应用概率统计 第 37 卷
第 4 期 2021 年 8 月

Chinese Journal of Applied Probability and Statistics
Aug., 2021, Vol. 37, No. 4, pp. 377-389
doi: 10.3969/j.issn.1001-4268.2021.04.004

Empirical Likelihood Test for Stationary Short Memory Time Series Models *

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Abstract: In this paper, we propose empirical likelihood method for parameter hypothesis test in short memory time series models. In practice, we may pay attention to not only the significance of all the parameters, but also the significance of some parameter in the models. So we construct different test statistics in these two situations, which are both shown to follow chi-square distributions asymptotically. In addition, our simulations investigate the power function for testing the concerned parameters and verify the validity of the proposed testing procedure.

Keywords: empirical likelihood; adjusted empirical likelihood; stationary ARMA models

2020 Mathematics Subject Classification: 62F12; 62M10

Citation: ZHANG X Z, LU Z P, LI M K, et al. Empirical likelihood test for stationary short memory time series models [J]. Chinese J Appl Probab Statist, 2021, 37(4): 377–389.

§1. Introduction

Based on the approximately independent periodogram ordinates, Monti [1] developed the frequency empirical likelihood (EL) for the parameters of ARMA models by deriving the estimating functions from the Whittle likelihood [2]. It is a breakthrough for applying EL to time series field. Following the way of [1], Yau [3] extended EL to long memory time series models by proving that there was a little portion of correlation among the periodogram for ARFIMA models [4,5]. However, these two works [1,3] considered the just-identified cases with the number of the estimating equations equivalent to the number of the unknown parameters in the moment models. In such case, the estimators of the parameters are less efficient than the estimators under the over-identified situations, that is, the number of parameters is larger than the number of equations. Accordingly, Nordman

*The project was supported by the Natural Science Foundation of Shanghai (Grant No. 17ZR1409000), the National Natural Science Foundation of China (Grant No. 11971171), the 111 Project (Grant No. B14019) and the Project of National Social Science Fund of China (Grant No. 15BTJ027).

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Received January 7, 2020. Revised May 22, 2020.

and Lahiri [6] constructed the new frequency EL which extended the application of EL to more general spectral parameters in time series model and developed new statistics for the over-identified cases.

In spite of its nice properties, the EL suffers from the problem of non-definition. For time series, its counterpart also inherits such undesirable features. Chen et al. [7] suggested the adjusted EL (AEL) to eliminate the problem. By adding a pseudo observation, the AEL ensures that the solution of the estimating equations exists. The processed convex hull of the sample makes sure the origin lies in it, which means the EL is always well-defined. For time series, there have been many works to construct confidence regions by EL and AEL. For example, Gamage et al. [8,9] introduced the AEL to construct confidence regions for parameters in short and long memory time series models. But little work has been done for hypothesis test by EL method. In this paper, we propose EL and AEL for testing the parameters of stationary ARMA models. First, we deduce the moment estimating equations to construct test statistics from Whittle likelihood. Then, the proposed EL and AEL test statistics for testing all the unknown parameters are shown to follow chi-square distributions, which is validate to calculate the power. However, in practice we may only be interest of the significance of part of the parameters, so it is necessary to build the test statistic for testing such vector subset of the parameters. And the proposed test statistic for the subset of parameters is also proved to converge to chi-square distribution. Finally, we verify the validity of the proposed statistics by simulation studies.

The rest of this paper is organised as follows. In Section 2, we present the test statistics of EL and AEL for time series. Simulation studies are conducted to verify our proposed tests in Section 3. The proofs of main results are deferred to the Appendix.

§2. Models and Proposed Test Procedure

In this section, we mainly present the frequency-domain EL and AEL test statistics for stationary ARMA models. The estimating equations, the foundation of EL and AEL inferences, are derived from Whittle likelihood, which is same as the way proposed by [1]. We start from introducing the Whittle likelihood for stationary short-memory model.

1) Whittle Likelihood for Stationary ARMA Models

A stationary ARMA model of [10; page 126] is denoted as

$$\Phi(B)X_t = \Theta(B)\varepsilon_t, \tag{1}$$

where $\{X_t\}$ is the time series, both $\Phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j$ and $\Theta(B) = 1 + \sum_{j=1}^{q} \theta_j B^j$ are operator polynomials such that there is no common root for the corresponding equations and all of the roots lie outside the unit circle, B is the backward operator satisfing $BX_t = X_{t-1}$ and ϕ_j and θ_j are constant coefficients, $\{\varepsilon_t\}$ is the white noise with mean zero and unknown variance σ^2. For such process, the spectral density of $\{X_t\}$ has the following closed form,

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{|\Theta(\mathrm{e}^{-i\omega})|^2}{|\Phi(\mathrm{e}^{-i\omega})|^2}, \qquad \omega \in [-\pi, \pi].$$

