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The Smallest g-Supersolution for BSDE with Jumps*

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Abstract

For Backward Stochastic Differential Equation with jumps and an increasing predictable RCLL process as its penalization term, we have defined g-supersolution for such a BSDE and obtained the limit theorem. As an application of the limit theorem, the existence and uniqueness of the smallest supersolution for BSDE with jumps and constraints on (Y, Z, q) is proved.

Keywords: Backward stochastic differential equation, g-supersolution, constraint.

AMS Subject Classification: 60H99, 60H30.

§ 1. Introduction

The idea of g-supersolution was introduced by Peng[2], and the limit theorem of a sequence of RCLL increasing g-supersolution for a BSDE with Brownian motion was investigated. As its application, the existence and uniqueness of the smallest g-supersolution with the constraint on solution (Y, Z) was obtained. By Lin (1999)[7] the same problems was discussed for BSDE without Lipschitz condition both on drift coefficient and constraint.

In the case of BSDE with jumps to consider these problems, such as introducing the conception of g-supersolution,monotonic limit theorem for g-supersolution, and the smallest g-supersolution with constraint, is still very interesting, but there are some key difficulties in solving these problems. We need a comparison theorem for BSDE with jumps, which the drift coefficient is $b(s, y, z, q)$. The paper is arranged as following: first, we obtain comparison theorem for this BSDE by Doléan-Dade formula and Itô formula for the case. Second, we obtained the limit theorem with jumps, constructed a sequence of g-supersolution for BSDE with jumps, proved that the limit of the sequences is still a g-supersolution. finally, we apply the limit theorem to derive the smallest g-supersolution for BSDE with jumps and constraint.

§ 2. Solution for BSDE with Jumps and Comparison Theorem

2.1 The definition of solution for BSDE with jumps

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Let $(\boldsymbol{\Omega},\mathcal{F},\mathsf{P})$ be a probability space, $(\mathcal{F}_t)_{0\le t\le T}$ be a subfiltration class of $\mathcal{F}$, which was generated by a Brownian motion and a poisson process, that is

$$\mathcal{F}_t=\sigma\Big[\int\int_{[0,s]\times U}N_p(\mathrm{d}l,\mathrm{d}x);s\le t,U\in\mathcal{B}(X)\Big]\vee\sigma[W(s);s\le t]\vee\mathcal{N},$$

where $\mathcal{N}$ denotes a total of P-null sets and $\sigma_1\vee\sigma_2$ denotes the σ field generated by σ_1 and σ_2, $(\mathbf{X},\mathcal{B}(\mathbf{X}))$ is a measurable space, $W(t)$ is a standard wiener process, and $p(\cdot)$ is a stationary $\mathcal{F}_t$-poisson process on X with the characteristic measure $\pi(\mathrm{d}x)$. We denote by $N_p(\mathrm{d}s,\mathrm{d}x)$ the counting measure introduced by $p(\cdot)$ and set $\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x)=N_p(\mathrm{d}s,\mathrm{d}x)-\pi(\mathrm{d}x)\mathrm{d}t$, $\widetilde{N}_p((0,t],U)$, $U\in\mathcal{B}(\mathbf{X})$, is a martingale.

We also use the following notation: $L^2_{p,\mathcal{F}}([0,T],R^d)$ the set of R^d-valued $\mathcal{F}_t$ adapted processes $h_t(x,\omega)$ such that

$$\mathsf{E}\int_0^T\int_X|h_t(x,\omega)|^2\pi(\mathrm{d}x)\mathrm{d}t<\infty,$$

$L^2_{\mathcal{F}}([0,T],R^d)$ the set of $l(t,w)$ $\mathcal{F}_t$-adapted R^d valued processes such that

$$\mathsf{E}\int_0^T|l_t(\omega)|^2\mathrm{d}t<+\infty$$

and $L^2_{\mathcal{F}}([0,T],R^{d\times r})$ defined similarly.

Also denoted by $L^2_{\pi(\cdot)}(R)$ the set of R-valued functions $q(x)$ on X, $\mathcal{B}(\mathbf{X})$ measureable, such that

$$\|q\|=\Big(\int_X|q(x)|^2\pi(\mathrm{d}x)\Big)^{1/2}<+\infty.$$

Consider BSDE

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s,q_s)\mathrm{d}s+A_T-A_t-\int_t^T Z_s\mathrm{d}W_s-\int_t^T\int_X q_s(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x),\qquad T\ge t\ge 0.\quad(2.1)$$

Definition 2.1 (Y_t,Z_t,q_t) is a solution for BSDE (2.1), if

(1) $(Y_t,Z_t,q_t)\in L^2_{\mathcal{F}}([0,T];R)\times L^2_{\mathcal{F}}([0,T];R^d)\times L^2_{p\mathcal{F}}([0,T];R)$;

(2) (Y_t,Z_t,q_t) satisfies (2.1) a.e..

