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On Algebraic and Exponential Transience for Continuous-Time Markov Chains *

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Abstract: In this paper, we investigate algebraic and exponential transience for continuous-time Markov chains (CTMCs). Equivalent relations of these transience are revealed between CTMCs and their jump chains and dual processes. The results are further applied to derive the criteria of these transience for general CTMCs, generalized Markov branching processes and birth-death processes.

Keywords: Markov chains; algebraic transience; exponential transience; jump chains; dual processes

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§1. Introduction

Let $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{R}_+$ be the set of nonnegative integers, positive integers and nonnegative real numbers, respectively. Let $X = \{X_t, t \in \mathbb{R}_+\}$ be an irreducible CTMC with transition function $P_{ij}(t)$ and regular q-matrix $Q = (q_{ij})$ on a countable state space E. For a nonnegative and nondecreasing function $r = r(t)$, $t \geqslant 0$, the chain X is said to be r-transient if

$$\int_0^\infty r(t) P_{ii}(t) \mathrm{d}t < \infty \tag{1}$$

for some state $i \in E$. In particular, when $r(t) = t^\ell$, $\ell \in \mathbb{N}$, the chain X is called ℓ-transient; when $r(t) = s^t$, $s > 1$, the chain X is called exponentially transient. It is worth noting that (1) holds for some $i \in E$ if and only if it holds for any $i \in E$ since the chain X is irreducible.

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Criteria for ℓ-transience and geometric transience for discrete-time Markov chains (DTMCs) have been presented in the literature. Meyn and Tweedie in Chapter 8 of [1] investigated the ordinary transience by establishing the appropriate drift condition. Then the equivalent drift criteria for both algebraic and geometric transience were given by Mao and Song [2]. Meanwhile, they first introduced the modified moments of the first return times to investigate algebraic and geometric transience. Later, Liu et al. [3] presented equivalent conditions for algebraic and geometric transience in terms of the modified moments of the first return times on finite non-empty sets, which were applied to investigate matrix-analytical models. In this paper, we will focus on algebraic and exponential transience for CTMCs, which has not been addressed well.

In the study of CTMCs, we usually hope to establish the relations with DTMCs so as to extend the results of DTMCs to the continuous time case. If the generator Q is bounded, we can analyze algebraic and exponential transience for CTMCs by their h-uniformized chains, see [3]. However, it fails when Q is unbounded. In the unbounded case, Tweedie [4] established the equivalent relations between CTMCs and their jump chains for ordinary transience. In Section 2, we will show that the relations are also valid for algebraic and exponential transience.

In Section 3, we will show that if a CTMC is stochastically monotone, then we can analyze its transience properties through its dual process. With the help of existing results about ergodicity [5–9], we are able to get the information about the absorption times of dual processes, which plays a key role in the analysis of transience for CTMCs. An important application of this method is to investigate algebraic and exponential transience for continuous-time birth and death processes.

In Section 4, we apply our results in Sections 2 and 3 to generalized Markov branching processes and continuous-time birth and death processes and derive explicit criteria for algebraic and exponential transience.

§2. Relation with Jump Chains

Let us define

$$J_n = \begin{cases} 0, & \text{if } n = 0; \\ \inf\{t > J_{n-1} \,|\, X_t \neq X_{t^-}\}, & \text{if } n \in \mathbb{N}, \end{cases}$$

and

$$Y_n = X_{J_n}, \qquad n \in \mathbb{Z}_+.$$

Then J_1 is the time of the first transition, and more generally, J_n is the time of the nth transition. As we know, the sequence of states $\{Y_n, n \in \mathbb{Z}_+\}$ visited by $\{X_t, t \in \mathbb{R}_+\}$ forms a DTMC, called the jump chain or the embedded Markov chain. This jump chain has transition matrix given by

$$P = (p_{ij}) = \begin{cases} \dfrac{q_{ij}}{q_i}, & \text{if } i \neq j \text{ and } q_i > 0; \\ 0, & \text{if } i = j \text{ and } q_i > 0; \\ \delta_{ij}, & \text{if } q_i = 0, \end{cases}$$

where $\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$

Theorem 1 Let $\ell \in \mathbb{N}$. The CTMC X is exponentially transient or ℓ-transient if and only if so is its jump chain.

Proof Since the chain X is irreducible, then we have $q_i > 0$ for every $i \in E$. Otherwise there would be an absorbing state, which contradicts the irreducibility of X. Now let $\mu < q_i$ for every $i \in C$. From the proof of Proposition 3.2 in Chapter 5 of [10], we have for $i \in E$,

$$\int_0^\infty \mathrm{e}^{\mu t} P_{ii}(t)\mathrm{d}t = \frac{1}{q_i - \mu}\Big[\sum_{m=0}^{\infty}\Big(\frac{q_i}{q_i - \mu}\Big)^m P_{ii}^{(m)}\Big], \tag{2}$$

where $P_{ii}^{(m)}$ is the element in row i and column i of the m-step transition matrix of the jump chain.