This kind of model is the most popular short memory models in time series. Let $X_1, X_2, \cdots, X_T$ be a realization of the process (1) and $\overline{X}$ be their mean. The periodogram ordinates are defined as

$$I(\omega_j) = \frac{1}{2\pi T}\Big\{\Big[\sum_{k=1}^{T}(X_k - \overline{X})\sin(\omega_j k)\Big]^2 + \Big[\sum_{k=1}^{T}(X_k - \overline{X})\cos(\omega_j k)\Big]^2\Big\},$$

for frequencies $\omega_j = 2\pi j/T$, $j = 1, 2, \cdots, T-1$. We restrict our attention to the first $n = [(T-1)/2]$ periodogram because of $I(\pi + \omega) = I(\pi - \omega)$. For ARMA models, EL works on the premise that the periodograms are asymptotical independent have proved by Monti [1].

2) EL and AEL Test Statistics

For the vector of parameters in ARMA models, we denote β as a vector of all unknown parameters. It is composed of the coefficients of AR, MA and the unknown variance of white noises, that is, $\beta = (\phi_1, \phi_2, \cdots, \phi_p, \theta_1, \theta_2, \cdots, \theta_q, \sigma^2)^{\mathsf{T}}$. Assume β belongs to a compact subset of the m-dimensional Euclidean space ($m = p+q+1$). Our main interests are to test the null hypothesis

$$H_0 : \beta = \beta_0, \tag{2}$$

moreover, we may only concern whether the part of MA or AR exist or not, that is to test the hypothesis such as

$$H_{01} : \theta = \theta_0 \qquad \text{or} \qquad H_{02} : \phi = \phi_0, \tag{3}$$

where $\theta = (\theta_1, \theta_2, \cdots, \theta_q)$ and $\phi = (\phi_1, \phi_2, \cdots, \phi_p)$, or other arbitrary subset of the unknown parameters.

In order to construct the test statistics, we first derive the based score equations from Whittle likelihood [2]. Taking derivative on Whittle likelihood with respect to β, we obtain the score equations as follow:

$$\sum_{j=1}^{n}\Big[\frac{I(\omega_j)}{f(\omega_j;\beta)} - 1\Big]\frac{\partial \ln\{f(\omega_j;\beta)\}}{\partial\beta} \equiv \sum_{j=1}^{n}\psi_j(I(\omega_j), \beta) = 0. \tag{4}$$

The EL test statistics for the vector β in ARMA models is defined as minus twice the empirical log-likelihood ratio (ELR), that is

$$R(\beta)=-2\sup\Big\{\sum_{j=1}^{n}\ln(np_j):\, p_j\geqslant 0,\ \sum_{j=1}^{n}p_j=1,\ \sum_{j=1}^{n}p_j\psi_j(I(\omega_j),\beta)=0\Big\}.$$

By Lagrange's multiplier method, we can express the adjusted ELR to be

$$R(\beta)=2\sum_{j=1}^{n}\ln\{1+\lambda^{\mathsf{T}}\psi_j(I(\omega_j),\beta)\},\tag{5}$$

where $\lambda:=\lambda(\beta)$ satisfies

$$0=\frac{1}{n}\sum_{j=1}^{n}\frac{\psi_j(I(\omega_j),\beta)}{1+\lambda^{\mathsf{T}}\psi_j(I(\omega_j),\beta)}.\tag{6}$$

If the EL estimator $\widetilde{\beta}$ for β can be obtained by minimizing $R(\beta)$, then the test statistic is defined by

$$R_1(\beta_0)=R(\beta_0)-R(\widetilde{\beta}).\tag{7}$$

And when the model is the just-identified, the term $R(\widetilde{\beta})$ will tend to be zero. As are proposed in [1], [8] and [9], the statistics are degenerated to be $R(\beta_0)$ under the just-identified cases. We will show that $R_1(\beta_0)$ follows χ^2_m asymptotically for the asymptotical independent periodogram ordinates of ARMA processes as n tends to infinity.

For simplicity, we use $\xrightarrow{\mathrm{d}}$ to mean convergence in distribution. Then we state our first result as follow:

Theorem 1 Under the assumptions of [1; Section 3] and the null hypothesis (2), $R_1(\beta_0)\xrightarrow{\mathrm{d}}\chi^2_m$, $n\to\infty$, where m is the dimension of the vector of the unknown parameter.