2.2 Comparison theorem for BSDE with jumps

Consider the BSDE

$$Y_t^1=\xi^1+\int_t^T g^1(s,Y_s^1,Z_s^1,q_s^1)\mathrm{d}s+A_T^1-A_t^1-\int_t^T Z_s^1\mathrm{d}W_s-\int_t^T\int_X q_s^1(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x)$$

and

$$Y_t^2=\xi^2+\int_t^T g^2(s,Y_s^2,Z_s^2,q_s^2)\mathrm{d}s+A_T^2-A_t^2-\int_t^T Z_s^2\mathrm{d}W_s-\int_t^T\int_X q_s^2(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x).$$

Theorem 2.1 suppose that the following assumption on data,

(1) $\xi^1\ge\xi^2$;

(2) $g^1(t,Y,Z,q)\ge g^2(t,Y,Z,q)$ $\forall Y\in R,\ Z\in R^d,\ q\in L^2_{\pi(\cdot)}(R)$;

(3) $A_t^1 - A_t^2$ is an increasing process;

(4)

$$-1 < \frac{g(s,y,z,q^1) - g(s,y,z,q^2)}{\displaystyle\int_X (q^1(x) - q^2(x))\pi(\mathrm{d}x)} \le C, \ \ \forall\, q^1, q^2 \in L^2_{\pi(\cdot)}(R), \ \text{ if } \int_X (q^1(x) - q^2(x))\pi(\mathrm{d}x) \neq 0,$$

then $Y_t^1 \ge Y_t^2$, $t \in [0,T]$, where $\pi(X) < \infty$.

Proof Set

$$a_s = \begin{cases} \dfrac{g^1(s,Y_s^1,Z_s^1,q_s^1) - g^1(s,Y_s^2,Z_s^1,q_s^1)}{Y_s^1 - Y_s^2} & Y_s^1 \neq Y_s^2; \\ 0 & \text{others}, \end{cases}$$

$$b_s = \begin{cases} \dfrac{g^1(s,Y_s^2,Z_s^1,q_s^1) - g^2(s,Y_s^2,Z_s^2,q_s^1)}{Z_s^1 - Z_s^2} & Z_s^1 \neq Z_s^2; \\ 0 & \text{others}, \end{cases}$$

$$c_s = \begin{cases} \dfrac{g^2(s,Y_s^2,Z_s^2,q_s^1) - g^2(s,Y_s^2,Z_s^2,q_s^2)}{\displaystyle\int_X (q_s^1(x) - q_s^2(x))\pi(\mathrm{d}x)} & \displaystyle\int_X (q_s^1(x) - q_s^2(x))\pi(\mathrm{d}x) \neq 0; \\ 0 & \text{others}. \end{cases}$$

Let $\widehat{Y}_t = Y_t^1 - Y_t^2$, $\widehat{Z}_t = Z_t^1 - Z_t^2$, $\widehat{q}_t = q_t^1 - q_t^2$, $\widehat{A}_t = A_t^1 - A_t^2$, then

$$\begin{aligned} Y_t^1 - Y_t^2 \ &= \ \xi^1 - \xi^2 + \int_t^T (g^1(s,Y_s^1,Z_s^1,q_s^1) - g^2(s,Y_s^2,Z_s^2,q_s^2))\mathrm{d}s + (A_T^1 - A_T^2) - (A_t^1 - A_t^2) \\ &\quad - \int_t^T (Z_s^1 - Z_s^2)\mathrm{d}W_s - \int_t^T\int_X (q_s^1 - q_s^2)\widetilde{N}(\mathrm{d}s, \mathrm{d}x). \end{aligned}$$

That is

$$\begin{aligned} \widehat{Y}_t \ &= \ \widehat{\xi} + \int_t^T (a_s\widehat{Y}_s + b_s\widehat{Z}_s + g_0(s))\mathrm{d}s + \int_t^T\int_X c_s\widehat{q}_s\pi(\mathrm{d}x)\mathrm{d}s - \int_t^T \widehat{Z}_s\mathrm{d}W_s \\ &\quad - \int_t^T\int_X \widehat{q}_s\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x) + \widehat{A}_T - \widehat{A}, \end{aligned}$$

where $g_0(s) = g^1(s,Y_s^2,Z_s^2,q_s^2) - g^2(s,Y_s^2,Z_s^2,q_s^2) \ge 0$. Consider the solution for the following linear BSDE:

$$\mathrm{d}Y_s = -(a_sY_s + b_sZ_s + g_0(s))\mathrm{d}s - \int_X c_sq_s\pi(\mathrm{d}x)\mathrm{d}s - \mathrm{d}A_s + Z_s\mathrm{d}W_s + \int_X q_s\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x), \quad Y_T = \xi,$$

where $\xi \ge 0$, $\mathrm{d}A_s \ge 0$, $g_0(s) \ge 0$. $L_t = 1 + \displaystyle\int_0^t L_{s-}\mathrm{d}X_s$ $(\star)$, where

$$X_t = \int_t^T a_s\mathrm{d}s + \int_t^T b_s\mathrm{d}W_s + \int_t^T\int_X c_s\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x).$$