We first prove the case of exponential transience. For some (then for all) state $i \in E$, if the jump chain is geometrically transient with

$$\sum_{m=0}^{\infty} s^m P_{ii}^{(m)} < \infty,$$

where $s > 1$, then we can choose $\mu = q_i - q_i/s > 0$ such that

$$\int_0^\infty \mathrm{e}^{\mu t} P_{ii}(t)\mathrm{d}t = \frac{s}{q_i}\sum_{m=0}^{\infty} s^m P_{ii}^{(m)} < \infty,$$

which shows the CTMC is exponentially transient. Conversely, suppose that the CTMC is exponentially transient, namely, for every $i \in E$ and some $\mu > 0$,

$$\int_0^\infty \mathrm{e}^{\mu t} P_{ii}(t)\mathrm{d}t < \infty.$$

By page 176 in [10], we know that the decay parameter λ_E is the convergence radius of the integral $\int_0^\infty \mathrm{e}^{\mu t} P_{ii}(t)\mathrm{d}t$ for any $i \in E$, which implies $\lambda_E \geqslant \mu > 0$. Without loss of generality, we can assume $\lambda_E > \mu$. Further, by Theorem 1.9 in Chapter 5 of [10] we have $\inf_{i\in E} q_i \geqslant \lambda_E > \mu$. Hence, for any fixed $i \in E$ we have $q_i > \mu$. Define $s_i = q_i/(q_i - \mu) > 1$, then by equation (2) we have

$$\sum_{m=0}^{\infty} s_i^m P_{ii}^{(m)} = (q_i - \mu) \int_0^{\infty} \mathrm{e}^{\mu t} P_{ii}(t)\mathrm{d}t < \infty,$$

from which, we know that the jump chain is geometrically transient.

Then we deal with the case of algebraic transience. It is worth noting that the above equation (2) also holds for $\mu \leqslant 0$. Now let

$$A(\mu) = \sum_{m=0}^{\infty} \Big(\frac{q_i}{q_i - \mu}\Big)^m P_{ii}^{(m)}$$

and its derivatives be given by

$$A^{(n)}(\mu) = \sum_{m=0}^{\infty} q_i^m \frac{m\cdots(m+n-1)}{(q_i-\mu)^{m+n}} P_{ii}^{(m)}, \qquad n \in \mathbb{N}.$$

Repeating taking the derivative of both sides of equation (2) with respect to μ, we can get

$$\int_0^{\infty} t^{\ell} \mathrm{e}^{\mu t} P_{ii}(t)\mathrm{d}t = \frac{\ell!}{(q_i-\mu)^{\ell+1}} A(\mu) + \frac{\ell!}{(q_i-\mu)^{\ell}} A^{(1)}(\mu) + \cdots + \frac{1}{q_i - \mu} A^{(\ell)}(\mu).$$

Let $\mu = 0$, then we have

$$\begin{aligned}
&\int_0^{\infty} t^{\ell} P_{ii}(t)\mathrm{d}t \\
&= \frac{\ell!}{q_i^{\ell+1}} A(0) + \frac{\ell!}{q_i^{\ell}} A^{(1)}(0) + \cdots + \frac{1}{q_i} A^{(\ell)}(0) \\
&= \frac{\ell!}{q_i^{\ell+1}} \sum_{m=0}^{\infty} P_{ii}^{(m)} + \frac{\ell!}{q_i^{\ell+1}} \sum_{m=0}^{\infty} m P_{ii}^{(m)} + \cdots + \frac{1}{q_i^{\ell+1}} \sum_{m=0}^{\infty} [m\cdots(m+\ell-1)] P_{ii}^{(m)}.
\end{aligned}$$

Since q_i and ℓ are determined, the above equation can be written as

$$\int_0^{\infty} t^{\ell} P_{ii}(t)\mathrm{d}t = a_0 \sum_{m=0}^{\infty} P_{ii}^{(m)} + a_1 \sum_{m=0}^{\infty} m P_{ii}^{(m)} + \cdots + a_{\ell} \sum_{m=0}^{\infty} m^{\ell} P_{ii}^{(m)}, \tag{3}$$

where a_i, $i = 0, 1, \cdots, \ell$ are positive constants dependent only on q_i and ℓ.

From equation (3), we can obtain the equivalent relation of algebraic transience between the CTMC and its jump chain immediately. If the jump chain is ℓ-transient, then for any fixed state i,

$$\sum_{m=0}^{\infty} m^{\ell} P_{ii}^{(m)} < \infty,$$

from which, we know that $\sum_{m=0}^{\infty} m^k P_{ii}^{(m)}$ are finite for $0 \leqslant k \leqslant \ell - 1$. By equation (3), we have

$$\int_0^{\infty} t^{\ell} P_{ii}(t)\mathrm{d}t < \infty,$$

which implies that the CTMC X is ℓ-transient. Conversely, if the CTMC X is ℓ-transient, that is, for any fixed state i,

$$\int_0^\infty t^\ell P_{ii}(t)\mathrm{d}t < \infty,$$

then by equation (3) again, we have

$$\sum_{m=0}^\infty m^\ell P_{ii}^{(m)} < \infty.$$

Hence, the jump chain is ℓ-transient. The proof is finished. $\square$

The drift criteria for algebraic and geometric transience for DTMCs have been given respectively by Theorem 2.2 and Theorem 3.2 in [2]. Together with Theorem 1, we can get the criteria for algebraic and exponential transience for CTMCs in terms of drift functions.

Corollary 2 The following statements are equivalent.

(i) The CTMC X is exponentially transient.

(ii) There exists some set $A \subset E$, constants $\lambda, b \in (0,1)$, and a function $W \geqslant I_A$ (with $W(i) < \infty$ for some $i \in E$) satisfying the drift condition

$$\sum_{j\neq i} \frac{q_{ij}}{q_i} W(j) \leqslant \lambda W(i) I_{A^c}(i) + b I_A(i), \qquad i \in E,$$

where $I(\cdot)$ is an indicator function.