This theorem allows us to calculate the power for testing the significance of parameters in ARMA models. Let $\chi^2_m(1-\alpha)$ be the $(1-\alpha)$ quantile of the chi-square distribution with m degree of freedom. Then the power of the test can be calculated according to

$$\mathsf{P}\{R_1(\beta)>\chi^2_m(1-\alpha)\}.$$

However, for a given β, the ELR function is well-defined if and only if the convex hull of $\widetilde{\Omega}_\beta=\{\psi_j(I(\omega_j),\beta),j=1,2,\cdots,n\}$ contains the origin. In practice, it often happen that the origin lies outside the convex hull in the case of the small sample size. In such situation, the ELR function has no definition and is defined to be infinity conventionally, however this convention does provide the relative plausibility of different parameter values. To address this problem, Chen et al.[7] proved AEL to be an easy-going but rather effective method to completely eliminate this dilemma. The key of AEL is to add a pseudo-observation such as

$$\psi_{n+1}(I(\omega_{n+1}),\beta)=-\frac{a}{n}\sum_{j=1}^{n}\psi_j(I(\omega_j),\beta):=-a\overline{\psi}_n,$$

to the origin observations, which make ELR always well-defined, where the adjustment level $a = o(n)$. The AEL test statistic of β is defined as

$$\begin{aligned}\widetilde{R}(\beta; a) &= -2 \sup \Big\{ \sum_{j=1}^{n+1} \ln(np_j) : p_j \geqslant 0, \sum_{j=1}^{n+1} p_j = 1, \sum_{j=1}^{n+1} p_j \psi_j(I(\omega_j), \beta) = 0 \Big\} \\ &= 2 \sum_{j=1}^{n+1} \ln\{1 + \lambda^{\mathsf{T}} \psi_j(I(\omega_j), \beta)\},\end{aligned}$$

where $\lambda := \lambda(\beta)$ is the solution to

$$\sum_{j=1}^{n+1} \frac{\psi_j(I(\omega_j), \beta)}{1 + \lambda^{\mathsf{T}} \psi_j(I(\omega_j), \beta)} = 0. \tag{8}$$

Chen et al. [7] showed that the asymptotic property of AEL is similar to one of EL for independent and identically distributed (iid) observations and any positive constant a of order n. We construct the AEL test statistic for parameters in time series model as

$$\widetilde{R}_1(\beta_0; a) = \widetilde{R}(\beta_0; a) - \widetilde{R}(\widetilde{\beta}; a).$$

We assert it also asymptotically follows χ^2_m under the null hypothesis (2).

Theorem 2 Under the assumptions in Theorem 1 and the null hypothesis (2), then $\widetilde{R}_1(\beta_0) \xrightarrow{\mathrm{d}} \chi^2_m$ as n goes to infinity.

The application of Theorem 2 is similar to the Theorem 1. However, in practice, there are always some nuisance parameters that we don't care about because they have no effect on the main characteristics of our study. So we only pay more attention to our concerned subset of the unknown parameter vector. Similar to the argument of [11], we state the result for testing hypothesis (3) about ARMA models. For simplicity, we unify the ETR notations of EL and AEL as $\widetilde{R}(\beta; a) := R(\beta)$.

Corollary 3 Suppose $\beta = (\theta_1, \theta_2)$, the dimensions of θ_1 and θ_2 are p and q respectively. For $H_{01} : \theta_1 = \theta_{01}$, the test statistic is defined as

$$R_2(\theta_{01}) = R(\theta_{01}, \widetilde{\theta}_2) - R(\widetilde{\theta}_1, \widetilde{\theta}_2).$$

It follows χ^2_p asymptotically as n goes to infinity, where $\widetilde{\theta}_2$ minimizes $R(\theta_{01}, \theta_2)$ with respect to θ_2.

A direct application of this corollary is to test the significance of the term of AR or MA in the short memory models. In the next section, we use simulation study to investigate the finite-sample performance of such EL and AEL based powers.

§3. Monte Carlo Experiment

In this section, we report the Monte Carlo simulation results of the power for testing the prameter in ARMA(p,q) process by our proposed EL and AEL test procedure.

We consider three stationary ARMA(p,q) processes with $p=q=0$, $p=0$, $q=1$ and $p=1$, $q=0$. We carry out the simulations under four kinds of innovation, including N$(0,1)$, t(5), $\chi^2(5)$ and exp(1) noises with mean zero. In all cases, 2 000 replications are generated to calculate the power. Significant level $\alpha=0.05$ and the adjustment level $a=\max(1,\ln(n)/2)$. Here it is notable that the number of the sample in each replication is $n=[(T-1)/2]$ even the size of the original time series $\{x_t\}$ is T. In Table 2, we study the hypothesis test of the part of AR in ARMA$(1,1)$ models and set the coefficient of MA to be 0.3. The powers for testing the significance of parameters are obtained by EL and AEL method under different series length and different hypothesized values and listed in Tables 1−2. In all figures, the horizontal line represents the significant level, the vertical line corresponds to the hypothesized values and the powers are calculated with the size of series $T=1\,000$. For Figure 3, we set the coefficient of AR to be 0.3 and test the significance of the part of MA.