It is easy to see $\triangle X_t = c_t > -1$, by Doléans-Dade exponential formula[8],

$$L_t = \exp\left\{X_t - X_0 - \frac{1}{2} < X >_t\right\} \prod_{0<s\le t}(1 + \triangle X_s)\exp\{-\triangle X_s\}$$

is the unique solution for SDE $(\star)$. Since $\triangle X_s > -1$, $L_t > 0$ a.s.. Applying Itô to L_tY_t[5], We have:

$$\begin{aligned}
\mathrm{d}Y_tL_t &= L_t\mathrm{d}Y_t + Y_t\mathrm{d}L_t + \mathrm{d}[L,Y]_t\\
&= L_t\Big[\Big(-a_tY_t - b_tZ_t - \int_X c_t(\omega)q_t\pi(\mathrm{d}x) - g_0(t)\Big)\mathrm{d}t + Z_t\mathrm{d}W_t + \int_X q_t\widetilde{N}_p(\mathrm{d}t,\mathrm{d}x) - \mathrm{d}A_t\Big]\\
&\quad +Y_t\Big[L_{t-}b_t\mathrm{d}W_t + L_{t-}a_t\mathrm{d}t + \int_X c_t(\omega)L_{t-}\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + Z_tL_{t-}b_t\mathrm{d}t\Big]\\
&\quad +\int_X c_t(\omega)L_{t-}q_t\pi(\mathrm{d}x)\mathrm{d}t,\\
L_tY_t &= \mathsf{E}^{\mathcal{F}_t}\Big[\int_t^T L_sg_0(s)\mathrm{d}s + \int_t^T L_s\mathrm{d}A_s + Y_TL_T\Big] \geq 0,
\end{aligned}$$

therefore $Y \geq 0$, we have $Y_t^1 \geq Y_t^2$. #

Corollary 2.1 Suppose that $\phi(s,y,z,q)$ satisfies Lipschitz condition with respect to

$$y,\ z,\ q,\ \phi(s,y,z,q) \geq 0, \qquad \forall\,(y,z) \in R^{1+d},\ q \in L^2_{\pi(\cdot)}(R),$$

and

$$0 < \frac{\phi(s,y,z,q^1) - \phi(s,y,z,q^2)}{\displaystyle\int_X (q^1-q^2)\pi(\mathrm{d}x)} \leq C$$

if $\int_X (q^1-q^2)\pi(\mathrm{d}x) \neq 0$ $(\star,\star)$, then $Y^i \leq Y^{i+1}$ a.s., where (Y^i, Z^i, q^i) is the solution for BSDE:

$$Y_t^i = \xi + \int_t^T g(s,Y_s^i,Z_s^i,q_s^i)\mathrm{d}s + i\int_t^T \phi(s,Y_s^i,Z_s^i,q_s^i)\mathrm{d}s - \int_t^T Z_s^i\mathrm{d}W_s - \int_t^T q_s^i\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x).$$

Proof Since $\phi(s,y,z,q) \geq 0$, $i\phi(s,y,z,q) < (1+i)\phi(s,y,z,q)$, and

$$\begin{aligned}
&\frac{g(s,y,z,q^1) + i\phi(s,y,z,q^1) - g(s,y,z,q^2) - i\phi(s,y,z,q^2)}{\displaystyle\int_X (q^1-q^2)\pi(\mathrm{d}x)}\\
=\ &\frac{g(s,y,z,q^1) - g(s,y,z,q^2)}{\displaystyle\int_X (q^1-q^2)\pi(\mathrm{d}x)} + \frac{i\phi(s,y,z,q^1) - i\phi(s,y,z,q^2)}{\displaystyle\int_X (q^1-q^2)\pi(\mathrm{d}x)}
\end{aligned}$$

if $\int_X (q^1-q^2)\pi(\mathrm{d}x) \neq 0$. By (iii) and $(\star,\star)$

$$-1 < \frac{g(s,y,z,q^1) + i\phi(s,y,z,q^1) - g(s,y,z,q^2) - i\phi(s,y,z,q^2)}{\displaystyle\int_X (q^1-q^2)\pi(\mathrm{d}x)} \leq C_2,$$

where $C_2 = C + C_1$, by comparison theorem, the result is obtained. #

2.3 g-supersolution decomposition

Thanks to the comparison theorem,we can define g-supersolution for BSDE with jumps, and since the existence and uniqueness of solution for the BSDE (2.1), we have the following proposition of uniqueness decomposition for g-supersolution.