Corollary 3 The following statements are equivalent.

(i) The CTMC X is ℓ-transient.

(ii) There exists some set $A \subset E$, constants $d \in (0,\infty)$, $b \in (0,1)$, and nonnegative functions W_n (with $W_n(i) < \infty$ for some $i \in E$), $n = 0, 1, \cdots, \ell$ satisfying for $n = 0, 1, \cdots, \ell$,

$$\begin{cases} \displaystyle\sum_{j\neq i} \frac{q_{ij}}{q_i} W_n(j) \leqslant W_n(i) - (\ell - n) W_{n+1}(i), & i \in A^c, \\ W_n(i) \geqslant 1, & i \in A, \\ \displaystyle\sum_{j\neq i} \frac{q_{ij}}{q_i} W_0(j) \leqslant d, & i \in A, \\ \displaystyle\sum_{j\neq i} \frac{q_{ij}}{q_i} W_\ell(j) \leqslant b, & i \in A, \end{cases}$$

where $W_{\ell+1} = 0$.

Next, we are going to present equivalent conditions for r-transience for CTMCs in terms of the modified moments of the first return times. For any non-empty subset $A \subset E$, we define the first return time on the set A as follows

$$\tau_A = \inf\{t \geqslant J_1 : X_t \in A\},$$

where J_1 is the first jump time of the CTMC X. Let

$$F_{iA} = \mathsf{P}\{\tau_A < \infty \,|\, X_0 = i\}$$

be the probability of X ever returning to A.

Proposition 4 For a function $r(t)$, where $r(t) = t^\ell$, $\ell \in \mathbb{N}$ or $r(t) = s^t$, $s \geqslant 1$, the following statements are equivalent.

(i) For some (then for all) $i \in E$, the CTMC X is r-transient.

(ii) For some (then for all) $i \in E$, $F_{ii} < 1$ and $\mathsf{E}_i[r(\tau_i) I_{\{\tau_i<\infty\}}] < \infty$.

(iii) For some (then for all) finite non-empty set $A \subset E$,

$$\max_{i\in A} \int_0^\infty r(t) P_{iA}(t) \mathrm{d}t < \infty.$$

(iv) For some (then for all) finite non-empty set $A \subset E$, $\max_{i\in A} \mathsf{E}_i[r(\tau_A) I_{\{\tau_A<\infty\}}] < \infty$, and $F_{jA} < 1$ for some $j \in A$.

Proof See Appendix in Section 5. □

Remark 5 It is worth noting that there is a result similar to Proposition 4 about DTMCs in [3], and here we extend it to the continuous time case. Proposition 4 provides us another criteria for algebraic and exponential transience for CTMCs in terms of the modified moments of the first return times on a single state or a finite non-empty set, whose proof uses Theorem 1.

§3. Relation with Dual Processes

In this part, we will introduce the concept of stochastic monotonicity and dual processes of CTMCs. Then the transience properties for CTMCs are investigated through the related dual processes.

Consider an irreducible CTMC $X^{(1)} = \{X_t^{(1)}, t \in \mathbb{R}_+\}$ on a countable state space $E = \{0, 1, 2, \cdots\}$ with the regular q-matrix $Q^{(1)}$ and the unique transition function $P_{ij}^{(1)}(t)$. Note that since $Q^{(1)}$ is regular, $P_{ij}^{(1)}(t)$ is honest, that is, $\sum\limits_{j\in E} P_{ij}^{(1)}(t) = 1$ for any $i \in E$.

Definition 6 $P_{ij}^{(1)}(t)$ is called stochastically monotone if $\mathsf{P}_i\{X_t^{(1)} \geqslant k\}$ is a nondecreasing function of i for every fixed k and t.

Lemma 7 [11] There exists another CTMC $X^{(2)} = \{X_t^{(2)}, t \in \mathbb{R}_+\}$ with transition function

$$P_{ij}^{(2)}(t) = \sum_{k=i}^{\infty}[P_{jk}^{(1)}(t) - P_{j-1,k}^{(1)}(t)], \qquad i, j \in E,\ t \geqslant 0, \tag{4}$$

where $P_{-1,k}^{(1)}(t) \equiv 0$ satisfying

$$\mathsf{P}_i\{X_t^{(2)} \leqslant j\} = \mathsf{P}_j\{X_t^{(1)} \geqslant i\}, \qquad i, j \in E,\ t \geqslant 0, \tag{5}$$

if and only if $P_{ij}^{(1)}(t)$ is stochastically monotone.

Then the chain $X^{(2)}$ with transition function $P_{ij}^{(2)}(t)$ defined by (4) is called the dual process of $X^{(1)}$. For dual processes, we have the following property.

Lemma 8 $P_{ij}^{(1)}(t)$ is honest if and only if $P_{00}^{(2)}(t) \equiv 1$, that is, state 0 is the absorbing state of chain $X^{(2)}$.

Proof From (5), we have

$$\mathsf{P}_i\{X_t^{(2)} \leqslant j\} = \mathsf{P}_j\{X_t^{(1)} \geqslant i\}, \qquad i, j \in E,\ t \geqslant 0.$$

When $i = j = 0$, it follows that

$$P_{00}^{(2)}(t) = \sum_{k=0}^{\infty} P_{0k}^{(1)}(t).$$

Hence $P_{00}^{(2)}(t) = 1$ if and only if $P_{ij}^{(1)}(t)$ is honest. $\square$

Inspired by Gong's master thesis [12] and the research on dual processes by Zhang and Chen [13], we investigate the algebraic and exponential transience for CTMCs with the help of their dual processes and have the following theorem.