Tabel 1 Powers for testing the parameter in AR(1) models

Model	method	n	hypotheses	true values				
			$\phi_0=0.2$	$\phi=0.10$	$\phi=0.15$	$\phi=0.20$	$\phi=0.25$	$\phi=0.35$
	EL	200		0.314	0.139	0.072	0.126	0.601
		500		0.602	0.218	0.063	0.224	0.923
		1 000		0.891	0.360	0.052	0.371	0.999
	AEL	200		0.297	0.132	0.066	0.116	0.587
		500		0.594	0.213	0.060	0.217	0.920
		1 000		0.890	0.355	0.050	0.367	0.999
$\varepsilon_t\sim$ N$(0,1)$			$\phi_0=0.5$	$\phi=0.35$	$\phi=0.45$	$\phi=0.50$	$\phi=0.55$	$\phi=0.60$
	EL	200		0.695	0.172	0.076	0.134	0.374
		500		0.961	0.272	0.063	0.260	0.740
		1 000		1.000	0.436	0.058	0.448	0.960
	AEL	200		0.678	0.163	0.069	0.126	0.359
		500		0.960	0.265	0.061	0.254	0.773
		1 000		1.000	0.431	0.056	0.444	0.959
			$\phi_0=0.2$	$\phi=0.10$	$\phi=0.15$	$\phi=0.20$	$\phi=0.25$	$\phi=0.35$
	EL	200		0.381	0.194	0.097	0.166	0.606
		500		0.629	0.245	0.082	0.210	0.946
		1 000		0.888	0.372	0.054	0.375	0.999
	AEL	200		0.362	0.184	0.091	0.155	0.592
		500		0.623	0.238	0.078	0.207	0.943
		1 000		0.885	0.367	0.050	0.368	0.999
$\varepsilon_t\sim\chi^2(5)-5$			$\phi_0=0.5$	$\phi=0.35$	$\phi=0.45$	$\phi=0.50$	$\phi=0.55$	$\phi=0.60$
	EL	200		0.741	0.220	0.109	0.165	0.421
		500		0.965	0.303	0.076	0.258	0.756
		1 000		0.999	0.453	0.068	0.473	0.960
	AEL	200		0.729	0.205	0.102	0.154	0.399
		500		0.965	0.300	0.073	0.249	0.751
		1 000		0.999	0.448	0.065	0.469	0.959

Tabel 2 Powers for testing the parameter in ARMA$(1,1)$ models

Model	method	n	hypotheses	true values				
			$\phi_0=0.2$	$\phi=0.10$	$\phi=0.15$	$\phi=0.20$	$\phi=0.25$	$\phi=0.35$
	EL	200		0.118	0.086	0.069	0.081	0.282
		500		0.147	0.083	0.055	0.092	0.517
		1 000		0.239	0.102	0.058	0.132	0.800
	AEL	200		0.109	0.076	0.061	0.074	0.260
		500		0.143	0.081	0.054	0.088	0.507
		1 000		0.236	0.099	0.055	0.130	0.797
$\varepsilon_t\sim \mathrm{N}(0,1)$			$\phi_0=0.5$	$\phi=0.35$	$\phi=0.45$	$\phi=0.50$	$\phi=0.55$	$\phi=0.60$
	EL	200		0.355	0.132	0.080	0.112	0.247
		500		0.605	0.171	0.061	0.154	0.517
		1 000		0.890	0.243	0.064	0.262	0.806
	AEL	200		0.338	0.122	0.070	0.097	0.232
		500		0.600	0.166	0.058	0.146	0.505
		1 000		0.887	0.240	0.062	0.259	0.802
			$\phi_0=0.2$	$\phi=0.10$	$\phi=0.15$	$\phi=0.20$	$\phi=0.25$	$\phi=0.35$
	EL	200		0.107	0.080	0.066	0.080	0.251
		500		0.150	0.084	0.050	0.096	0.514
		1 000		0.257	0.100	0.062	0.149	0.789
	AEL	200		0.100	0.074	0.061	0.073	0.238
		500		0.146	0.082	0.047	0.093	0.507
		1 000		0.255	0.098	0.060	0.146	0.787
$\varepsilon_t\sim\chi^2(5)-5$			$\phi_0=0.5$	$\phi=0.35$	$\phi=0.45$	$\phi=0.50$	$\phi=0.55$	$\phi=0.60$
	EL	200		0.335	0.133	0.071	0.113	0.257
		500		0.614	0.167	0.063	0.167	0.502
		1 000		0.879	0.230	0.052	0.268	0.798
	AEL	200		0.345	0.134	0.066	0.109	0.247
		500		0.618	0.163	0.054	0.160	0.493
		1 000		0.878	0.228	0.050	0.264	0.793

From Tables $1-2$ and Figures $1-3$, we find the test has reasonable asymptotic power properties. We list the observations as follows. First, for all cases when the parameter takes the hypothesized value, the powers of test get close to the significance level. Second, when the value that the parameter take is far away from the hypothesized value, the power will get large. Third, comparing the powers in Tables $1-2$ and Figures $2-3$, we find that the power of the test is larger when there are less unknown parameters in the model. Finally, we find the powers obtained from AEL is a bit smaller than those obtained from EL, which does not against the usefulness of adjustment. That is, the AEL always achieves the high-order coverage probability by adjustment. In conclusion, these results demonstrate that our proposed testing procedures are useful.