Let BSDE be as the following:

$$Y_t=\xi+\int_{t\wedge\tau}^{\tau}g(s,Y_s,Z_s,q_s)\mathrm{d}s+A_\tau-A_{t\wedge\tau}-\int_{t\wedge\tau}^{\tau}Z_s\mathrm{d}W_s-\int_{t\wedge\tau}^{\tau}\int_X q_s(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x),\quad 0\le t\le\tau,\tag{2.2}$$

where τ is a given stopping time, $\xi\in L^2(\mathbf{\Omega},\mathcal{F}_\tau,\mathsf{P})$, A is a given RCLL predictable increasing process with $A_0=0$ and $\mathsf{E}A_\tau^2<\infty$.

Definition 2.2 If Y_t is a solution BSDE (2.2). then we call Y_t a g-supersolution on $[0,\tau]$. If $A_t\equiv 0\ \ \forall\, t\in[0,\tau]$, we call Y_t a g-solution on $[0,\tau]$.

Proposition 2.1 Given Y_t a g-supersolution on $[0,\tau]$, there exists a unique $Z_t\in L^2_{\mathcal{F}}([0,\tau],R^d)$, a unique increasing RCLL process A_t on $[0,\tau]$, with $A_0=0$ and $\mathsf{E}A_\tau^2<+\infty$, $q\in L^2_{p,\mathcal{F}}([0,\tau],R)$, such that the fourfold (Y,Z,A,q) satisfy (2.2).

Proof Suppose that there exists two triple A_t^1,Z_t^1,q_t^1; A_t^2,Z_t^2,q_t^2 satisfying (2.2). We have

$$Y_t=\xi+\int_{t\wedge\tau}^{\tau}g(s,Y_s^1,Z_s^1,q_s^1)\mathrm{d}s-\int_{t\wedge\tau}^{\tau}Z_s^1\mathrm{d}W_s+A_\tau^1-A_{t\wedge\tau}^1-\int_{t\wedge\tau}^{\tau}\int_X q_s^1(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x)$$

and

$$Y_t=\xi+\int_{t\wedge\tau}^{\tau}g(s,Y_s^2,Z_s^2,q_s^2)\mathrm{d}s-\int_{t\wedge\tau}^{\tau}Z_s^2\mathrm{d}W_s+A_\tau^2-A_{t\wedge\tau}^1-\int_{t\wedge\tau}^{\tau}\int_X q_s^2(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x).$$

By Itô formula[5],

$$\mathsf{E}\int_0^t|Z_s^1-Z_s^2|^2\mathrm{d}s+\int_0^t\int_X|q_s^1-q_s^2|^2\lambda(\mathrm{d}x)\mathrm{d}t+\sum_{0<s\le t}[\triangle(A_s^1-A_s^2)]^2=0,$$

that is $Z^1=Z^2$, $q^1=q^2$ a.e. and from this it follows $A_t^1=A_t^2$. #

Definition 2.3 Let Y_t be a g-supersolution on $[0,\tau]$ and let (Y_t,Z_t,A_t,q_t) be the related unique fourfold in the sense of preposition 2.1, then we call (A_t,Z_t,q_t) the unique decomposition of Y_t.

§3. Monotonic Limits Theorem for g-Supersolution with Jumps

Let Y_t^i be an increasing process which converges to Y_t, with $\mathsf{E}\sup\limits_{0\le t\le T}|Y_t|^2<+\infty$. It is clear, that

$$\mathsf{E}\sup_{0\le t\le T}|Y_t^i|^2\le C,\qquad \mathsf{E}\int_0^T|Y_t^i-Y_t|^2\mathrm{d}s\longrightarrow 0,\tag{3.1}$$

where the constant C is independent of i, and A_t^i is a continuous increasing process with

$$\mathsf{E}|A_T^i|^2<+\infty.\tag{H1}$$

Lemma 3.1 Under the assumpation of (3.1) and (H1), there exists a constant C, which is independent of i, such that:

$$\mathsf{E}\int_0^T|Z_s^i|^2\mathrm{d}s\le C,\qquad \mathsf{E}(A_T^i)^2\le C,\qquad \mathsf{E}\int_0^T\int_X|q_s^i(x)|^2\pi(\mathrm{d}x)\mathrm{d}t\le C,$$

where Y^i, A^i, Z^i, q^i satisfy the following BSDE:

$$Y_t^i = \xi + \int_t^T g(s, Y_s^i, Z_s^i, q_s^i)\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}W_s + A_T^i - A_t^i - \int_t^T\int_X q_s^i(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x).$$