Theorem 9 Assume that the irreducible CTMC $X^{(1)}$ is honest and stochastically monotone. Let T be the absorption time of its dual process $X^{(2)}$, then

(i) $X^{(1)}$ is transient if and only if $\mathsf{E}_1[T] < \infty$;

(ii) $X^{(1)}$ is ℓ-transient if and only if $\mathsf{E}_1[T^{\ell+1}] < \infty$;

(iii) $X^{(1)}$ is exponentially transient if and only if there exists a positive number α such that $\mathsf{E}_1[\mathrm{e}^{\alpha T}] < \infty$.

Proof According to (5), we have

$$\mathsf{P}_j\{X_t^{(1)} \geqslant i\} = \mathsf{P}_i\{X_t^{(2)} \leqslant j\}, \qquad i, j \in E,\ t \geqslant 0.$$

In particular, when $i = j = 0$, we can obtain

$$\mathsf{P}_0\{X_t^{(1)} \geqslant 0\} = \mathsf{P}_0\{X_t^{(2)} \leqslant 0\} = P_{00}^{(2)}(t); \tag{6}$$

when $i = 1, j = 0$, it follows that

$$\mathsf{P}_0\{X_t^{(1)} \geqslant 1\} = \mathsf{P}_1\{X_t^{(2)} \leqslant 0\} = P_{10}^{(2)}(t). \tag{7}$$

By subtracting equation (7) from equation (6), we can get

$$P_{00}^{(1)}(t) = P_{00}^{(2)}(t) - P_{10}^{(2)}(t) = 1 - P_{10}^{(2)}(t) = \mathsf{P}_1\{T > t\}, \tag{8}$$

where the second equality follows from Lemma 8 and the fact that $Q^{(1)}$ is regular. Hence

$$\int_0^\infty P_{00}^{(1)}(t)\mathrm{d}t = \int_0^\infty \mathsf{P}_1\{T > t\}\mathrm{d}t = \mathsf{E}_1[T].$$

Furthermore, let $F(t) := \mathsf{P}_1\{T \leqslant t\}$ be the probability distribution function of T starting from state 1, then from equation (8) we have

$$\begin{aligned}
\mathsf{E}_1[T^{\ell+1}] &= \int_0^\infty t^{\ell+1}\mathrm{d}F(t) \\
&= \int_0^\infty \Big[\int_0^t (\ell+1)x^\ell \mathrm{d}x\Big]\mathrm{d}F(t) \\
&= \int_0^\infty \Big[\int_x^\infty \mathrm{d}F(t)\Big](\ell+1)x^\ell \mathrm{d}x \\
&= (\ell+1)\int_0^\infty x^\ell \mathsf{P}_1\{T > x\}\mathrm{d}x \\
&= (\ell+1)\int_0^\infty x^\ell P_{00}^{(1)}(x)\mathrm{d}x.
\end{aligned}$$

Since $\ell + 1$ is fixed and positive, then chain $X^{(1)}$ is ℓ-transient for state 0 if and only if $\mathsf{E}_1[T^{\ell+1}] < \infty$.

Similar to the above analysis, we can have

$$\begin{aligned}
\mathsf{E}_1[\mathrm{e}^{\alpha T}] &= \int_0^\infty \sum_{n=0}^\infty \frac{(\alpha t)^n}{n!}\mathrm{d}F(t) \\
&= \sum_{n=0}^\infty \frac{\alpha^n}{n!}\int_0^\infty \Big(\int_0^t n x^{n-1}\mathrm{d}x\Big)\mathrm{d}F(t) \\
&= \sum_{n=0}^\infty \frac{\alpha^n}{n!}\int_0^\infty \Big[\int_x^\infty \mathrm{d}F(t)\Big] n x^{n-1}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_0^{\infty} n x^{n-1} \mathsf{P}_1\{T > x\} \mathrm{d}x \\
&= \int_0^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} n x^{n-1} \mathsf{P}_1\{T > x\} \mathrm{d}x \\
&= \alpha \int_0^{\infty} \mathrm{e}^{\alpha x} \mathsf{P}_1\{T > x\} \mathrm{d}x \\
&= \alpha \int_0^{\infty} \mathrm{e}^{\alpha x} P_{00}^{(1)}(x) \mathrm{d}x.
\end{aligned}
$$

Since α is fixed and positive, then the chain $X^{(1)}$ is exponentially transient for state 0 if and only if $\mathsf{E}_1[\mathrm{e}^{\alpha T}] < \infty$. Finally, since $X^{(1)}$ is irreducible, then the transience for state 0 is equivalent to the transience for any state in E. So the proof is complete. □

Remark 10 (i) of Theorem 9 is not new, which was first presented in [12]. Here we extend it to the algebraic and exponential transience. Meanwhile, it is worth noting that (ii) is actually valid for any $\ell \in \mathbb{R}_+$.