§4. Appendix

In this section, we give the proof of the Theorem 1, Theorem 2 and the Corollary 3. For simplicity, we note $\psi_j(I(\omega_j),\beta)=\psi(x_j,\beta)$ and $\|\cdot\|$ denotes Euclidean norm.

The Proof of Theorem 1 For the true value β_0, we just consider the parameter

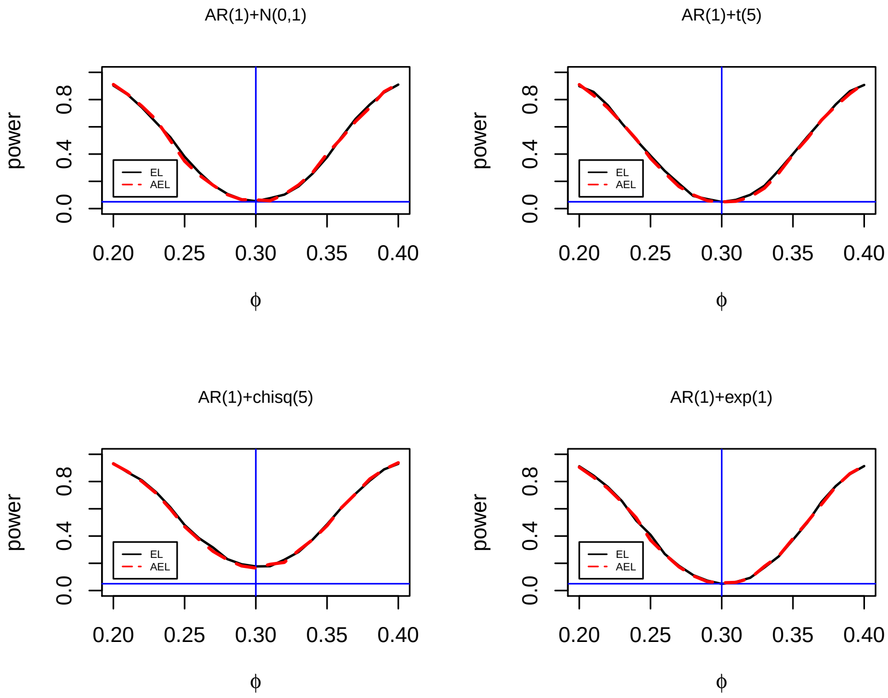

Figure 1 The power for AR(1) model $(\phi = 0.3)$

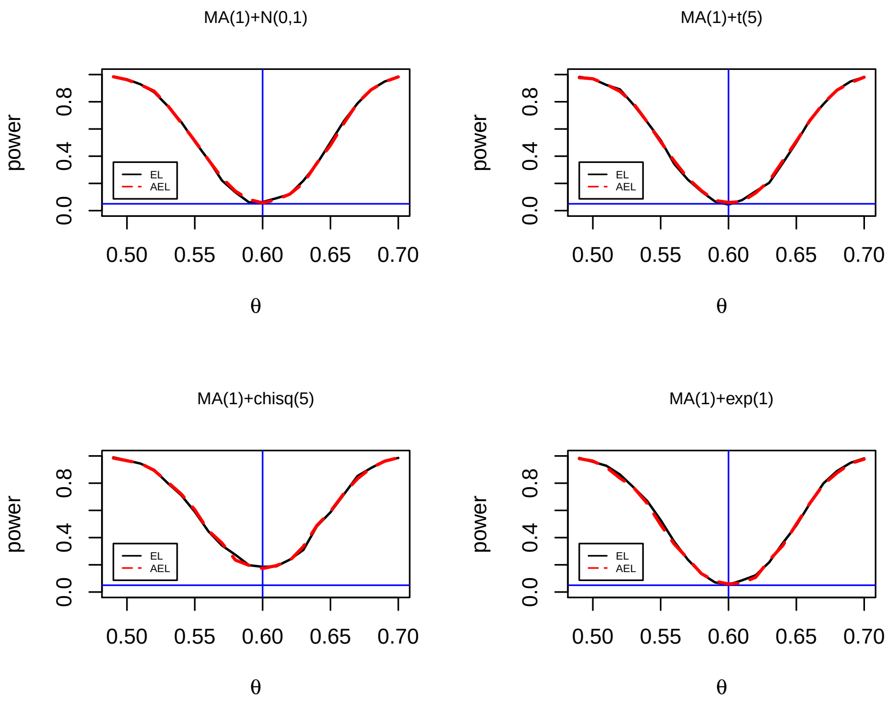

Figure 2 The power for MA(1) model $(\theta = 0.6)$

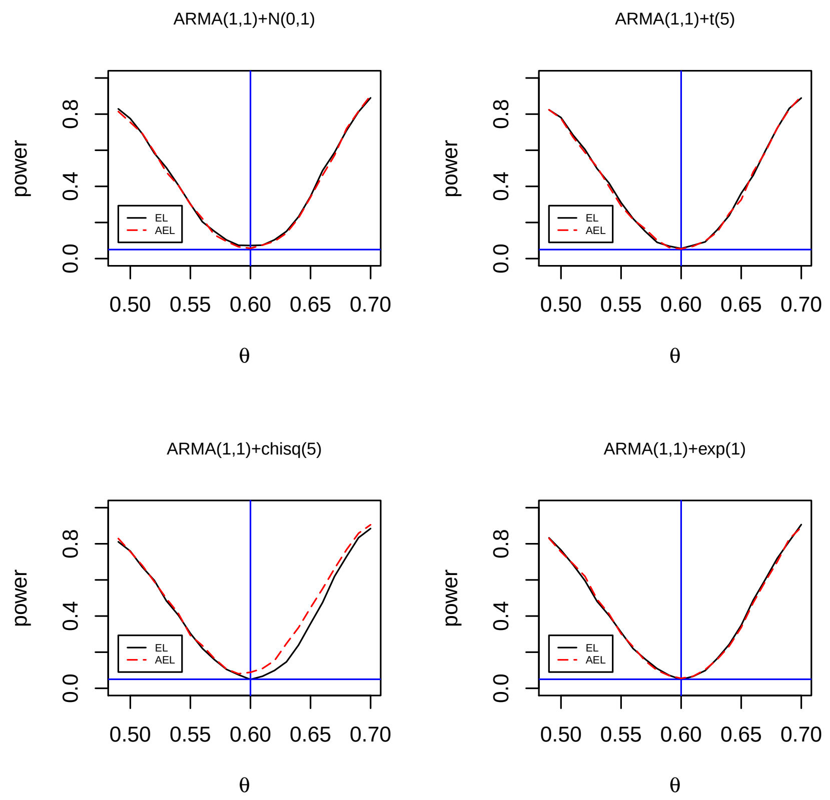

Figure 3 The power for ARMA$(1,1)$ model ($\theta = 0.6$)

β in the ball $B_\beta = \{\beta \,|\, \|\beta - \beta_0\| \leqslant n^{-1/2}\}$.

First we will show $\lambda_\beta = O_p(n^{-1/2})$. In fact, $\max\limits_{1\leqslant j\leqslant n} \|\psi_j(x_j,\beta)\| = o_p(n^{1/2})$. This is obtained directly by dominated convergence theorem, when

$$\mathsf{E}\Big\{\frac{1}{n}\sum_{j=1}^{n} \|\psi_j(x_j,\beta)\|^2\Big\} < \infty.$$

Note $\lambda = \rho u$, where $\rho = \|\lambda\|$, so $\|u\| = 1$, then timing the term u^{T}, the equation (6) gets the form as

$$\begin{aligned}
0 &= u^{\mathsf{T}}\frac{1}{n}\sum_{j=1}^{n}\frac{\psi_j(x_j,\beta_0)}{1+\lambda^{\mathsf{T}}\psi_j(x_j,\beta_0)}\\