Proof

$$A_T^i = Y_0^i - Y_T^i - \int_0^T g(s, Y_s^i, Z_s^i, q_s^i)\mathrm{d}s + \int_t^T Z_s^i \mathrm{d}W_s + \int_t^T\int_X q_s^i(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x),$$

by (3.1), we have

$$\begin{aligned} \mathsf{E}|A_T^i|^2 &\le C_1 + C_2\mathsf{E}\int_0^T |Z_s^i|^2\mathrm{d}s + C_3\mathsf{E}\int_0^T\int_X |q_s^i(x)|^2\pi(\mathrm{d}x)\mathrm{d}s \\ &\le C_1 + C_4\Big(\mathsf{E}\int_0^T |Z_s^i|^2\mathrm{d}s + \mathsf{E}\int_0^T \|q_s^i\|^2\mathrm{d}s\Big), \end{aligned}$$

where $C_4 = \max(C_2, C_3)$, on the other hand, by Itô formula, we have:

$$\mathsf{E}|Y_0^i|^2 + \mathsf{E}\int_0^T |Z_s^i|^2\mathrm{d}s + \mathsf{E}\int_0^T \|q_s^i\|^2\mathrm{d}s \le \mathsf{E}|Y_T^i|^2 + 2\int_0^T Y_s^i g(s, Y_s^i, Z_s^i, q_s^i)\mathrm{d}s + 2\mathsf{E}\int_0^T Y_s^i\mathrm{d}A_s^i,$$

so

$$\mathsf{E}\int_0^T |Z_s^i|^2\mathrm{d}s + \mathsf{E}\int_0^T \|q_s^i\|^2\mathrm{d}s \le C_5 + \frac{1}{2C_4}\mathsf{E}|A_T^i|^2,$$

from this $\mathsf{E}|A_T^i|^2 \le C$, finelly

$$\mathsf{E}\int_0^T |Z_s^i|^2\mathrm{d}s \le C, \qquad \mathsf{E}\int_0^T \|q_s^i\|^2\mathrm{d}s \le C.$$

We have proved Lemma 3.1. By the lemma, $g^i = g(s, Y_s^i, Z_s^i, q_s^i)$, Z^i, q^i are bounded in $L^2_{\mathcal{F}}([0,T], R)$, $L^2_{\mathcal{F}}([0,T], R^d)$, $L^2_{p,\mathcal{F}}([0,T], R)$, so they have weak limits g^0, Z, q in these spaces respectively[6], so

$$\begin{gathered} \int_0^t g_s^i\mathrm{d}s = \int_0^t g(s, Y_s^i, Z_s^i, q_s^i)\mathrm{d}s - \int_0^t Z_s^i\mathrm{d}W_s - \int_0^t\int_X q_s^i(x)\widetilde{N}(\mathrm{d}s, \mathrm{d}x), \\ \int_0^t\int_X q_s^i(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x) \longrightarrow \int_0^t\int_X q_s(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x) \end{gathered}$$

converges weakly. Since

$$\begin{aligned} Y_t^i &= Y_0^i - \int_0^t g(s, Y_s^i, Z_s^i, q_s^i)\mathrm{d}s - A_t^i + \int_0^t\int_X q_s^i(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x) + \int_0^t Z_s^i\mathrm{d}W_s, \\ A_t^i &= -Y_t^i + Y_0^i - \int_0^t g(s, Y_s^i, Z_s^i, q_s^i)\mathrm{d}s + \int_0^t\int_X q_s^i(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x) + \int_0^t Z_s^i\mathrm{d}W_s. \end{aligned}$$

We have:

$$A_t^i \to A_t = -Y_t + Y_0 - \int_0^t g_s^0\mathrm{d}s + \int_0^t\int_X q_s(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x)$$

weakly in $L^2(\boldsymbol{\Omega}, \mathcal{F}_t, \mathsf{P})$. If Z_s^i, q_s^i converges strongly in $L^p_{\mathcal{F}}([0,T], R)$, $L^p_{p,\mathcal{F}}([0,T], R)$ $(1 \le p < 2)$ respectively then $g^0 = g(s, Y_s, Z_s, q_s)$. #

Lemma 3.2 Let

$$Y_t = Y_0 + M_t + A_t + \int_0^t \int_X q_s(x) \widetilde{N}_p(\mathrm{d}s, \mathrm{d}x).$$