Consider a CTMC $X^{(1)}$ on the state space E with the q-matrix $Q^{(1)} = (q_{ij}^{(1)})$. Define the matrix $Q^{(2)}$ by

$$
q_{ij}^{(2)} = \sum_{k=i}^{\infty} (q_{jk}^{(1)} - q_{j-1,k}^{(1)}), \qquad i, j \in E, \tag{9}
$$

where $q_{-1,k}^{(1)} \equiv 0$. From Proposition 4.2 in Chapter 7 of [10], we know that $Q^{(2)}$ is the q-matrix of dual process $X^{(2)}$. As shown in Lemma 8, if $X^{(1)}$ is honest, then $X^{(2)}$ is transient and state 0 is an absorbing state. It is usually not intuitive to get the absorption information directly from $Q^{(2)}$. However, if the set $E' = E \backslash \{0\}$ is communicating, then we may get the information about the absorption time of $X^{(2)}$ through the known ergodicity results. The details are explain as follows.

Define

$$
Q^{(3)} = (q_{ij}^{(3)}) = \begin{pmatrix} -1 & 1 & 0 & \cdots \\ q_{10}^{(2)} & q_{11}^{(2)} & q_{12}^{(2)} & \cdots \\ q_{20}^{(2)} & q_{21}^{(2)} & q_{22}^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},
$$

then the Markov chain with generator $Q^{(3)}$ (denoted by $X^{(3)}$) is irreducible. If we denote by $\mathsf{E}_i[\tau_j]$ the expected time until the process enters state j for the first time starting in i, then $\mathsf{E}_1[\tau_0]$ of $X^{(3)}$ is equal to the mean absorption time of $X^{(2)}$ starting from state 1, which is what we focus on. Hence, through the ergodicity results of $X^{(3)}$, we can easily obtain the information about the absorption times of $X^{(2)}$.

§4. Applications

Now we apply the results in Sections 2 and 3 to investigate the transient properties of generalized Markov branching processes and birth-death processes.

1) Generalized Markov Branching Processes

Example 11 Consider a generalized Markov branching process $X = \{X_t, t \in \mathbb{R}_+\}$ (see [14]) on a countable state space $E = \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ with q-matrix $Q = (q_{ij})$ give by

$$q_{ij} = \begin{cases} b_j, & i = 0; \\ r_i a_{j-i+1}, & i \geqslant 1,\ j \geqslant i-1; \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\begin{cases} -b_0 = \sum\limits_{j=1}^{\infty} b_j < \infty, & b_j \geqslant 0,\ j \geqslant 1; \\ -a_1 = \sum\limits_{j \neq 1}^{\infty} a_j < \infty, & a_j \geqslant 0,\ j \neq 1. \end{cases}$$

Suppose X is irreducible, equivalently, P is irreducible. Then the jump chain of X has transition matrix given by

$$P_{ij} = \begin{cases} \dfrac{b_j}{-b_0}, & i = 0,\ j \geqslant 1; \\ \dfrac{a_{j-i+1}}{-a_1}, & i \geqslant 1,\ j \geqslant i-1,\ j \neq i; \\ 0, & \text{otherwise}. \end{cases}$$

Obviously, the jump chain is a M/G/1-type Markov chain. From the proof of Theorem 1 in [3], we know that if a M/G/1-type Markov chain is transient, then it will be geometrically transient. Hence from Theorem 1 in [3] and our Theorem 1, we can get the following proposition.

Proposition 12 The generalized Markov branching process X is transient if and only if $\sum\limits_{k \neq 1} k a_k > a_1$. Moreover, X is exponentially transient if it is transient.

2) Birth and Death Processes

Next, we consider the classical birth and death processes. We will illustrate the transience properties through their corresponding dual processes.

Example 13 Consider a continuous-time birth and death process $X^{(1)}$ on the state space $E=\mathbb{Z}_+$ with q-matrix $Q^{(1)}=(q_{ij}^{(1)})$ given by

$$q_{ij}^{(1)}=\begin{cases}\lambda_i, & \text{if } j=i+1,\ i\geqslant 0;\\ \mu_i, & \text{if } j=i-1,\ i\geqslant 1;\\ -\lambda_0, & \text{if } j=i=0;\\ -(\lambda_i+\mu_i), & \text{if } j=i,\ i\geqslant 1.\end{cases} \tag{10}$$

According to the analysis in Section 3, we can determine the dual process $X^{(2)}$ of $X^{(1)}$ with q-matrix

$$Q^{(2)}=(q_{ij}^{(2)})=\begin{pmatrix}0 & 0 & 0 & 0 & \cdots\\ \lambda_0 & -\lambda_0-\mu_1 & \mu_1 & 0 & \cdots\\ 0 & \lambda_1 & -\lambda_1-\mu_2 & \mu_2 & \cdots\\ 0 & 0 & \lambda_2 & -\lambda_2-\mu_3 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Define the CTMC $X^{(3)}$ with q-matrix

$$Q^{(3)}=(q_{ij}^{(3)})=\begin{pmatrix}-1 & 1 & 0 & 0 & \cdots\\ \lambda_0 & -\lambda_0-\mu_1 & \mu_1 & 0 & \cdots\\ 0 & \lambda_1 & -\lambda_1-\mu_2 & \mu_2 & \cdots\\ 0 & 0 & \lambda_2 & -\lambda_2-\mu_3 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Let $\mu_0=1$ and $\beta_0=1$, $\beta_i=(\mu_0\mu_1\cdots\mu_{i-1})/(\lambda_0\lambda_1\cdots\lambda_{i-1})$, $i\geqslant 1$. According to [15] or [8], we have for the chain $X^{(3)}$ and for $k\geqslant 1$,