&= u^{\mathsf{T}}\frac{1}{n}\sum_{j=1}^{n}\frac{\psi_j(x_j,\beta_0)+\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0)\lambda-\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0)\lambda}{1+\lambda^{\mathsf{T}}\psi_j(x_j,\beta_0)}\\
&= u^{\mathsf{T}}\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)-\frac{1}{n}\sum_{j=1}^{n}\frac{u^{\mathsf{T}}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0)u}{1+\rho u^{\mathsf{T}}\psi_j(x_j,\beta_0)}\rho,
\end{aligned}$$

because

$$\Big|u^{\mathsf{T}}\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\Big| \geqslant \Big\{\Big[u^{\mathsf{T}}\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0)u\Big]\Big/\Big[1+\rho\max_{1\leqslant j\leqslant n}|u^{\mathsf{T}}\psi_j(x_j,\beta_0)|\Big]\Big\}\rho,$$

$$\max_{1\leqslant j\leqslant n}|u^{\mathsf{T}}\psi_j(x_j,\beta)|\leqslant\|u\|\max_{1\leqslant j\leqslant n}\|\psi_j(x_j,\beta)\|=o_p(n^{1/2}),$$
$$\Big|u^{\mathsf{T}}\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\Big|=O_p(n^{-1/2})$$

and

$$\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0)\to V>0$$

in probability proved by Motin [1], where V is a covariance matrix which is elaborated in [1]. Then we can obtain $\rho=O_p(n^{-1/2})$ and $\lambda=O_p(n^{-1/2})$.