If A_t and $N(\mathrm{d}t, \mathrm{d}x)$ are not jump at the same time, then

$$\begin{aligned} Y_t^2 &= Y_0^2 + 2\int_0^t Y_s \mathrm{d}M_s + 2\int_0^t Y_{s-} \mathrm{d}A_s + \sum_{0<s\le t} (\Delta A_s)^2 \\ &\quad + \int_0^t \mathrm{d}\langle M, M\rangle_s + \int_0^t \int_X q_s^2(x)\pi(\mathrm{d}x)\mathrm{d}s + 2\int_0^{t+} \int_X Y_{s-} q_s(x) \widetilde{N}_p(\mathrm{d}s, \mathrm{d}x), \end{aligned}$$

where M_t is a continuous martingale, A_t is RCLL predictable increasing process, $N_p(\mathrm{d}t, \mathrm{d}x)$ is the poisson counting measure defined by $p(\cdot)$ with compensator $\pi(\mathrm{d}x)\mathrm{d}t$, $\widetilde{N}_p(\mathrm{d}x, \mathrm{d}t)$ is the martingale such that

$$\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x) = N_p(\mathrm{d}s, \mathrm{d}x) - \pi(\mathrm{d}x)\mathrm{d}t.$$

By Ito's formula, we can get the result.

Lemma 3.3 Suppose that A_t is RCLL predicable increasing process, $p(\cdot)$ is a poisson process, there exists a sequence of strictly positive predictable stopping time τ_i such that

$$\sum_{0<s\le T} \Delta A_s = \sum_{\tau_i} A_{\tau_i} 1_{\tau_i \le T}.$$

Then $\tau_i \ne \sigma_j$; $i, j = 1, 2\cdots$, where

$$\sigma_i \in D_{p_n}, \quad i = 1, 2, \cdots \qquad \text{and} \qquad \sigma_1 < \sigma_2 < \cdots < \sigma_n < \cdots;$$
$$D_{p_n} = \{s \in D_p, p(s) \in U_n\};$$

and D_p is a countable subset of $(0, \infty)$, $U_n \in \mathcal{B}(\mathbf{X})$, $U_n \subset U_{n+1}$, $n = 1, \cdots, \cdots$, $\cup U_n = X$.

Proof Since τ_i is a predictable stopping time, we say that σ_i is an unpredictable stopping time. If it is not so, we have ([8] theorem 5.34)

$$\mathsf{E}[N_{\sigma_i} - \lambda\sigma_i - (N_{\sigma_i -} - \lambda\sigma_i -)] = 0,$$

but

$$\mathsf{E}[N_{\sigma_i} - \lambda\sigma_i - (N_{\sigma_i -} - \lambda\sigma_i -)] = \mathsf{E}[1] = 1,$$

so $\sigma_i \ne \tau_j$ a.e., that is A and $N_p(\mathrm{d}t, \mathrm{d}x)$ do not jumps at the same time.

Let

$$Y_t = Y_0 - \int_0^t g_s \mathrm{d}s - A_s + \int_0^t Z_s \mathrm{d}W_s + \int_0^t \int_X q_s(x) \widetilde{N}_p(\mathrm{d}s, \mathrm{d}x),$$

by Itô formula (Lemma 3.2)

$$\begin{aligned} Y_t^2 &= Y_0^2 - 2\int_0^t Y_{s-} g_s \mathrm{d}s - 2\int_0^t Y_{s-} \mathrm{d}A_s + 2\int_0^t Y_s Z_s \mathrm{d}W_s + \int_0^t Z_s^2 \mathrm{d}s + \int_0^t \int_X q_s^2(x)\pi(\mathrm{d}x)\mathrm{d}t \\ &\quad + 2\int_0^t Y_{s-} \int_X q_s(x) \widetilde{N}_p(\mathrm{d}s, \mathrm{d}x) - \sum_{0<s\le t} (\Delta A_s)^2, \end{aligned} \tag{3.2}$$

$$\mathsf{E}Y_T^2+2\mathsf{E}\int_t^T Y_s g_s \mathrm{d}s+2\mathsf{E}\int_t^T Y_s \mathrm{d}A_s+\sum_{t<s\le T}(\Delta A_s)^2=\mathsf{E}Y_t^2+\int_t^T Z_s^2\mathrm{d}s+\mathsf{E}\int_t^T\int_X q_s^2(x)\pi(\mathrm{d}x)\mathrm{d}s,$$

set

$$Y_t^i=\xi+\int_t^T g(s,Y_s^i,Z_s^i,q_s^i)\mathrm{d}s+A_T^i-A_t^i-\int_t^T Z_s^i\mathrm{d}W_s-\int_t^T\int_X q_s^i(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x),$$

we have:

