$

we have (16) implies (5). This completes the proof of the assertion. □

Corollary 6 [10] Let $\overrightarrow{X}$ be a nonhomogeneous Markov chain taking values in χ with the sequence of transition matrices $P_n = \{p_n(x, y),\, x, y \in \chi\}$. Assume that $\{f_n(x, y)\}$ is a collection of bivariate functions defined in χ^2. Write $f_k = f_k(X_{k-1}, X_k)$. If there exists a constant $b > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}(f_k^2 \mathrm{e}^{b|f_k|} \mid X_{k-1}) \leqslant C_2(\omega) < \infty \qquad \text{a.e.}, \tag{18}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \{f_k(X_{k-1}, X_k) - \mathsf{E}[f_k(X_{k-1}, X_k) \mid X_{k-1}]\} = 0 \qquad \text{a.e.} \tag{19}$$

Proof Suppose that random environment $\overrightarrow{\xi}$ takes a fixed sequence $\overrightarrow{c} = \{c_n,\, n = \cdots, -1, 0, 1, \cdots\}$. Let $p_n(x, y) = p(c_n; x, y)$, then $\overrightarrow{X}$ is a Markov chain in random environment with constant vector $\overrightarrow{c}$. In this case, $\mathscr{F}_n = \sigma\{\overrightarrow{X}_0^n, \overrightarrow{c}\} = \sigma(X_0^n)$. By Markov property,

$$\begin{aligned} &\mathsf{E}(f_k^2 \mathrm{e}^{b|f_k|} \mid \mathscr{F}_{k-1}) = \mathsf{E}(f_k^2 \mathrm{e}^{b|f_k|} \mid X_{k-1}); \\ &\mathsf{E}[f_k(X_{k-1}, X_k) \mid \mathscr{F}_{k-1}] = \mathsf{E}[f_k(X_{k-1}, X_k) \mid X_{k-1}], \end{aligned} \tag{20}$$

this assertion follows. □

Remark 7 Liu has discussed a strong limit theorem for the average of ternary functions of Markov chains in bi-infinite random environment (See, Theorem 3.1 of [10]). Corollary 5 generalizes the result of [10], and corollary 6 is a special case of that results for Markov chains in bi-infinite random environment when $\overrightarrow{\xi}$ only take a fixed sequence $\overrightarrow{c} = \{c_n,\, n = \cdots, -1, 0, 1, \cdots\}$.

Theorem 8 Let $\overrightarrow{X}$ be a Markov chain in bi-infinite random environment $\overrightarrow{\xi}$ that contains transition matrix $P_\theta = \{p(\theta; x, y),\, \theta \in \Theta,\, x, y \in \chi\}$. Assume that

$$\min_{\theta\in\Theta,\, x,y\in\chi} p(\theta; x, y) = a > 0,$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [p^{-1}(\xi_{k-1}; X_{k-1}, X_k) - N] = 0 \qquad \text{a.e.}$$

Proof Let $f_{k+1}(\overrightarrow{\theta}; x, y) = p^{-1}(\theta_k; x, y)$, $\theta_k \in \Theta$ in Theorem 4. In this case,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}(\mathrm{e}^{b|f_k(\overrightarrow{\xi}; X_{k-1}, X_k)|} \mid \mathscr{F}_{k-1}) \\
=\ &\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}(\mathrm{e}^{b|p^{-1}(\xi_{k-1}; X_{k-1}, X_k)|} \mid \mathscr{F}_{k-1}) \\
\leqslant\ &\mathrm{e}^{b/a} < \infty.
\end{aligned}$$

From Theorem 4, (6) holds. Since

$$\begin{aligned}
&\mathsf{E}[f_k(\overrightarrow{\xi}; X_{k-1}, X_k) \mid \mathscr{F}_{k-1}] \\
=\ &\mathsf{E}[p^{-1}(\xi_{k-1}; X_{k-1}, X_k) \mid \mathscr{F}_{k-1}] \\
=\ &\sum_{y=1}^{N} p^{-1}(\xi_{k-1}; X_{k-1}, y) p(\xi_{k-1}; X_{k-1}, y) \\
=\ &N.
\end{aligned}$$

Therefore, this theorem can be obtained by Theorem 4. □

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双无限随机环境下马氏链的一类强极限定理

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摘 要: 本文的目的是要研究双无限随机环境下马氏链的一个强极限定理. 作为推论得到了非齐次马氏链的一个强大数定律. 最后, 得到双无限随机环境中马氏链的随机转移概率调和平均的强极限定理.

关键词: 强极限定理; 调和平均; 马氏链; 随机环境

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