Next we will show $R_1(\beta_0)\xrightarrow{\mathrm{d}}\chi_m^2$. For Whittle estimator $\widetilde{\beta}$ minimizes $R(\beta)$, we have the following equations:

$$0=\frac{1}{n}\sum_{j=1}^{n}\frac{\psi_j(x_j,\widetilde{\beta})}{1+\widetilde{\lambda}^{\mathsf{T}}\psi_j(x_j,\widetilde{\beta})},\qquad 0=\frac{1}{n}\sum_{j=1}^{n}\frac{\widetilde{\lambda}^{\mathsf{T}}\psi_{j1}(x_j,\widetilde{\beta})}{1+\widetilde{\lambda}^{\mathsf{T}}\psi_j(x_j,\widetilde{\beta})},\tag{9}$$

where $\psi_{j1}(x_j,\beta)=\partial\psi_j(x_j,\beta)/\partial\beta^{\mathsf{T}}$. By Taylor expansion, we expand the above equations at $(\beta,\lambda)=(\beta_0,0)$. Then we have

$$0=\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)+\frac{1}{n}\sum_{j=1}^{n}\psi_{j1}(x_j,\beta_0)(\widetilde{\beta}-\beta_0)-\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0)\widetilde{\lambda}+\delta_n o_p(1),$$
$$0=\frac{1}{n}\sum_{j=1}^{n}\psi_{j1}^{\mathsf{T}}(x_j,\beta_0)\widetilde{\lambda}+\delta_n o_p(1),$$

where $\delta_n=\|\widetilde{\beta}-\beta_0\|+\|\widetilde{\lambda}\|=o_p(n^{-1/2})$.

Note

$$A=\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0),$$
$$S_{11}=\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0),\qquad S_{12}=-\frac{1}{n}\sum_{j=1}^{n}\psi_{j1}(x_j,\beta_0)=S_{21}^{\mathsf{T}}.$$

Then we can obtain

$$\begin{pmatrix}A\\0\end{pmatrix}=\begin{pmatrix}S_{11}&S_{12}\\S_{21}&0\end{pmatrix}\begin{pmatrix}\widetilde{\lambda}\\\widetilde{\beta}-\beta_0\end{pmatrix}+o_p(n^{-1/2}).$$

If we note

$$\begin{pmatrix}S_{11}&S_{12}\\S_{21}&0\end{pmatrix}^{-1}=\begin{pmatrix}S^{11}&S^{12}\\S^{21}&S^{22}\end{pmatrix},$$

then

$$\begin{pmatrix}\widetilde{\lambda}\\\widetilde{\beta}-\beta_0\end{pmatrix}=\begin{pmatrix}S^{11}&S^{12}\\S^{21}&S^{22}\end{pmatrix}\begin{pmatrix}A\\0\end{pmatrix}+o_p(n^{-1/2}).$$

By Taylor expansion, we obtain

$$\begin{aligned}
R(\widetilde{\beta}) &= 2\sum_{j=1}^{n} \ln\{1+\widetilde{\lambda}^{\mathsf{T}}\psi_j(x_j,\widetilde{\beta})\} \\
&= 2n\begin{pmatrix} A^{\mathsf{T}} & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\lambda} \\ \widetilde{\beta}-\beta_0 \end{pmatrix} - n\begin{pmatrix} \widetilde{\lambda}^{\mathsf{T}} & (\widetilde{\beta}-\beta_0)^{\mathsf{T}} \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\lambda} \\ \widetilde{\beta}-\beta_0 \end{pmatrix} \\
&\quad + o_p(1) \\
&= n\begin{pmatrix} A^{\mathsf{T}} & 0 \end{pmatrix}\begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix} = nA^{\mathsf{T}}S^{11}A + o_p(1).
\end{aligned}$$

Another hand, by

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\psi_j(x_j,\beta_0)}{1+\lambda_{\beta_0}^{\mathsf{T}}\psi_j(x_j,\beta_0)} = 0,$$

we have $\lambda_{\beta_0} = S_{11}^{-1}A + o_p(1)$, and $R(\beta_0) = nA^{\mathsf{T}}S_{11}^{-1}A + o_p(1)$. So the log-empirical likelihood ratio testing statistic

$$R_1(\beta_0) = nA^{\mathsf{T}}(S_{11}^{-1} - S^{11})A + o_p(1) = -nA^{\mathsf{T}}(S_{11}^{-1}S_{12}S^{22}S_{21}S_{11}^{-1})A + o_p(1).$$