$$\begin{aligned}&\mathsf{E}|Y_t-Y_t^i|^2+\mathsf{E}\int_t^T|Z_s-Z_s^i|^2\mathrm{d}s+\mathsf{E}\int_t^T\int_t^T|q_s(x)-q_s^i(x)|^2\lambda(\mathrm{d}x)\mathrm{d}t\\ =\;&2\mathsf{E}\int_t^T(Y_s-Y_s^i)(g_s^0-g(s,Y_s^i,Z_s^i,q_s^i))\mathrm{d}s+2\mathsf{E}\int_t^T(Y_s-Y_s^i)\mathrm{d}(A_s-A_s^i)\\ &+\sum_{t<s\le T}[\Delta(A_s-A_s^i)]^2+\mathsf{E}|Y_T-Y_T^i|^2,\end{aligned}\tag{3.3}$$

$$2\mathsf{E}\int_t^T|Y_s-Y_s^i||g_s^0-g(s,Y_s^i,Z_s^i,q_s^i)|\mathrm{d}s\le C\Big(\mathsf{E}\int_0^T|Y_s-Y_s^i|^2\mathrm{d}s\Big)^{1/2}\longrightarrow 0.\tag{3.4}$$

(Since $|Y_s^i-Y_s|\le|Y_s^1-Y_s|$, the result above follows by Lebesques dominant convergence theorem.) Since

$$\mathsf{E}\int_0^T|Y_s-Y_s^i|\mathrm{d}A_s\le\Big(\mathsf{E}\sup_{0\le s\le T}|Y_s-Y_s^1|^2\Big)^{1/2}(\mathsf{E}(A_T)^2)^{1/2}\mathsf{E}\int_{[0,T]}|Y_s-Y_s^i|\mathrm{d}A_s\longrightarrow 0.\tag{3.5}$$

If

$$\sum[\Delta(A_s-A_s^i)]^2\longrightarrow 0,$$

then $Z^i\to Z$, $q^i\to q$ stongly, but the case is not so. Since $\Delta A_s^i=0$ for all t and i, A is a RCLL predictable increasing process,

$$\sum_{0<s\le t}[\Delta(A_s-A_s^i)]^2=\sum_{0<s\le T}(\Delta A_s)^2.\qquad\#$$

Proposition 3.1 If Z^i converges to Z in measure on $[0,T]\times\mathbf{\Omega}$, and $\mathsf{E}\int_0^T|Z_s^i|^2\mathrm{d}s$ be bound uniformly, then Z^i converges to Z strongly in $L^p_{\mathcal{F}}([0,T],R)$, $p\in[1,2)$[2].

Proposition 3.2 Z^i and q^i converges to Z, q in measure respectively[2].

By proposition 3.1 and proposition 3.2, $Z^i\longrightarrow Z$, $q^i\longrightarrow q$ strongly $L^p_{\mathcal{F}}([0,T],R)$, $L^p_{p,\mathcal{F}}([0,T],R)$ respectively $p\in[1,2)$.

§ 4. Smallest g-Supersolution with Constraint on (Y,Z,q)

In this section we consider the smallest g-supersolution with jumps and with constranits on (Y,Z,q), ie. Y,Z,q stisfying following BSDE:

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s,q_s)\mathrm{d}s+A_T-A_t-\int_t^T Z_s\mathrm{d}W_s-\int_t^T\int_X q_s(x)\widetilde{N}_p(\mathrm{d}s,\mathrm{d}x)\tag{4.1}$$

and Y, Z, q must be in the set of following:

$$K_t(\omega) = \{(Y, Z, q,) \in R \times R^d \times R_+, \phi(t, Y, Z, q,) = 0\}, \tag{4.2}$$

where g, ϕ satisfy Lip. condition.

$$\phi : R_+ \times R \times R^d \times L^2_{p,\mathcal{F}}(R) \longrightarrow R_+.$$

Definition 4.1 A g-supersolution Y_t on $[0, T]$ with the decomposition (Y, Z, q) is said to be the smallest g-supersolution subjet to the constraints (4.2), if it satisfies (4.2) and $Y_t \le Y'_t$, a.e., for any g-supersolution (Y') on $[0, T]$ with the decomposition (Z', A', q') subject to $\phi(t, Y'_t, Z'_t, q'_t) \equiv 0$.

Standing assumption there exists a g-supersolution $\widehat{Y}$ with its decomposition $\widehat{Z}$, $\widehat{A}$, $\widehat{q}$ satisfying BSDE (4.1) and $\phi(\cdot, \widehat{Y}, \widehat{Z}, \widehat{q}) = 0$, with $\mathsf{E} \sup\limits_{0\le t\le T} |\widehat{Y}_t|^2 < +\infty$.

Theorem 4.1 Suppose that standing assumption holds. Then there exists a unique smallest g-supersolution subject to the constraint (4.1).