$$\begin{aligned}\mathsf{E}_k[\tau_0]&=\sum_{j=0}^{k-1}\frac{1}{\beta_j\mu_j}\sum_{i=j+1}^{\infty}\beta_i,\\ \mathsf{E}_k[\tau_0^n]&=n\sum_{j=0}^{k-1}\frac{1}{\beta_j\mu_j}\sum_{i=j+1}^{\infty}\beta_i\mathsf{E}_i[\tau_0^{n-1}], \qquad n>1,\end{aligned}$$

where $\mathsf{E}_i[\tau_0^n]$ represents the n-order moment of the time until the chain $X^{(3)}$ enters state 0 for the first time starting in i. Further, from [5] or Theorem 1.1 and Remark 2.2 in [9] we know that the chain $X^{(3)}$ is exponentially ergodic, namely, there exists a positive number α such that $\mathsf{E}_1[\mathrm{e}^{\alpha\tau_0}]<\infty$, if and only if

$$\sup_{k>0}\sum_{j=0}^{k-1}\frac{1}{\beta_j\mu_j}\sum_{i=k}^{\infty}\beta_i<\infty.$$

Hence, from Theorem 9 and the above analysis we know that the chain $X^{(1)}$ is transient if and only if

$$\mathsf{E}_1[\tau_0] < \infty, \quad \text{that is,} \quad \sum_{i=1}^{\infty} \frac{\mu_0\mu_1\cdots\mu_{i-1}}{\lambda_0\lambda_1\cdots\lambda_{i-1}} < \infty.$$

In addition, for algebraic and exponential transience we have the following proposition.

Proposition 14 Let $\ell \in \mathbb{N}$. Suppose that the continuous-time birth and death process is irreducible. Then

(i) $X^{(1)}$ is ℓ-transient if and only if

$$\sum_{k=1}^{\infty} \beta_k \mathsf{E}_k[\tau_0^{\ell}] < \infty,$$

where

$$\mathsf{E}_k[\tau_0^n] = \begin{cases} \sum_{j=0}^{k-1} \frac{1}{\beta_j\mu_j} \sum_{i=j+1}^{\infty} \beta_i, & n = 1; \\ n\sum_{j=0}^{k-1} \frac{1}{\beta_j\mu_j} \sum_{i=j+1}^{\infty} \beta_i \mathsf{E}_i[\tau_0^{n-1}], & n > 1. \end{cases}$$

(ii) $X^{(1)}$ is exponentially transient if and only if

$$\sup_{k>0} \sum_{j=0}^{k-1} \frac{1}{\beta_j\mu_j} \sum_{i=k}^{\infty} \beta_i < \infty.$$

Remark 15 Above we have obtained the criteria for algebraic and exponential transience for continuous-time birth-death processes. Furthermore, by the method of h-approximation chain we can also obtain the criteria for algebraic and geometric transience for discrete-time birth-death processes.

Consider an irreducible DTMC $Z = \{Z_n, n \in \mathbb{Z}_+\}$ with transition matrix

$$P = (p_{ij}) = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ a_1 & r_1 & b_1 & 0 & \cdots \\ 0 & a_2 & r_2 & b_2 & \cdots \\ 0 & 0 & a_3 & r_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can define the corresponding CTMC $X = \{X_t, t \in \mathbb{R}_+\}$ with generator $Q = P - I$. Since Q is bounded, by Liu et al.[3] we know that X is transient, ℓ-transient or exponentially transient if and only if so is Z. By Proposition 14, we can get the criteria for algebraic and geometrical transience for discrete-time birth-death process Z directly.

§5. Appendix: Proof of Proposition 4

Proof Since the chain X is irreducible, it is direct to show that $\int_0^\infty r(t)P_{ii}(t)\mathrm{d}t$ for some $i \in E$ if and only if $\int_0^\infty r(t)P_{jj}(t)\mathrm{d}t$ for any $j \in E$. Hence, we only need to prove the following statements are equivalent for any fixed $i \in E$ and any fixed finite set A containing i:

(i′) $\displaystyle\int_0^\infty r(t)P_{ii}(t)\mathrm{d}t < \infty$,

(ii′) $F_{ii} < 1$, $\mathsf{E}_i[r(\tau_i)I_{\{\tau_i<\infty\}}] < \infty$,

(iii′) $\displaystyle\max_{i\in A}\int_0^\infty r(t)P_{iA}(t)\mathrm{d}t < \infty$,

(iv′) $\displaystyle\max_{i\in A}\mathsf{E}_i[r(\tau_A)I_{\{\tau_A<\infty\}}] < \infty$ and $F_{jA} < 1$ for some j.

Similar to the proof of Proposition 1 in [3], we will establish the equivalent relationship by showing that (ii′) $\Rightarrow$ (i′) $\Rightarrow$ (iii′) $\Rightarrow$ (iv′) $\Rightarrow$ (ii′).