By Slutsky theorem we have

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\widetilde{\beta})}{\partial\beta^{\mathsf{T}}} \to S_{12}$$

in probability which have been proved in [1]. Then

$$A = \frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0) = \frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\widetilde{\beta})}{\partial\beta^{\mathsf{T}}}(\beta_0-\widetilde{\beta}) + o_p(n^{1/2}) = -S_{12}(\beta_0-\widetilde{\beta}) + o_p(n^{1/2})$$

and

$$\begin{aligned}
-nA^{\mathsf{T}}(S_{11}^{-1}S_{12}S^{22}S_{21}S_{11}^{-1})A &= n(\beta_0-\widetilde{\beta})^{\mathsf{T}}S_{21}S_{11}^{-1}S_{12}(\beta_0-\widetilde{\beta}) \\
&= n(\beta_0-\widetilde{\beta})^{\mathsf{T}}\widehat{V}^{-1}(\beta_0-\widetilde{\beta}) + o_p(1),
\end{aligned}$$

where $\widehat{V} = S_{21}S_{11}^{-1}S_{12}$. S_{12} and S_{11} converge to V in probability, see [1]. So $\widehat{V}^{-1}$ converges to V^{-1} in probability and the rank of V is m. In addition, Dzhaparidze[12] pointed out under regular conditions $\sqrt{n}(\widetilde{\beta}-\beta) \xrightarrow{\mathrm{d}} \mathrm{N}(0,V)$, then $R_1(\beta_0) \xrightarrow{\mathrm{d}} \chi_m^2$. □

The Proof of Theorem 2 Similar to the proof of Theorem 1, we only show the framework of the proof. First $\psi_{n+1} = a\overline{\psi}_n = o_p(n^{1/2})$, then $\max\limits_{1\leqslant j\leqslant n+1}\|\psi_j(x_j,\beta)\| = o_p(n^{1/2})$. Second, from equation (8), we also can obtain $\lambda = O_p(n^{-1/2})$. Third, $\widetilde{R}_1(\beta_0;a)$ can expand as the following equality:

$$\widetilde{R}_1(\beta_0;a) = n(\beta_0-\widetilde{\beta})^{\mathsf{T}}(\widehat{V}^*)^{-1}(\beta_0-\widetilde{\beta}) + o_p(1),$$

where $\widehat{V}^* = S_{21}^*(S_{11}^*)^{-1}S_{12}^*$,

$$S_{11}^* = \frac{1}{n}\sum_{j=1}^{n+1}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0) \qquad \text{and} \qquad S_{12}^* = -\frac{1}{n}\sum_{j=1}^{n+1}\frac{\partial\psi_j(x_j,\beta_0)}{\partial\beta}.$$

Because

$$\frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0) = O_p(n^{-1/2}),$$

then

$$\begin{aligned}
S_{11}^* &= \frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0) + \frac{1}{n}\psi_{n+1}(x_{n+1},\beta_0)\psi_{n+1}^{\mathsf{T}}(x_{n+1},\beta_0)\\
&= \frac{1}{n}\sum_{j=1}^{n}\psi_j(x_j,\beta_0)\psi_j^{\mathsf{T}}(x_j,\beta_0) + o_p(1)
\end{aligned}$$

and

$$S_{12}^* = \frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\beta_0)}{\partial\beta} + o_p(1).$$

So we can obtain $\widetilde{R}_1(\beta_0;a)$ converges to the χ_m^2 distribution asymptotically as n goes to infinity. $\square$

The Proof of Corollary 3 By Taylor expansion we have

$$R_2(\theta_{01}) = nA^{\mathsf{T}}(-S_{11})^{-1/2}S_{11}^{-1/2}(D-M)S_{11}^{-1/2}(-S_{11})^{-1/2}A + o_p(1),$$

where the notations of A and S_{11} are same as the proof of Theorem 1 and extend to the proof of Theorem 2.

$$\begin{aligned}
D &= \Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\beta)}{\partial\beta}\Big)\Big[\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\theta)}{\partial\beta}\Big)^{\mathsf{T}}S_{11}^{-1}\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\beta)}{\partial\beta}\Big)\Big]^{-1}\\
&\quad\times\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\beta)}{\partial\beta}\Big)^{\mathsf{T}},\\
M &= \Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\beta)}{\partial\theta_1}\Big)\Big[\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\theta)}{\partial\theta_1}\Big)^{\mathsf{T}}S_{11}^{-1}\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\beta)}{\partial\theta_1}\Big)\Big]^{-1}\\
&\quad\times\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\psi_j(x_j,\beta)}{\partial\theta_1}\Big)^{\mathsf{T}}.
\end{aligned}$$

By the result in [13], if $D-M$ is nonnegative defined, then $R_2(\theta_{01}) \xrightarrow{\mathrm{d}} \chi_p^2$, where p is the dimension of subvector θ_1 of the unknown parametes. $\square$

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平稳自回归模型的经验似然检验

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摘　要: 这篇文章中, 我们对平稳短记忆时间序列模型中的参数假设检验运用经验似然方法, 在实际中, 我们可能不仅关心所有参数的显著性而且更关心模型中的某一部分参数是否存在, 因而我们除了建立检验所有参数的统计量, 还建立了检验部分参数显著与否的统计量, 这些统计量被证实都渐近服从卡方分布, 另外, 我们的模拟研究了检验的势函数并证实了提出的检验程序的有效性.

关键词: 经验似然; 调整经验似然; 平稳自回归滑动平均模型

中图分类号: O212.7