Proof Consider BSDE as following:

$$Y^i_t = \xi + \int_t^T g(s, Y^i_s, Z^i_s, q^i_s)\mathrm{d}s + i\int_t^T \phi(s, Y^i_s, Z^i_s, q^i_s)\mathrm{d}s - \int_t^T Z^i_s \mathrm{d}W_s - \int_t^T\int_X q^i_s(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x). \tag{4.3}$$

For any fixed i, BSDE (4.3) has a unique solution (Y^i, Z^i, q^i). By corollary 2.1, we have $Y^{i+1} > Y^i$, $i = 1, 2, \cdots$. Since there exists a solution $\widehat{Y}$, $\widehat{Z}$, $\widehat{q}$, $\widehat{A}$ satisfies the following BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, q_s)\mathrm{d}s + A_T - A_t - \int_t^T Z_s\mathrm{d}W_s - \int_t^T\int_X q_s(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x)$$

subjet to $\phi(s, Y, Z, q,) = 0$, also by comparison theorem (corollary 2.1). We have $Y^i \le \widehat{Y}$ $\forall i$, and Y^i converges to some Y increasingly with $\mathsf{E}\sup\limits_{0\le t\le T}|Y_t|^2 \le C$. Set $A^i_t = i\int_0^t \phi(s, Y^i_s, Z^i_s, q^i_s)\mathrm{d}s$, equation(4.3) can be written as following:

$$\begin{aligned}
&Y^i_t = \xi + \int_t^T g(s, Y^i_s, Z^i_s, q^i_s)\mathrm{d}s + A^i_T - A^i_t - \int_0^T Z^i_s\mathrm{d}W_s - \int_t^T\int_X q^i_s(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x),\\
&|A^i_T|^2 = \Big| - \xi + Y^i_0 - \int_0^T g(s, Y^i_s, Z^i_s, q^i_s)\mathrm{d}s + \int_0^T Z_s\mathrm{d}W_s - \int_0^T\int_X q^i_s(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x)\Big|^2,\\
&\mathsf{E}\Big|\int_0^T Z^i_s\mathrm{d}W_s\Big|^2 \le C^1_T\mathsf{E}\int_0^T |Z^i_s|^2\mathrm{d}s,\\
&\mathsf{E}\Big|\int_0^t\int_X q^i_s(x)\widetilde{N}_p(\mathrm{d}s, \mathrm{d}x)\Big|^2 \le C^2_T\mathsf{E}\int_0^T\int_X |q^i_s(x)|^2\pi(\mathrm{d}x)\mathrm{d}s,
\end{aligned}$$

we have

$$\mathsf{E}|A^i_T|^2 \le C_3 + C_T\mathsf{E}\int_0^T |Z^i_s|^2\mathrm{d}s + C_T\mathsf{E}\int_0^T \|q^i_s\|^2\mathrm{d}s.$$

By Itô formula, we can get the following result:

$$\mathsf{E}\int_0^T |Z^i_s|^2\mathrm{d}s \le C_4 + (4C_T)^{-1}\mathsf{E}|A^i_T|^2, \qquad \mathsf{E}\int_0^T \|q_s\|^2\mathrm{d}s \le C_5 + (4C_T)^{-1}\mathsf{E}|A^i_T|^2,$$

so $\exists C$ such that: $\mathsf{E}\int_0^T |Z_s^i|^2 \mathrm{d}s \le C$, $\mathsf{E}|A_T^i|^2 \le C$, $\mathsf{E}\int_0^T \|q_s^i\|^2 \mathrm{d}s \le C$ i.e. $\mathsf{E}|A_T^i|^2$, $\mathsf{E}\int_0^T |Z_s^i|^2 \mathrm{d}s$, $\mathsf{E}\int_0^T \|q_s^i\|^2 \mathrm{d}s$ are bounded uniformly, by the monotonic g-supersolution limits theorem with respect to jumps, Y^i converges to some Y and it is also a g-supersolution. that is, there exists a RCLL predictable increasing process A, and $Z \in L_{\mathcal{F}}^p([0,T], R)$, $p \in [1,2)$, $q \in L_{p,\mathcal{F}}^p([0,T], R)$, such that (Y, Z, A, q) satisfies BSDE (4.3). By the uniqueness decomposition of g-supersolution, Z, A, q is determined uniquely by Y. Notice:

$$\mathsf{E}|A_T^i|^2 = \mathsf{E}\Big|i\int_0^T \phi(s, Y_s^i, Z_s^i, q_s^i)\mathrm{d}s\Big|^2 \le C.$$

It is easy to see Y is the smallest g-supersolution subject to constraint $\phi(s, Y_s, Z_s, q_s) = 0$. #

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带跳倒向随机微分方程最小 g- 上解

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对带跳和一个右连左极的增过程作为惩罚项的倒向随机微分方程定义 g- 上解，并得到极限定理，作为其应用，在变量 (y,z,q) 受限条件下，讨论该方程的最小 g- 上解存在唯一性.

关 键 词: 倒向随机微分方程， g- 上解，受限条件.

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