(ii′) $\Rightarrow$ (i′). For $r(t) = t^\ell$, $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
\int_0^\infty t^\ell P_{ii}(t)\mathrm{d}t &= \int_0^\infty t^\ell \Big[\mathrm{e}^{-q_i t} + \int_0^t P_{ii}(t-u)\mathrm{d}F_{ii}(u)\Big]\mathrm{d}t \\
&= \frac{\ell!}{q_i^{\ell+1}} + \int_0^\infty\int_0^\infty (v+u)^\ell P_{ii}(v)\mathrm{d}v\mathrm{d}F_{ii}(u) \\
&= \frac{\ell!}{q_i^{\ell+1}} + \int_0^\infty\int_0^\infty \Big(v^\ell + u^\ell + \sum_{k=1}^{\ell-1} C_\ell^k v^k u^{\ell-k}\Big) P_{ii}(v)\mathrm{d}v\mathrm{d}F_{ii}(u) \\
&= \frac{\ell!}{q_i^{\ell+1}} + F_{ii}\int_0^\infty v^\ell P_{ii}(v)\mathrm{d}v + \mathsf{E}_i\big[\tau_i^\ell I_{\{\tau_i<\infty\}}\big]\int_0^\infty P_{ii}(v)\mathrm{d}v \\
&\quad + \sum_{k=1}^{\ell-1} C_\ell^k \int_0^\infty u^{\ell-k}\mathrm{d}F_{ii}(u)\int_0^\infty v^k P_{ii}(v)\mathrm{d}v \\
&\leqslant \frac{\ell!}{q_i^{\ell+1}} + F_{ii}\int_0^\infty v^\ell P_{ii}(v)\mathrm{d}v + \mathsf{E}_i\big[\tau_i^\ell I_{\{\tau_i<\infty\}}\big]\int_0^\infty P_{ii}(v)\mathrm{d}v \\
&\quad + \big(1 + \mathsf{E}_i\big[\tau_i^\ell I_{\{\tau_i<\infty\}}\big]\big)\sum_{k=1}^{\ell-1} C_\ell^k \int_0^\infty v^k P_{ii}(v)\mathrm{d}v,
\end{aligned}$$

where the last inequality holds since

$$\begin{aligned}
\int_0^\infty u^{\ell-k}\mathrm{d}F_{ii}(u) &= \int_0^1 u^{\ell-k}\mathrm{d}F_{ii}(u) + \int_1^\infty u^{\ell-k}\mathrm{d}F_{ii}(u) \\
&\leqslant 1 + \int_1^\infty u^\ell \mathrm{d}F_{ii}(u) \leqslant 1 + \int_0^\infty u^\ell\mathrm{d}F_{ii}(u) \\
&= 1 + \mathsf{E}_i\big[\tau_i^\ell I_{\{\tau_i<\infty\}}\big].
\end{aligned}$$

Hence we can obtain

$$\begin{aligned}\int_0^\infty t^\ell P_{ii}(t)\mathrm{d}t \leqslant & \Big[\frac{\ell!}{q_i^{\ell+1}} + \sum_{k=1}^{\ell-1} C_\ell^k \int_0^\infty v^k P_{ii}(v)\mathrm{d}v\Big]/(1-F_{ii}) \\ & + \frac{\mathsf{E}_i\big[\tau_i^\ell I_{\{\tau_i<\infty\}}\big]}{1-F_{ii}}\Big[\int_0^\infty P_{ii}(v)\mathrm{d}v + \sum_{k=1}^{\ell-1} C_\ell^k \int_0^\infty v^k P_{ii}(v)\mathrm{d}v\Big].\end{aligned}$$

From the above inequality, we can use an induction argument to yield that $\int_0^\infty t^\ell P_{ii}(t)\mathrm{d}t < \infty$.

For $r(t)=s^t$, $s>1$, from the assumption that $F_{ii}<1$ and $\mathsf{E}_i[s^{\tau_i} I_{\{\tau_i<\infty\}}] = \int_0^\infty s^t \mathrm{d}F_{ii}(t) < \infty$, we have that there exists some $\widehat{s} > 1$ such that $\widehat{s} < \mathrm{e}^{q_i}$ and $\mathsf{E}_i[\widehat{s}^{\tau_i} I_{\{\tau_i<\infty\}}] < 1$. In fact, we have

$$\begin{aligned}\int_0^\infty \widehat{s}^t P_{ii}(t)\mathrm{d}t &= \int_0^\infty \widehat{s}^t \Big[\mathrm{e}^{-q_i t} + \int_0^t P_{ii}(t-u)\mathrm{d}F_{ii}(u)\Big]\mathrm{d}t \\ &= \int_0^\infty (\widehat{s}\,\mathrm{e}^{-q_i})^t \mathrm{d}t + \int_0^\infty \int_0^\infty \widehat{s}^{(u+v)} P_{ii}(v)\mathrm{d}v\mathrm{d}F_{ii}(u) \\ &= \frac{1}{q_i - \ln \widehat{s}} + \int_0^\infty \widehat{s}^u \mathrm{d}F_{ii}(u) \int_0^\infty \widehat{s}^v P_{ii}(v)\mathrm{d}v.\end{aligned}$$

That is,

$$\int_0^\infty \widehat{s}^t P_{ii}(t)\mathrm{d}t = \Big\{(q_i - \ln \widehat{s})\Big[1 - \int_0^\infty \widehat{s}^u \mathrm{d}F_{ii}(u)\Big]\Big\}^{-1}.$$

Since $\mathsf{E}_i[\widehat{s}^{\tau_i} I_{\{\tau_i<\infty\}}] = \int_0^\infty \widehat{s}^u \mathrm{d}F_{ii}(u) < 1$, we can obtain $\int_0^\infty \widehat{s}^t P_{ii}(t)\mathrm{d}t < \infty$.

Specially, when $s = 1$, it is easy to show that

$$\int_0^\infty P_{ii}(t)\mathrm{d}t = \frac{1}{q_i(1-F_{ii})} < \infty.$$

(i′) $\Rightarrow$ (iii′). Since the chain is irreducible, for any state j such that $j \neq i$ and $j \in A$, there exists a positive number $u_j > 0$ such that $P_{ji}(u_j) > 0$. Then for $r(t) = t^\ell$, $\ell \in \mathbb{N}$ or $r(t) = s^t$, $s \geqslant 1$, we have

$$\begin{aligned}\int_0^\infty r(t)P_{ii}(t)\mathrm{d}t &\geqslant \int_{u_j}^\infty r(t-u_j+u_j)P_{ii}(t)\mathrm{d}t \\ &\geqslant \int_{u_j}^\infty r(t-u_j)P_{ii}(t)\mathrm{d}t \\ &\geqslant P_{ji}(u_j)\int_{u_j}^\infty r(t-u_j)P_{ij}(t-u_j)\mathrm{d}t \\ &= P_{ji}(u_j)\int_0^\infty r(v)P_{ij}(v)\mathrm{d}v. \qquad (11)\end{aligned}$$

It follows that $\int_0^\infty r(t)P_{ij}(t)\mathrm{d}t < \infty$ from the assumption that $\int_0^\infty r(t)P_{ii}(t)\mathrm{d}t < \infty$. Since A is a finite set, we have for any $i \in A$,

$$\begin{aligned}\int_0^\infty r(t)P_{iA}(t)\mathrm{d}t &= \int_0^\infty \sum_{j\in A} r(t)P_{ij}(t)\mathrm{d}t \\ &= \int_0^\infty r(t)P_{ii}(t)\mathrm{d}t + \sum_{j\in A, j\neq i}\int_0^\infty r(t)P_{ij}(t)\mathrm{d}t < \infty. \end{aligned} \tag{12}$$

(iii′) ⇒ (iv′). Assume that (iii′) holds. Obviously, for any $i \in A$,

$$\mathsf{E}_i\big[r(\tau_A)I_{\{\tau_A<\infty\}}\big] \leqslant \int_0^\infty r(t)P_{iA}(t)\mathrm{d}t < \infty.$$

Now we prove that $F_{jA} < 1$ for some $j \in A$. By Theorem 1, we have known that the transience properties for the chain X are equivalent to those for its jump chain. Denote the jump chain by $Y = \{Y_n, n \in \mathbb{Z}_+\}$ and let its transition matrix, the first return time on the set A and the probability ever returning to A respectively be $\overline{P}$, $\overline{\tau}_A$ and $\overline{F}_{iA}$. According to (11) and (12), we can get for any $i \in A$,

$$\int_0^\infty r(t)P_{iA}(t)\mathrm{d}t \leqslant \Big[1 + \sum_{j\in A, j\neq i}\frac{1}{P_{ji}(u_j)}\Big]\int_0^\infty r(t)P_{ii}(t)\mathrm{d}t.$$

Since A is a finite set, then

$$1 + \sum_{j\in A, j\neq i}\frac{1}{P_{ji}(u_j)} < \infty.$$

It follows that for any $i \in A$,

$$\int_0^\infty r(t)P_{iA}(t)\mathrm{d}t < \infty$$

from the assumption that $\max\limits_{i\in A}\int_0^\infty r(t)P_{ii}(t)\mathrm{d}t < \infty$. By Theorem 1 and Proposition 1 in [3], we have

$$\sum_{n=0}^\infty r(n)\overline{P}_{iA}^{(n)} < \infty$$

and then $\overline{F}_{jA} < 1$ for some $j \in A$, which implies that

$$\mathsf{P}\{\overline{\tau}_A = \infty \,|\, Y_0 = j\} > 0.$$

Since the state transitions in the CTMC X are the same as the state transitions in the jump chain Y and X is non-explosive, then we have

$$\mathsf{P}\{\tau_A = \infty \,|\, X_0 = j\} > 0,$$

which shows that $F_{jA} < 1$ for some $j \in A$.

(iv′) ⇒ (ii′). It is obvious that $F_{jj} \leqslant F_{jA} < 1$ for some $j \in A$, which implies that $F_{ii} < 1$ for any $i \in E$ according to the irreducibility of X. Since

$$\max_{i\in A} \mathsf{E}_i\big[r(\tau_A)I_{\{\tau_A<\infty\}}\big] < \infty,$$

from Lemma 3.3 in [16], there exits a subset $A' \subset A$ such that

$$\max_{i\in A} \mathsf{E}_i\big[r(\tau_{A'})I_{\{\tau_{A'}<\infty\}}\big] < \infty.$$

By irreducibility, in a manner similar to Remark (2) on page 210 of [10], we have for any $j \in A$,

$$\max_{i\in A} \mathsf{E}_i\big[r(\tau_j)I_{\{\tau_j<\infty\}}\big] < \infty.$$

Then, from Remark 2.2 in [17], we can obtain that for any i,

$$\mathsf{E}_i\big[r(\tau_i)I_{\{\tau_i<\infty\}}\big] < \infty.$$

Hence the proof is finished. □

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连续时间马氏链的代数及指数非常返性

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摘 要: 本文研究了连续时间马氏链的代数非常返性和指数非常返性, 揭示了连续时间马氏链与其跳跃链和对偶过程之间的等价关系. 运用所得结果, 我们进一步得到了广义分支过程和生灭过程等连续时间马氏链的非常返性判别准则.

关键词: 马氏链; 代数非常返; 指数非常返; 跳跃链; 对偶过程

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